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On Astumian's Paradox

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An *Astumian game* is defined by a finite Markov chain with state space S with precisely two absorbing states, the *winning* and the *losing* state. The other states are transient, one of them is the starting position. The game is said to be *losing* (respectively fair respectively *winning*) if the probability to be absorbed at the winning state is smaller than 0.5 (respectively = 0.5 respectively larger than 0.5). Astumian's paradox states that there are losing games on the same state space S a stochastic mixture of which is winning. (By "stochastic mixture" we mean that in each step one decides with the help of a fair coin whether to use the transition probabilities of the first or the second game.)

Most of our results concern *fair* games. Mixtures are systematically investigated. Rather surprisingly, the winning probability of the mixture of fair games can be arbitrarily close to zero (or to one). Even more counter-intuitive are examples of *definitely losing games* (this means that the winning probability is exactly zero) such that the winning probability of the mixture is arbitrarily close to one. We show, however, that such extreme examples are possible only if one tolerates huge running times of the game.

As a natural generalization one can also consider *arbitrary mixtures*: the fair coin is replaced by a biased one, with probability λ respectively $1 - \lambda$ one plays with the first respectively the second game. It turns out that fair games exist such that – depending on the choice of λ – the λ-mixture can be fair, losing or winning.

Keywords: Random games; Astumian's paradox; Parrondo paradox; Markov chain.

1. Introduction

The human brain is not prepared well to understand probabilistic phenomena. Even some elementary facts, e.g., in connection with the Bayes theorem, seem to be paradoxical. Since the nineties of the last century many scientists have studied a new class of paradoxes, the most famous of them is Parrondo's (see [7]). They have been invented to illustrate certain seemingly paradoxical situations on the microscopic scale. The essential feature is that one can invent certain losing games such that suitable stochastic mixtures give rise to a winning game. The paradoxical behavior can be explained by using the theory of finite Markov chains (see [5]), it has to be admitted, however, that many natural questions are still open.

R. D. Astumian has presented in [2] another example of this type (see [1] and [2] for the connection with Brownian motors). In a slightly more general setting than in [2] his construction reads as follows. One considers a Markov chain on the state space $S = \{1, \ldots, s\}$, where $s \geq 3$. Three states play a particular role: state 1 is called the *winning state*, state 2 is the *losing state*, and state 3 is the *starting position*. We assume that the winning and the losing state are absorbing and that all other states are transient[1]. To state it otherwise: The stochastic matrix $P = (p_{ij})_{i,j=1,\ldots,s}$ which defines the chain has the form

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3s} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{s1} & p_{s2} & p_{s3} & \cdots & p_{s4} \end{pmatrix} =: \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ r_{11} & r_{12} & q_{11} & \cdots & q_{1,s-2} \\ \vdots & \vdots & \vdots & & \vdots \\ r_{s-2,1} & r_{s-2,2} & q_{s-2,1} & \cdots & q_{s-2,s-2} \end{pmatrix},$$

where $Q := (p_{ij})_{i,j=3,\ldots,s}$ satisfies $\lim_{n\to\infty} Q^n = 0$.

Note that Q describes the transitions among the transient states wheras R contains the probabilities to jump from an $i \geq 3$ to the winning and the losing state. Here is a simple example: if P is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0.1 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0.5 \\ 0.5 & 0.4 & 0.1 & 0 & 0 \end{pmatrix},$$

then one has

$$R = \begin{pmatrix} 0.1 & 0 \\ 0 & 0 \\ 0.5 & 0.4 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0.9 & 0 \\ 0.2 & 0.3 & 0.5 \\ 0.1 & 0 & 0 \end{pmatrix}.$$

Such stochastic matrices will be called *Astumian matrices*, the collection of all $(s \times s)$-Astumian matrices will be denoted by $\mathcal{A}_s$.

The walk will start at state 3. Sooner or later it will be absorbed in $\{1, 2\}$, and one says that the game is won respectively lost if it is absorbed at 1 respectively 2. Let $p(P)$ be the probability to win. Astumian has constructed two such games in the case $s = 5$ with stochastic matrices P_1 and P_2 where both $p(P_1)$ and $p(P_2)$ are smaller than 0.5, but $p\big(0.5 \cdot (P_1 + P_2)\big) > 0.5$. Since $0.5 \cdot (P_1 + P_2)$ corresponds to a game where in each round one decides stochastically (with equal probability) whether to play with P_1 or with P_2 this is really a paradox of the above type.[2]

In order to be able to investigate also the quantitative aspects of this situation our approach will be slightly different from Astumian's. We will call an Astumian

[1] For the elementary notions in connection with Markov chains we refer the reader to [4].

[2] There has been a controversy (cf. [3], [9]) whether Astumian's example really behaves as claimed. It has been shown in [6] that his analysis is correct.

game *fair* (respectively *losing* respectively *winning*) if $p\,(P) = 0.5$ (respectively < 0.5 respectively > 0.5). We will consider situations where the stochastic mixture of two fair games is winning. One should note that every such pair gives rise to a situation as considered by Astumian, i.e., two losing games may generate a winning game: one only has to replace in both matrices the nonzero p_{i1} (respectively p_{i2}) by $p_{i1} - \varepsilon$ (respectively $p_{i2} + \varepsilon$) for $i = 3\ldots, s$, where ε is a small positive number (cf. Lemma 5.3 below).

Our investigation will start in Sec. 2 with a review of some facts concerning finite Markov chains. The following sections contain the discussion of certain natural questions in connection with this paradox: How large (or how small) can the winning probability $p\left(0.5 \cdot (P_1 + P_2)\right)$ of the mixture be? What about games without selfloops, i.e., chains where the p_{ii} vanish for $i \geq 3$? What happens if only chains are considered where the expected running time until absorption is bounded by a constant K? Is it possible to observe a paradox if the games are "very fair" (the winning probabiliy is 0.5 for *all* starting positions $i \in \{3, \ldots, s\}$)? Can one observe new phenomena in the case of more general mixtures (not only $0.5 \cdot (P_1 + P_2)$ but $\lambda P_1 + (1-\lambda)P_2$ for $\lambda \in [\,0, 1\,]$)? Is paradoxical behavior exceptional? Can extreme phenomena also occur if one works with Astumian matrices where the winning probability is exactly zero?

