Banach J. Math. Anal. 10 (2016), no. 3, 482–494
http://dx.doi.org/10.1215/17358787-3599741
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

ON CERTAIN UNIFORMLY OPEN MULTILINEAR MAPPINGS

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Communicated by G. Popescu

Abstract. We obtain two results stating the uniform openness of bilinear operators and multilinear functionals. The first result deals with Banach spaces $L^p := L^p_{\mathbb{K}}$ (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and pointwise multiplication from $L^p \times L^q$ to L^r (where $1/p + 1/q = 1/r$). The second result is concerned with the nontrivial n-linear functionals from the product $X_1 \times \cdots \times X_n$ of normed spaces (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) to the field $\mathbb{K}$.

1. Introduction

Assume that X and Y are metric spaces. A mapping $T\colon X \to Y$ is called *open* if it sends every open set in X to an open set in Y. For $x_0 \in X$ we say that T is *(locally) open* at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $B(T(x_0), \delta) \subset T[B(x_0, \varepsilon)]$ (see [8, Section 13, XIII]). Here $B(z, r)$ stands for the open ball with center z and radius $r > 0$ in a given space. Note that in the above definition one can use closed balls instead of open balls. Obviously, T is open if and only if it is open at every $x \in X$. Note that the notion of local openness is still interesting for various mappings in topology (see, e.g., [5]).

In [1], T is called *uniformly open* if T is locally open at every x and, for given ε, the δ can be chosen such that it does not depend on x. An open mapping need not be uniformly open; the function arctan serves as an easy example.

The classical Banach open mapping principle states that every linear continuous surjection between two Banach spaces is an open mapping. In fact, it is

Received Jul. 13, 2015; Accepted Oct. 2, 2015.
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2010 *Mathematics Subject Classification.* Primary 46B25; Secondary 47A06, 47A07, 54C10.
Keywords. multiplication, uniformly open mapping, L_p spaces, multilinear functionals.

uniformly open since its openness at the origin implies the openness at the remaining points with the same δ. It is known that the counterpart of this principle for bilinear continuous surjections is false. The problem concerned with this fact was initiated by Rudin [11] (see also [4], [6] for the solutions with nonopenness at zero). Note that there are several generalizations of the Banach open mapping principle, but they do not discuss bilinear maps (see [9]). The following simple counterexample can be found in Rudin's book [12, Chapter 2, Exercise 11].

Example. Let $T\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(t,x) := tx$. This map is bilinear and continuous, and it fails to be locally open at the elements of $(\{0\} \times \mathbb{R}^2) \setminus \{(0,(0,0))\}$.

Pointwise multiplication is a natural example of a bilinear continuous surjective operator for several function spaces X. However, in general it need not be open. A well-known counterexample is the Banach algebra $C[0,1]$ of real-valued continuous functions on $[0,1]$ endowed with the supremum norm (see [2]). The set of points where pointwise multiplication in $C[0,1]$ is not open forms a nowhere dense subset of $C[0,1] \times C[0,1]$; this follows from the characterization due to Behrends [3] where products of more than 2 factors were also examined.

The general case of the Banach algebra CK of real-valued continuous functions on a compact topological space K was investigated by Komisarski [7]. He proved that multiplication in CK is open if and only if the topological dimension $\dim K$ is 0. If $\dim K = 1$, then multiplication in CK is not open but is weakly open, which means that the interior of $B(f,\varepsilon) \cdot B(g,\varepsilon)$ is nonempty for all $f, g \in CK$ and $\varepsilon > 0$. Finally, if $\dim K > 1$, then multiplication in CK is not weakly open. In particular, multiplication is not open on a nonempty open subset of $CK \times CK$.

Remarkably, in the case of $C_{\mathbb{C}}[0,1]$, the respective Banach algebra of complex-valued continuous functions, the multiplication is open (see [3]). In [1], one can find other examples where multiplication in function spaces is open or even uniformly open.

In the present paper, we investigate *pointwise multiplication on spaces of measurable functions* and *scalar-valued multilinear maps*. Our main results state that there—in contrast to the case of continuous functions and of vector-valued multilinear maps—uniform openness can always be guaranteed.

Given a measure space $(\Omega, \mathcal{A}, \mathbb{P})$, consider the Banach space $L^p_{\mathbb{K}}(\Omega)$ (abbreviated as L^p) over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $p, q, r \in [1, \infty]$ be numbers such that $1/p + 1/q = 1/r$. (Here we adopt the convention that $1/\infty = 0$ so that, e.g., $p = r$, $q = \infty$, even $p = q = r = \infty$, are admissible choices.) It is well known that the pointwise multiplication from $L^p \times L^q$ to L^r is a well-defined continuous map.

