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A note on the preceding paper by Piotrowski and Sladkowski and the response of Astumian

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We point out that the analysis of Piotrowski and Sladkowski of the paradox described by Astumian has a flaw since they consider a different game.

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It is due to Parrondo that certain paradoxical phenomena in connection with stochastic games have attracted the attention of many scientists in different fields during the last years. (For a description of Parrondo's paradox see [4].)

In [1] Astumian has provided another paradox: he describes two losing games a stochastic mixture of which gives rise to a winning game. Piotrowski and Sladkowski state in [5] that his analysis is wrong and that a paradoxical behaviour cannot be observed in this situation. As an answer Astumian claims in [2] that this criticism is not justified. Let me explain why Astumian is right.

The aim of study here are Markov chains on the state space $\{1, 2, 3, 4, 5\}$. As usual we will describe such chains by stochastic matrices $P = (p_{ij})_{i,j=1,\ldots,5}$. In the present situation the states 1 and 5 are absorbing. A walk starts at state 3, and if it is absorbed at 5 (resp. at 1) the game is won (resp. lost).

Astumian's game 1 (AG1) is given by the following matrix:

$$P_1 = \frac{1}{36} \begin{pmatrix} 36 & 0 & 0 & 0 & 0 \\ 4 & 24 & 8 & 0 & 0 \\ 0 & 5 & 29 & 2 & 0 \\ 0 & 0 & 4 & 24 & 8 \\ 0 & 0 & 0 & 0 & 36 \end{pmatrix}.$$

With the help of elementary linear algebra one can determine the winning and losing probabilities[1]. The idea is simple. First note that in order to calculate these probabilities the steps $i \to i$ for the transient i can be neglected provided the transition probabilities $i \to j$ (for $j \neq i$) are adjusted properly: one has to replace p_{ii} by 0 and p_{ij} by $p_{ij}/(1-p_{ii})$ for $i = 2, 3, 4$ and $j \neq i$. Call the new matrix *the reduced matrix* P'. In the case of AG1 one gets

$$P_1' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 5/7 & 0 & 2/7 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote, for $i = 1, 2, 3, 4, 5$, by w_i the probability that the walk will be absorbed in state 5 if it starts in i (so that w_3 is the probability to win the game). Then, as a consequence of the Markov property, one has $w_i = \sum_j p_{ij} w_j$ for every i. Together with $w_1 = 0$ and $w_5 = 1$ this allows the calculation of the w_i in closed form. If one works with the reduced matrix P' one obtains

$$w_3 = \frac{p_{34}' p_{45}'}{1 - p_{23}' p_{32}' - p_{34}' p_{43}'}.$$

For the game AG1 this number equals $w_3^{(1)} := 4/9$, i.e., AG1 is a losing game.

Similarly one can analyze Astumian's game 2 (AG2):

$$P_2 = \frac{1}{36}\begin{pmatrix} 36 & 0 & 0 & 0 & 0 \\ 5 & 29 & 2 & 0 & 0 \\ 0 & 4 & 24 & 8 & 0 \\ 0 & 0 & 5 & 29 & 2 \\ 0 & 0 & 0 & 0 & 36 \end{pmatrix}, \quad P_2' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5/7 & 0 & 2/7 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 5/7 & 0 & 2/7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and for the corresponding winning probability $w_3^{(2)}$ one obtains again 4/9: also AG2 is a losing game.

Now comes the crucial step. Astumian considers a third game which is derived from AG1 and AG2 by switching between these two games randomly (with equal probability). The correct analysis is as follows:

- The new game is governed by the stochastic matrix $P_3 := (P_1 + P_2)/2$.
- For the calcualtion of the winning probability it suffices to consider the reduced matrix P_3'.
- The winning probabilities can simply be derived from the entries of the matrix P_3'.

[1] Much more can be said: see Chapter 5 in [3].

This is done in [2]: Astumian correctly calculates

$$P_3 = \frac{1}{72} \begin{pmatrix} 72 & 0 & 0 & 0 & 0 \\ 9 & 53 & 10 & 0 & 0 \\ 0 & 9 & 53 & 10 & 0 \\ 0 & 0 & 9 & 53 & 10 \\ 0 & 0 & 0 & 0 & 72 \end{pmatrix},$$

and from the correct P_3' one gets $w_3^{(3)} = 100/181$. This means that random mixing has given rise to a winning game.

In [5] instead the authors work with

$$(P_1' + P_2')/2 = \frac{1}{21} \begin{pmatrix} 21 & 0 & 0 & 0 & 0 \\ 11 & 0 & 10 & 0 & 0 \\ 0 & 11 & 0 & 10 & 0 \\ 0 & 0 & 11 & 0 & 10 \\ 0 & 0 & 0 & 0 & 21 \end{pmatrix},$$

and from this matrix one really obtains the (false) winning probability 100/221: also the mixture seems to be a losing game.

Summing up, the error in [5] is that it is tacitly assumed there that

$$\left(\frac{1}{2}(P_1 + P_2)\right)' = \frac{1}{2}(P_1' + P_2').$$

But this is in general – and in particular in the case under consideration – *not* true. This fact is paradoxical, but traps of that kind are rather common in probability.

References

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