The proofs will be elementary, only linear algebra and elementary calculus are needed. It will turn out that the basic reason why paradoxical behavior is observed is nonlinearity: the probabilities and runnung times of importance here are nonlinear functions of the matrices under consideration, this yields a strange behavior when passing to stochastic mixtures.

2. The Expected Running Time and the Absorption Probabilities

Rather than to work with P it will be convenient to deal with the submatrices R and Q directly. Q governs the walk in the transient states before absorption and R contains the probabilities to jump from a transient state to state 1 or 2. It is known that elementary linear algebra suffices to calculate from R and Q the various probabilities and expected running times which are of interest. A crucial role will play the fact that $I - Q$ is invertible (I denotes the $(s-2) \times (s-2)$ identity matrix), the inverse of this matrix is often called the *fundamental matrix*.

Proposition 2.1. *Denote, for $i = 3, \ldots, s$ and $j = 1, 2$ by $\tilde{p}_{ij}$ the probability that a walk which starts at i will leave the transient states by a jump to j; in particular we have $\tilde{p}_{31} = p(P)$. Then*

$$\tilde{P} := \begin{pmatrix} \tilde{p}_{31} & \tilde{p}_{32} \\ \vdots & \vdots \\ \tilde{p}_{s1} & \tilde{p}_{s2} \end{pmatrix} = (I - Q)^{-1} R.$$

Proof. A proof can be found in Theorem 5.3 of [4], there it is also shown that $(Id - Q)^{-1}$ can be written as $Id + Q + Q^2 + \cdots$ (in analogy to the well-known formaula $(1-q)^{-1} = 1 + q + q^2 + \cdots$ for the geometric series).

One can also give a direct argument. Consider, for example, the $\tilde{p}_{i1}$, $i = 3, \ldots, s$. If we put $\tilde{p}_{11} := 1$ and $\tilde{p}_{21} := 0$ then the $\tilde{p}_{i1}$ are the probabilities to win the game when starting in state i. Now one has to observe that $\tilde{p}_{i1} = \sum_j p_{ij}\tilde{p}_{j1}$ for every state i: if one starts in i, one will land in j with probability p_{ij}, and this guarantees a winning probability of $\tilde{p}_{j1}$. Thus $P\mathbf{v} = \mathbf{v}$, where $\mathbf{v}$ is the vector $(\tilde{p}_{11}, \ldots, \tilde{p}_{s1})^\top$ (the symbol "$\top$" denotes transposition). Because of $\tilde{p}_{11} = 1$ and $\tilde{p}_{21} = 0$ this means that

$$p_{i1} + \sum_{j \geq 3} p_{ij}\tilde{p}_{j1} = \tilde{p}_{i1}$$

for $i \geq 3$. Similar remarks apply to the losing probabilities. This yields the equation $R + Q\tilde{P} = \tilde{P}$ from which our assertion follows immediately (provided one believes that $(Id - Q)^{-1}$ exists). □

Proposition 2.2. *Denote, for $i, j = 3, \ldots, s$, by w_{ij} the expected number of j-visits of a walk which starts at i before it is absorbed in 1 or 2. (By definition the starting step counts as the first visit of i so that $w_{ii} \geq 1$.) Then $W := (w_{ij})_{ij=3,\ldots,s} = (Id-Q)^{-1}$. Therefore the components of the vector $(Id-Q)^{-1}(1, 1, \ldots, 1)^\top$ are the expected values of the running times of walks starting at i for $i = 3, \ldots, s$ until the end of the game. We define the* expected running times *$\tau_i(P)$ by*

$$(\tau_3(P), \ldots, \tau_s(P))^\top := (Id - Q)^{-1}(1, 1, \ldots, 1)^\top.$$

Of particular interest will be $\tau_3(P)$ which will be called $\tau(P)$ in the sequel. This number is the expected running time of the Astumian game described by P.

Proof. This is also proved in Theorem 5.3 of [4], also this time a direct argument is possible which omits the technical details (Markov property, existence of the expected values, ...). One has to note that $W = Id + QW$, a formula which can be derived from an analysis of the first step. (If $i \neq i'$ then the first step will be to j with probability p_{ij} and thus $w_{ii'} = \sum_j p_{ij}w_{ji'}$; for $i = i'$ one has to add one since "starting" counts as a step.) □

3. How Paradoxical Can the Mixture Behave?

First note that "losing" and "winning" play a symmetric role: losing and winning probabilities change their places if we exchange the columns of R. Thus examples with high winning probabilities immediately give rise to examples with small ones. We start our investigations with the surprising fact that mixtures of fair games can have an arbitrarily high winning probability if s is not too small. More precisely:

Proposition 3.1. *Let P_1 and P_2 be Astumian matrices. We assume that the associated Astumian games are fair, that is*

$$p(P_1) = p(P_2) = 0.5.$$

(i) *If $s = 3$, then the mixture $0.5 \cdot (P_1 + P_2)$ is also fair.*
(ii) *The winning probability for $0.5 \cdot (P_1 + P_2)$ lies strictly between 0 and 1.*

Better general results are not possible: For every $\varepsilon > 0$ one can find examples of fair games on $S = \{1, 2, 3, 4\}$ such that the winning probability of the mixture is smaller than ε (and others where this probability is bigger than $1 - \varepsilon$).

Proof. (i) A game associated with a (3×3)-matrix P is fair iff $p_{31} = p_{32}$, and this equality passes from two matrices to convex combinations

(ii) Let P be such that the winning probability $p(P)$ is strictly positive. This means that there exists an $i_0 \in \{3, \ldots, s\}$ such that $p_{i_0 1} > 0$ and a path from state 3 to state i_0 is possible. If this is the case then this property will also hold for $0.5 \cdot (P + \overline{P})$ where $\overline{P}$ is any other stochastic matrix. In particular the mixture of two fair games will have a positive winning probability. A similar argument shows that this probability is necessarily strictly smaller than one.

Let a "small" number $\eta > 0$ be given, it will be specified later. We consider the following two Astumian matrices:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \eta & \eta & 1-2\eta & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \eta & \eta & 0 & 1-2\eta \\ \eta & \eta & 1-2\eta & 0 \end{pmatrix}. \tag{3.1}$$

Both are obviously fair, for the first one it is important to note that no walk starting at 3 will ever be at state 4. For the mixture one has

$$\frac{1}{2}\big(P_1 + P_2\big) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \eta & \eta & (1-2\eta)/2 & (1-2\eta)/2 \\ \eta/2 & (1+\eta)/2 & (1-2\eta)/2 & 0 \end{pmatrix}.$$

Qualitatively is clear what will happen: With overwhelming probability a walk starting in 3 will stay at 3 or move to 4. There it has a large probability to be absorbed at 2 or to return to 3. This happens again and again, it is rather unlikely to be absorbed at 1.

Of course one can make this precise. With $\alpha := (1-2\eta)/2$ it follows from Proposition 2.1 that the winning probability for the mixture is $(\eta + a\eta/2)/(1 - \alpha - \alpha^2)$. Since this expression tends to zero with $\eta \to 0$ one can find for given $\varepsilon > 0$ an η such that $p\big(0.5 \cdot (P_1 + P_2)\big) < \varepsilon$ for our games.

This proves the first part of the remaining assertion, for the second one it is, as already noted only necessary to interchange the columns of the R-matrices in P_1 and P_2. □

4. "Fast" Astumian Games

Let P be an Astumian matrix. In order do decide whether the associated game is fair the p_{ii} for $i = 3, \ldots, s$ seem to be rather unimportant. If one wants to decide whether the walk is finally absorbed in state 1 or in state 2 selfloops will not count. To state it otherwise: if one modifies the matrix such that the p_{ii} are set to zero and the proportions of the p_{ij} remain the same then the winning probability should not change.

We define $P^* = (p^*_{ij})$ by $p^*_{11} = p^*_{22} = 1$, $p^*_{ii} = 0$ for $i \geq 3$ and $p^*_{ij} = p_{ij}/(1 - p_{ii})$ for $i \geq 3$ and all j. This is again an Astumian matrix, and $p(P) = p(P^*)$ as expected.

To verify this we consider $\tilde{P}$ and $\tilde{P^*}$ for P and P^* as in the proof of Proposition 2.1. From the definition of P^* it follows immediately that the system of equations $(Id - Q)\tilde{P} = R$ transforms to the system $(Id - Q^*)\tilde{P^*} = R^*$ if the i'th row multiplied with $1/(1-p_{i+2,i+2})$ for all i. Hence both systems must have the same solution.

Thus one can save time by playing with P^* instead of playing with P. It is necessary, however, to note that the transformation $P \mapsto P^*$ is nonlinear. In particular it is in general *not* true that $0.5 \cdot (P_1^* + P_2^*) = \big(0.5 \cdot (P_1 + P_2)\big)^*$ so that when investigating Astumian's paradox one must not neglect the p_{ii}. (Cf. [3], [9] and [6] where this trap played an important role[3]).

In Astumian's original example and in the "extreme" games of the preceding section at least one of the matrices under consideration does not satisfy the condition $p_{ii} = 0$ for $i \geq 3$. Thus one might suspect that in order to observe the paradox it is necessary that some p_{ii} are strictly positive. This is not true, even for $s = 4$ (the smallest possible case) one can find "fast" paradoxical games. Here is an example: if one defines

$$P_1 = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}, \quad P_2 = \frac{1}{12}\begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 4 & 0 & 8 \\ 5 & 1 & 6 & 0 \end{pmatrix},$$

then P_1 and P_2 are fair, but $p\big(0.5 \cdot (P_1 + P_2)\big) = 16/33 \neq 0.5$. (The necessary calculations here use Proposition 2.1(i).)

Is it possible to find fair Astumian matrices with $p_{ii} = 0$ for $i \geq 3$ where $p\big(0.5 \cdot (P_1 + P_2)\big)$ is extremely small? The answer is "yes and no":

Proposition 4.1. *Call an Astumian matrix P* fast *if $p_{ii} = 0$ for $i \geq 3$.*

(i) *For $\varepsilon > 0$ there are two fair Astumian matrices P_1 and P_2 which are fast such that $p\big(0.5 \cdot (P_1 + P_2)\big) \leq \varepsilon$ (and other matrices where $p\big(0.5 \cdot (P_1 + P_2)\big) \geq 1 - \varepsilon$).*

(ii) *There exists a positive δ_0 such that*

$$p\big(0.5 \cdot (P_1 + P_2)\big) \in [\,\delta_0, 1 - \delta_0\,]$$

whenever P_1, P_2 are fair and fast Astumian 4×4-matrices.

Proof. (i) As noted at the beginning of Sec. 3 it suffices to provide matrices such that $p\big(0.5 \cdot (P_1 + P_2)\big)$ is small. Let $\eta > 0$ be given. We define

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta & \eta & 0 & 1-2\eta & 0 \\ \eta & \eta & 1-2\eta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta & \eta & 0 & 0 & 1-2\eta \\ 0 & 0 & 0 & 0 & 1 \\ \eta & \eta & 1-2\eta & 0 & 0 \end{pmatrix}.$$

[3] It has been pointed out to us by D. Abbott that also in [8] the analysis of Astumian's original game has a flaw by the same reason.

Note that P_1 and P_2 are the matrices defined in (3.1) in disguise. To avoid the diagonal one needs an extra state. As above it suffices to choose η sufficiently small in order to arrive at winning probabilities for the mixture which are ε-close to zero.

(ii) The matrices which we are going to investigate here have the form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ r_{11} & r_{12} & 0 & q_{12} \\ r_{21} & r_{22} & q_{21} & 0 \end{pmatrix}.$$

In this simple case it is easy to calculate everything explicitly: one has

$$Id - Q = \begin{pmatrix} 1 & -q_{12} \\ -q_{21} & 1 \end{pmatrix}, \; (Id - Q)^{-1} = \frac{1}{1 - q_{12}q_{21}} \begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix}$$

and consequently

$$p\,(P) = \frac{r_{11} + r_{21}q_{12}}{1 - q_{12}q_{21}}.$$

Now let two fair $P_1, P_2 \in \mathcal{A}_4$ of this type be given, we will use the superscripts "(1)" and "(2)" for their entries (the entries of the mixture will be called r_{ij} and q_{ij} as before).

Case 1: $q_{12}^{(1)} \le 1/4$.
In this case we have

$$\frac{1}{2} = p\,(P_1) = \frac{r_{11}^{(1)} + r_{21}^{(1)}q_{12}^{(1)}}{1 - q_{12}^{(1)}q_{21}^{(1)}} \le \frac{r_{11}^{(1)} + 0.25}{0.75},$$

i.e., $r_{11}^{(1)} \ge 1/8$. Then one will have $p\,(P) \ge r_{11} \ge 1/16$ for every $P = 0.5 \cdot (P_1 + \overline{P})$ with $\overline{P} \in \mathcal{A}_4$. (It is not necessary here that $\overline{P}$ is fair or fast.)

Case 2: $1/4 \le q_{12}^{(1)} \le 1/2$.
This time we observe that

$$\frac{1}{2} = p\,(P_1) = \frac{r_{11}^{(1)} + r_{21}^{(1)}q_{12}^{(1)}}{1 - q_{12}^{(1)}q_{21}^{(1)}} \le \frac{r_{11}^{(1)} + r_{21}^{(1)}}{1 - 0.5}.$$

Consequently $r_{11}^{(1)} + r_{21}^{(1)} \ge 1/4$ so that $r_{11}^{(1)} \ge 1/8$ or $r_{21}^{(1)} \ge 1/8$.

If $r_{11}^{(1)} \ge 1/8$ holds we argue as in case 1 above to conclude that $p\,(P) \ge 1/16$ for every mixture $P = 0.5 \cdot (P_1 + \overline{P})$. Suppose that $r_{21}^{(1)} \ge 1/8$. Then, if $P = 0.5 \cdot (P_1 + \overline{P})$ is a mixture of P_1 with some $\overline{P}$, we will have $r_{21} \ge 1/16$ and $q_{12} \ge 1/8$ so that $p\,(P) \ge 1/128$.

Case 3: $q_{12}^{(1)} \ge 1/2$ and $q_{21}^{(1)} \le 1/2$.
Under this assumption one can prove similarly as in case 2 that $p\,(P) \ge 1/64$ for every mixture P.

The same analysis applies if the conditions in case 1, case 2 or case 3 are valid for P_2. Thus there only remains

Case 4: $q^{(1)}_{21},\ q^{(2)}_{21},\ q^{(1)}_{12},\ q^{(2)}_{12} \geq 1/2$.

We put $\overline{q}^{(1)}_{12} := 1 -\ q^{(1)}_{12},\ \overline{q}^{(2)}_{12} := 1 -\ q^{(2)}_{12}$ and $\overline{q}^{(1)}_{21} := 1 -\ q^{(1)}_{21},\ \overline{q}^{(2)}_{21} := 1 -\ q^{(2)}_{21}$. Then

$$\frac{1}{2} = \frac{r^{(1)}_{11} + r^{(1)}_{21} q^{(1)}_{12}}{1 - q^{(1)}_{12} q^{(1)}_{21}} = \frac{r^{(1)}_{11} + r^{(1)}_{21} q^{(1)}_{12}}{\overline{q}^{(1)}_{12} + \overline{q}^{(1)}_{21} - \overline{q}^{(1)}_{12}\overline{q}^{(1)}_{21}}.$$

If one observes that $\alpha + \beta - \alpha\beta \geq (\alpha + \beta)/2$ for $\alpha, \beta \in [\,0, 1\,]$ one can continue this estimate with

$$\ldots \leq 2\frac{r^{(1)}_{11} + r^{(1)}_{21} q^{(1)}_{12}}{\overline{q}^{(1)}_{12} + \overline{q}^{(1)}_{21}} \leq 2\frac{r^{(1)}_{11} + r^{(1)}_{21}}{\overline{q}^{(1)}_{12} + \overline{q}^{(1)}_{21}}$$

so that

$$\overline{q}^{(1)}_{12} + \overline{q}^{(1)}_{21} \leq 4(r^{(1)}_{11} + r^{(1)}_{21}).$$

Similarly one obtains

$$\overline{q}^{(2)}_{12} + \overline{q}^{(2)}_{21} \leq 4(r^{(2)}_{11} + r^{(2)}_{21}).$$

It follows that

$$\overline{q}_{12} + \overline{q}_{21} \leq 4(r_{11} + r_{21}),$$

where $r_{ij} := 0.5 \cdot (r^{(1)}_{ij} + r^{(2)}_{ij})$, $\overline{q}_{ij} := 1 - q_{ij}$ (with $q_{ij} := 0.5 \cdot (q^{(1)}_{ij} + q^{(2)}_{ij})$). Consequently,

$$\begin{aligned} p\big(0.5 \cdot (P_1 + P_2)\big) &= \frac{r_{11} + r_{21}q_{12}}{1 - q_{12}q_{21}} \\ &= \frac{r_{11} + r_{21}q_{12}}{\overline{q}_{12} + \overline{q}_{21} - \overline{q}_{12}\overline{q}_{21}} \\ &\geq \frac{1}{4}\frac{r_{11} + r_{21}}{\overline{q}_{12} + \overline{q}_{21}} \\ &\geq \frac{1}{16}. \end{aligned}$$

This completes the proof, we have seen that $\delta_0 = 1/128$ is an admissible choice. □

Remarks: 1. The preceding analysis shows why the q_{12}-value of the Astumian matrix P_2 in the proof of Proposition 3.1 has to be close to one.

2. Note that the essential idea in the proof of (ii) was to transform the *nonlinear* condition $p\,(P) = 1/2$ into a family of inequalities between *linear* combinations of the r_{ij} and the q_{ij}. In this way it was possible to derive properties of convex combinations of two Astumian matrices from the fairness assumption.

5. The Paradox in the Presence of Bounded Running Times

For the examples of fair games in section 3 which give rise to extremely small probabilities for $0.5 \cdot (P_1 + P_2)$ the matrices $(Id - Q)^{-1}$ have large entries. By Proposition 2.2 this means that the expected running time of the associated games is a huge number. Is this essential to observe the "extreme" paradox? The answer is "yes":

Proposition 5.1. *Let $s \in \mathbb{N}$ and $K > 0$ be given. There exists a positive constant δ_0 (depending on s and K) such that the following holds: whenever P_1 is a fair Astumian $(s \times s)$-matrix such that the expected running time of the game is bounded by K, then the winning probability of a mixture $0.5 \times (P_1 + P_2)$ of P_1 with any other Astumian $(s \times s)$-matrix lies in $[\,\delta_0, 1 - \delta_0\,]$.*
Thus very small winning probabilities for mixtures of fair games are only possible if the running time is huge for both games.

For the proof we need some preliminary results, a number $K > 0$ and an integer s will be fixed.

Lemma 5.2. *Let $P = (p_{ij})$ be a fair Astumian matrix such that the running time $\tau(P)$ is bounded by K. Then there exists an $i \in \{3, \ldots, s\}$ such that $p_{i1} \geq 1/2K(s-2)$.*
To state it otherwise: if all p_{i1} satisfy $p_{i1} < \delta_1 := 1/2K(s-2)$, then P cannot be fair unless $\tau(P) > K$.

Proof. If $\tau(P) \leq K$, then, by Proposition 2.2, all entries in the first row of $(Id - Q)^{-1}$ are bounded by K. It now follows from Proposition 2.1 that $0.5 = p(P) \leq K \sum_{i=3}^{s} p_{i1} \leq K(s-2) \max_{i=3,\ldots,s} p_{i1}$ and this proves the claim. □

Lemma 5.3. *Let $P \in \mathcal{A}_s$ and $i_0, j_0 \in \{3, \ldots, s\}$; we suppose that $p_{i_0 j_0} > 0$. Let $\varepsilon \in [\,0, p_{i_0 j_0}\,]$ be given. We define an Astumian matrix $P_\varepsilon = (p^\varepsilon_{ij})$ by*

$$p^\varepsilon_{i_0 1} := p_{i_0 1} + \varepsilon,\ p^\varepsilon_{i_0 j_0} := p_{i_0 j_0} - \varepsilon$$

and $p^\varepsilon_{ij} := p_{ij}$ for the other i, j.
Then $p(P_\varepsilon) \geq p(P)$ and $\tau_i(P_\varepsilon) \leq \tau_i(P)$ for $i = 3, \ldots, s$.

+ **Remark:** Intuitively it is clear that these inequalities should hold: if one replaces a step from i_0 to j_0 – from where one might win or not – by a step to the winning state the winning probability should be higher and the game should be shorter. It is surprisingly complicated to verify this obvious fact.

Proof. We denote by R_ε and Q_ε the R- and Q-submatrix of P_ε as in Proposition 2.1, and similarly we will write $\tilde{P}_\varepsilon$ for the matrix of winning and losing probabilities when starting at the $i = 3, \ldots, s$.

By Proposition 2.1 we know that $\tilde{P}_\varepsilon = (Id - Q_\varepsilon)^{-1} R_\varepsilon$, or $\tilde{P}_\varepsilon - Q_\varepsilon \tilde{P}_\varepsilon = R_\varepsilon$. This holds for $\varepsilon \in [\,0, p_{i_0 j_0}\,]$. By differentiation we get

$$\tilde{P}'_\varepsilon - Q'_\varepsilon \tilde{P}_\varepsilon - Q_\varepsilon \tilde{P}'_\varepsilon = R'_\varepsilon,$$

where the derivative of a matrix function $\varepsilon \mapsto f_\varepsilon$ is denoted by f'_ε; it is meant componentwise.
$Q'_\varepsilon \tilde{P}_\varepsilon$ and R'_ε are easily determined, it follows that

$$\tilde{P}'_\varepsilon = (Id - Q_\varepsilon)^{-1} M,$$

where M is an $(s-2) \times 2$-matrix where the only nonvanishing row is the i_0'th with entries $1 - \tilde{p}_{j_0 1}, -\tilde{p}_{j_0 2}$. Since $1 - \tilde{p}_{j_0 1}$ and all entries of $(Id - Q_\varepsilon)^{-1} = Id + Q_\varepsilon + Q_\varepsilon^2 + \cdots$

are nonnegative it follows that $(\tilde{p}^{\varepsilon}_{i1})' \geq 0$ for $i = 3, \ldots, s$. Therefore the function $\varepsilon \mapsto \tilde{p}^{\varepsilon}_{i1}$ is increasing.

For the remaining part of the proof we argue similarly. We put $F_\varepsilon := (Id - Q_\varepsilon)^{-1}$, and from $F_\varepsilon - Q_\varepsilon F_\varepsilon = Id$ we conclude that $(Id - Q_\varepsilon)F'_\varepsilon = Q'_\varepsilon F_\varepsilon$. Thus all entries of F'_ε must be nonpositive since this matrix equals $(Id - Q_\varepsilon)^{-1} Q'_\varepsilon$, where $(Id - Q_\varepsilon)^{-1}$ is nonnegative and Q'_ε is nonpositive. □

Lemma 5.4. *There is a $\delta_2 > 0$ with the following property: whenever $P = (p_{ij})$ is a fair $(s \times s)$-Astumian matrix such that $p_{31}, p_{41} \ldots, p_{s_0 1} \leq \delta_2$ and $p_{ij} \leq \delta_2$ for $3 \leq i \leq s_0$ and $s_0 < j \leq s$ for a suitable $s_0 \leq s$, then $\tau(P) > K$.*

Remark: The assertion of the lemma is the formal version of the following plausible result: A large running time is needed if one wants to have a fair game under the condition that there is only a tiny probability to win directly from a state in $\{3, \ldots, s_0\}$ and also only a small chance to leave this subset.

Proof. Define $P^* := (p^*_{ij})$ by

$$\begin{aligned} p^*_{i1} &:= p_{i1} + p_{i,s_0+1} + \cdots + p_{is}, \\ p^*_{ij} &:= 0, \end{aligned}$$

for $i = 3, \ldots, s_0$ and $j = s_0 + 1, \ldots, s$ (and $p^*_{ij} := p_{ij}$ for the other i, j). In P^* there is no chance to leave $\{3, \ldots, s_0\}$ and thus we can work with the $(s_0 \times s_0)$-matrix $P^{**} := (p^*_{ij})_{i,j=1,\ldots,s_0}$ instead of working with P^*.

But for P^{**} the assumptions of Lemma 5.3 are satisfied provided we know that the p^*_{i1} are bounded by δ_1 for $i = 3, \ldots, i_0$. This will be true if we define δ_2 by $\delta_2 := \delta_1/s$. □

Lemma 5.5. *Let s and K be given as before. There is a $\delta_3 > 0$ with the following property: for every fair Astumian $(s \times s)$-matrix P such that $\tau(P) \leq K$ one has $p_{31} \geq \delta_3$ or there are an $l \leq s - 2$ and indices $i_1, \ldots, i_l \in \{3, \ldots, s\}$ such that*

$$p_{3,i_1},\, p_{i_1,i_2},\, p_{i_2,i_3}, \ldots, p_{i_{l-1},i_l},\, p_{i_l,1} \geq \delta_3.$$

Proof. Let a fair P be given. With the usual notation the boundedness assumption means that $\sum_{i=1}^{s-2} \kappa_i \leq K$, where $\kappa_1, \ldots, \kappa_{s-2}$ are the entries of the first row of $(Id - Q)^{-1}$ (cf. Proposition 2.2). In particular all κ_i are bounded by K.

But, by Proposition 2.1, $p(P) = \sum_{i=1}^{s-2} r_{i1}\kappa_i = 1/2$ so that there must exist indices $i \geq 3$ with

$$r_{i1} \geq \delta_1 = \frac{1}{2K(s-2)}.$$

Let δ_3 be the minimum of δ_1 and the number δ_2 from Lemma 5.4. It might happen that $p_{31} \geq \delta_3$, then we are done. If this is not the case we may assume (without loss of generality) that the i with $r_{i1} \geq \delta_3$ are the $i = s_1 + 1, \ldots, s$ for a suitable

$s_1 < s$. This means that P has the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \circ & * & p_{33} & \cdots & p_{3,s_1} & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \circ & * & p_{s_1,3} & \cdots & p_{s_1,s_1} & * & \cdots & * \\ \diamond & * & * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \diamond & * & * & \cdots & * & * & \cdots & * \end{pmatrix},$$

where the entries $\diamond$ (respectively $\circ$) are bounded from below (respectively from above) by δ_3.

Now suppose that all p_{ij} for $i = 3, \ldots, s_1$ and $j = s_1+1, \ldots, s$ are bounded from above by δ_3. This would lead to a contradiction since $\delta_3 \le \delta_2$ so that by Lemma 5.4 the running time would be too large.

Thus there are certain $i \in \{3, \ldots, s_1\}$ for which there exists a $j > s_1$ with $p_{ij} \ge \delta_3$. Suppose first that $i = 3$ is admissible: there is an $i_1 > s_1$ with $p_{3,i_1} \ge \delta_3$. Then the proof of the lemma is complete since $p_{i_3,1} \ge \delta_3$ by construction.

If this is not the case we arrange the states $i \in \{3, \ldots, s_1\}$ for which a j with "large" p_{ij} exists such that they constitute the set $\{s_2 + 1, \ldots, s_1\}$ for some s_2. Then P looks as follows:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \circ & * & p_{33} & \cdots & p_{3,s_2} & * & \cdots & * & \circ & \cdots & \circ \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \circ & * & p_{s_2,3} & \cdots & p_{s_2,s_2} & * & \cdots & * & \circ & \cdots & \circ \\ \circ & * & * & \cdots & * & * & \cdots & * & \sharp & \cdots & \sharp \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \circ & * & * & \cdots & * & * & \cdots & * & \sharp & \cdots & \sharp \\ \diamond & * & * & \cdots & * & * & \cdots & * & * & \vdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \diamond & * & * & \cdots & * & * & \cdots & * & * & \vdots & * \end{pmatrix}.$$

Here the $\diamond$ and the $\circ$ have the same meaning as before, and in every line with $\sharp$-entries there is somewhere an element among the $\sharp$'s which is bounded from below by δ_3.

Can it happen that $p_{ij} \le \delta_3$ for all $i = 3, \ldots, s_2$ and all $j = s_1 + 1, \ldots, s_2$? Surely not, this would violate Lemma 5.4. Thus we can continue our construction, sooner or later (at most after $s-2$ steps) we will arrive at a situation where one can find a sequence of jumps each of which having a probabibility of at least δ_3 which end in the winning state 1. $\square$

Proof of Proposition 5.1 Suppose that P_1 is fair and that the expected running time $\tau(P)$ is bounded by K. Further let P_2 be another Astumian matrix, by $P =$

$(p_{ij}) = 0.5 \times (P_1 + P_2)$ we denote the mixture. As before we will distinguish the entries of the matrices P_1 and P_2 by superscripts.

By the last lemma there exist a $\delta_3 > 0$ (which does not depend on P_1) and indices $i_1, \ldots, i_l \in \{3, \ldots, s\}$ such that

$$p^{(1)}_{1,i_1},\, p^{(1)}_{i_1,i_2},\, p^{(1)}_{i_2,i_3}, \ldots,\, p^{(1)}_{i_{l-1},i_l},\, p^{(1)}_{i_l,1} \geq \delta_3.$$

It follows that

$$p_{1,i_1},\, p_{i_1,i_2},\, p_{i_2,i_3}, \ldots, p_{i_{l-1},i_l},\, p_{i_l,1} \geq \delta_3/2$$

which means that in the game associated with P the probability to win is at least

$$p_{1,i_1} \cdot p_{i_1,i_2} \cdot p_{i_2,i_3} \cdots p_{i_{l-1},i_l} \cdot p_{i_l,1} \geq \left(\frac{\delta_3}{2}\right)^{l+1} \geq \left(\frac{\delta_3}{2}\right)^{s-1}.$$

If already $p^{(1)}_{31} \geq \delta_3$ should hold the argument is much simpler: then $p_{31} \geq \delta_3/2$ so that $p\,(P) \geq \delta_3/2$.

This shows that our assertion is true with $\delta_0 := (\delta_3/2)^{s-1}$. □

Remark: We have assumed that the expected running time *when starting the game in state* 1 is bounded. One can give a much simpler proof under the assumption that this holds for every starting position: We consider the collection $\mathcal{A}^{f,K}_s$ of fair $s \times s$ Astumian matrices P such that *all* $\tau_i(P)$ are bounded by K, and we want to show that the numbers $p\,\big(0.5 \times (P_1 + P_2)\big)$ are bounded away from 0 and from 1.

$\mathcal{A}^{f,K}_s$ is a subset of the set $\mathcal{S}$ of the stochastic $s \times s$-matrices, the topology will be the euclidean metric of $\mathbb{R}^{s^2}$. Since $Q \mapsto (Id - Q)^{-1}$ is a continuous function on the set of substochastic matrices Q with $Q^n \to 0$ it follows that $\mathcal{A}^{f,K}_s$ is a closed and thus compact subset of $\mathcal{S}$.

Now Proposition 3.1(ii) comes into play. If $\phi : \mathcal{A}^{f,K}_s \times \mathcal{A}^{f,K}_s \to \mathbb{R}$ denotes the map which associates with a pair (P_1, P_2) the winning probability of $0.5 \times (P_1 + P_2)$, then the range of ϕ lies by this proposition in $]\,0, 1\,[$. But ϕ continuous, and since the domain is compact its range must lie in some interval $[\,\delta_0, 1 - \delta_0\,]$.

6. Is there a "Strong" Paradox?

Up to now the starting point of our games always was state 3, and we have called a game fair if the winning probability is 0.5. Call a game *very fair* if the winning probability is 0.5 regardless of the starting position: the player may choose any $i \in \{3, \ldots, s\}$, the chances to lose or to win always balance.

Is there an Astumian paradox for very fair games? The answer is "no":

Proposition 6.1. *Suppose that* P_1 *and* P_2 *are very fair. Then* $0.5 \cdot (P_1 + P_2)$ *also has this property.*

Proof. We adopt the notation of section 2, the Q- and R-submatrices of P_1, P_2 and $0.5 \cdot (P_1 + P_2)$ are called Q_1, R_1, Q_2, R_2 and Q, R, respectively.
By assumption (and by Proposition 2.1) we know that

$$(Id - Q_1)Z = R_1,\ (Id - Q_1)Z = R_1,$$

where Z is the $\big((s-2)\times 2\big)$-matrix all entries of which equal 0.5. But then also $(Id-Q)Z=R$ holds, hence the result.

(The proof shows more than claimed: If P_1 and P_2 have the same "winning matrix", i.e., $\tilde{P} := \tilde{P_1} = \tilde{P_2}$, then $\tilde{P}$ is also the winning matrix for $0.5\cdot(P_1+P_2)$.) □

7. Arbitrary Mixtures

Now we are going to investigate arbitrary mixtures of two Astumian matrices: for $\lambda \in [\,0,1\,]$ we consider $\lambda P_1 + (1-\lambda)P_2$. This matrix corresponds to an Astumian game where in each round one decides with a biased coin whether to play with P_1 or with P_2: with probability λ (respectively $1-\lambda$) one plays with P_1 (respectively P_2).

Suppose that both P_1 and P_2 are fair, what can be said about $\lambda P_1 + (1-\lambda)P_2$? We will call $p(\lambda)$ the winning probability of the mixed game[4]. Then $p : [\,0,1\,] \to \mathbb{R}$ is a continuous function such that $p(0) = p(1) = 0.5$, and we know that there are cases where $p(0.5)$ is close to zero or close to one. But one can observe even more spectacular phenomena if s is not too small:

Proposition 7.1.

(i) *If there are more than $s-4$ different $\lambda \in \,]\,0,1\,[$ such that $p(\lambda)=0.5$, then $p(\lambda)=0.5$ for all λ. This means that all mixtures are fair in this case.*

(ii) *Suppose that $s=4$. Then there are only three possibilities:*

- *All λ-mixtures with $\lambda \in \,]\,0,1\,[$ are losing;*
- *all λ-mixtures with $\lambda \in \,]\,0,1\,[$ are winning;*
- *all λ-mixtures with $\lambda \in \,]\,0,1\,[$ are fair.*

(iii) *For $s \geq 5$ it might happen that there are $\lambda_1, \lambda_2, \lambda_3$ in $]\,0,1\,[$ such that the λ_1-mixture is losing, the λ_2-mixture is winning and the λ_3-mixture is fair.*

Proof. (i) Let us have a closer look at $\big(Id - (\lambda Q_1 + (1-\lambda)Q_2)\big)^{-1}$. This is a $(s-2)\times(s-2)$-matrix each entry of which is a rational function such that the nominator has (at most) degree $s-3$ and the denominator has (at most) degree $s-2$; this follows at once from Cramer's rule.

By Proposition 2.1 $p(\lambda)$ is the matrix product of the first row of the matrix $\big(Id - (\lambda Q_1 + (1-\lambda)Q_2)\big)^{-1}$ with the first column of $\lambda R_1 + (1-\lambda)R_2$, therefore one has $p(\lambda) = A(\lambda)/B(\lambda)$ where A and B are polynomials of degree at most $s-2$. We know that $A(0) = 0.5B(0)$ and $A(1) = 0.5B(1)$ since P_1 and P_2 are fair. Thus $C := A - 0.5\cdot B$ has the two roots 0 and 1. Since the degree of C is bounded by $s-2$ it follows that there can exist at most $s-4$ further roots unless C is identically zero. This proves the claim.

(ii) follows immediately from (i).

[4]As usual the game starts at state 3.

(iii) For the following two matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0,01035 & 0,00464 & 0,37869 & 0,23085 & 0,37545 \\ 0,00853 & 0,00646 & 0,76454 & 0,21577 & 0,00468 \\ 0,00209 & 0,01290 & 0,38094 & 0,21694 & 0,38710 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0,0070 & 0,0079 & 0,3532 & 0,4049 & 0,2267 \\ 0,0090 & 0,0059 & 0,1326 & 0,4275 & 0,4247 \\ 0,0058 & 0,0091 & 0,5196 & 0,3572 & 0,1081 \end{pmatrix}$$

one obtains a p-function for which the values at $\lambda = k/10$, $k = 0, \ldots, 10$ are sometimes smaller and sometimes larger than 0.5:

λ :	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$p(\lambda)$:	0,5000	0,4979	0,4971	0,4972	0,4980	0,4992	0,5004	0,5015	0,5019	0,5016	0,5000

Thus one can find losing, winning and fair games among the mixtures. (It follows by interpolation that the λ with $p(\lambda) = 0.5$ is close to 0.566.) The example has been found with the help of a Delphi program which generates random Astumian matrices.

8. Is Paradoxical Behavior Exceptional?

There is a number of sufficient conditions which imply that Astumian's paradox will not occur. One has been mentioned in Sec. 6, another is the assumption that $Q_1 = Q_2$ holds. In this section we will give a heuristic argument that paradoxical behavior is rather the rule than the exception.

Suppose that P_1 and P_2 are two fair Astumian matrices which have been chosen by some random procedure from the set of all such matrices. Then, as it has been observed in the proof of Proposition 7.1, the associated p-function is – as a function of λ – the quotient of two polynomials A, B of degree at most $s-2$. It is rather unlikely that A and B are proportional so that it will be very exceptional that p is a constant function. Therefore there is an overwhelming probability that among the mixtures one can find winning or losing games.

In fact one can show that in the collection of pairs of Astumian matrices which will be provided with the topology of $\mathbb{R}^{2s^2}$ the subsets of pairs where the p-function is not constant is an open dense subset. The proof is elementary but rather lengthy, it is therefore omitted here.

9. Mixtures of Definitely Losing Games Might be Almost Definitely Winning

Up to now we have investigated mixtures of fair games. This approach was chosen for the sake of symmetry.

In this final section we turn from fair games to more "extreme" situations: an Astumian matrix P and the associated game are called *definitely losing* (respectively *definitely winning*) if $p(P) = 0$ (respectively $p(P) = 1$) holds.

Mixtures of such games can behave rather surprisingly:

Proposition 9.1.

(i) *If $s = 3$ and P_1, P_2 are definitely losing, then also $0.5 \cdot (P_1 + P_2)$ has this property. The same assertion holds for definitely winning games.*

(ii) *For $s \geq 4$ mixtures of definitely losing games can be almost definitely winning. More precisely: for $\varepsilon > 0$ there are definitely losing P_1, P_2 such that*

$$p\big(0.5 \cdot (P_1 + P_2)\big) \geq 1 - \varepsilon.$$

Similarly one can find definitely winning games for which the mixture is almost definitely losing.

Proof. (i) This is obvious: an Astumian matrix $P = (p_{ij})_{i,j=1.2.3}$ is definitely losing iff $p_{31} = 0$, and this property passes to mixtures.

(ii) As before it suffices to treat the first part. Let a small positive number η be given. We define

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \eta & 1-\eta & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \eta & 1-\eta & 0 \end{pmatrix}.$$

Driven by P_1 (respectively P_2) a typical walk will stay a long time at state 3 (respectively will move immediately to state 4 where it is caught for many steps) until it is finally absorbed by the losing state 2.

The mixture looks like this:

$$P := \frac{1}{2}\big(P_1 + P_2\big) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \eta/2 & (1-\eta)/2 & 1/2 \\ 1/2 & \eta/2 & (1-\eta)/2 & 0 \end{pmatrix}.$$

Now there is a fifty percent chance to move to state 4 where it is very likely to win the game; with nearly fifty percent one has another chance.

Of course this can also be made precise by using the results of Propostion 2.1. It turns out that $p(P) = 1/(1 + 3\eta)$, and this number is arbitrarily close to one if η is sufficiently small.

(*Remark:* By the same argument as in the proof of Proposition 3.1(ii) it follows that $p(P) = 1$ is not possible for definitely losing games.) $\square$

It should be noted that many of our results for mixtures of fair games have an analogue for mixtures of definitely losing (or winning) games or more general situations. As a sample result we state

Proposition 9.2. *Let $K, \pi_0 > 0$ and an integer $s \geq 3$ be given. Then there exists a positive $\delta_0 = \delta_0(K, \pi_0, s)$ such that the following holds: whenever $P_1 \in \mathcal{A}_s$ is an Astumian matrix such that $\tau(P_1) \leq K$ and $p(P_1) \geq \pi_0$, then $p\big(0.5 \cdot (P_1 + P_2)\big) \geq \delta_0$ for any $P \in \mathcal{A}_s$.*

Proof. It is only necessary to repeat the above argument in the proof of Proposition 5.1. The inequality $p_{i1} \geq 1/2K(s-2)$ (for a suitable $i \geq 3$) from Lemma 5.2 is now replaced by $p_{i1} \geq \pi_0/K(s-2)$, the following steps are the same. □

10. Summary

We have shown how linear algebra can be used to investigate Astumian's paradox and to discuss examples. The main results of the present paper are:

- The mixture $0.5 \cdot (P_1 + P_2)$ of two fair games P_1, P_2 can have a winning probability which is arbitrarily close to zero (or to one).

- This can also happen if the games are "fast" (= no selfloops) provided there are at least five states.

- No extreme paradoxical situations (i.e., arbitrarily low or arbitrarily high winning probabilities for mixtures of fair games) will occur if the expected running time of the games under consideration is bounded by a constant K.

- If two games are "very fair" (i.e., the winning probability associated with any starting position is 0.5) then the same is true for the mixtures. The reason is that when investigating very fair games one deals with expressions which are stable with respect to convex combinations.

- There are fair games which, depending on the mixing probabilities, give rise to losing, fair and winning games.

- Paradoxical situations abound, they are rather the rule than the exception.

- Even games with winning probability zero can be mixed such that "almost winning" games result.

A number of questions remain open. For example, there are several results which guarantee the existence of positive constants with certain properties (e.g., Proposition 4.1 or Proposition 5.1). The proofs provide numbers which are certainly too small. What are the optimal values?

Also it would be desirable to understand better the strange behavior in connection with arbitrary mixtures. E.g.: are there, for given $\varepsilon > 0$, two fair Astumian matrices for which one can find $\lambda, \lambda' \in \,]\,0, 1\,[$ such that

$$p\big(\lambda P_1 + (1-\lambda) P_2\big) \leq \varepsilon,\ p\big(\lambda' P_1 + (1-\lambda') P_2\big) \geq 1 - \varepsilon\ ?$$

And can this happen arbitrarily often?

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