Theorem 1.1. *Given any numbers $p, q, r \in [1, \infty]$ with $1/p + 1/q = 1/r$, the pointwise multiplication from $L^p \times L^q$ to L^r is a uniformly open map.*

Theorem 1.2. *Let $X_1, \dots, X_n$ be normed spaces over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let T from $X_1 \times \cdots \times X_n$ to $\mathbb{K}$ be a nontrivial n-linear functional. Then T is uniformly open.*

The products $L^p \times L^q$ (in Theorem 1.1) and $X_1 \times \cdots \times X_n$ (in Theorem 1.2) are provided with the maximum norm.

Note that Theorem 1.1 extends the result of [1], where the openness of the map was shown for $r = 1$ and $\mathbb{K} = \mathbb{R}$. The uniform openness in this case (which solves the problem from [1]) was proved in [10]. Here we present a different proof in a general case. The dissertation [10] contains the result of Theorem 1.2 in the case of $\mathbb{K} = \mathbb{R}$ and $n = 2$ with a different proof.

Theorem 1.1 will be proved in Section 2, and Theorem 1.2 will be proved in Section 3. Section 4 contains some supplements.

2. Pointwise multiplication on spaces of measurable functions

Let $f, g : \Omega \to \mathbb{K}$ be measurable functions. In order to prove that multiplication is locally open at (f, g), one has to find for a given "small" function d two "small" functions d_1, d_2 such that $(f + d_1)(g + d_2) = fg + d$; that is, $fd_2 + gd_1 + d_1 d_2 = d$. To state it otherwise, given numbers $a, b, c \in \mathbb{K}$, we aim at finding $x, y \in \mathbb{K}$ such that

- $ax + by + xy = c$;
- x, y depend measurably on a, b, c;
- for "small" c the x, y are also "small."

We will show in the next proposition that a suitable x-y-choice is in fact possible, and this will be the key preparation to prove Theorem 1.1.

Proposition 2.1.

(i) *Suppose that $p, q, r \in]1, \infty[$ are such that $1/p + 1/q = 1/r$. Then there are Borel-measurable maps $\phi_{p,q,r;1}, \phi_{p,q,r;2} : \mathbb{K}^3 \to \mathbb{K}$ with the following property: for all $(a, b, c) \in \mathbb{K}^3$ one has $ax + by + xy = c$ and $|x| \le |c|^{r/p}$, $|y| \le |c|^{r/q}$ (here x stands for $\phi_{p,q,r;1}(a, b, c)$ and y for $\phi_{p,q,r;2}(a, b, c)$).*

(ii) *Let ε be a positive number. Then there are Borel-measurable maps $\phi_{\varepsilon;1}, \phi_{\varepsilon;2} : \mathbb{K}^3 \to \mathbb{K}$ with the following property: for all $(a, b, c) \in \mathbb{K}^3$ one has $ax+by+xy = c$ and $|x| \le \varepsilon$, $|y| \le |c|/\varepsilon$ (here $x := \phi_{\varepsilon;1}(a, b, c)$ and $y := \phi_{\varepsilon;2}(a, b, c)$).*

Proof.

The case $\mathbb{K} = \mathbb{R}$, assertion (i). Denote by C_1 the set $\{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}, \alpha \ge 0\}$. We claim that there is a Borel-measurable map $\phi : C_1 \to [-1, 1]$ such that one has, with $t = \phi(\alpha, \beta)$, $t\alpha + t^2\beta = \beta$ for all α, β. Such a map can be defined as follows:

- $\phi(\alpha, 0) := 0$ for all α;
- $\phi(\alpha, \beta) := (-\alpha + \sqrt{\alpha^2 + 4\beta^2})/(2\beta)$ if $\beta \ne 0$.

It is easy to see that ϕ has the claimed properties.

For the definition of the $\phi_{p,q,r;1}, \phi_{p,q,r;2}$, we partition $\mathbb{R}^3$ into four measurable sets and construct these mappings there separately. The partition is defined as follows:

$$P_1 := \big\{(a, b, c) \mid c \ge 0, ac^{r/p} + bc^{r/q} \ge 0\big\};$$
$$P_2 := \big\{(a, b, c) \mid c \ge 0, ac^{r/p} + bc^{r/q} < 0\big\};$$
$$P_3 := \big\{(a, b, c) \mid c < 0, a|c|^{r/p} - b|c|^{r/q} \ge 0\big\};$$
$$P_4 := \big\{(a, b, c) \mid c < 0, a|c|^{r/p} - b|c|^{r/q} < 0\big\}.$$

Let us deal with P_1 first. We are looking for x, y such that $ax + by + xy = c$ and $|x| \leq |c|^{r/p}, |y| \leq |c|^{r/q}$. As a first approach we try it with $x_0 := c^{r/p}, y_0 := c^{r/q}$. Then $ax_0 + by_0 + x_0y_0 = ax_0 + by_0 + c$, a number that might be larger than c since $\alpha := ax_0 + by_0$ is nonnegative by assumption. But with $t := \phi(\alpha, c)$ we would have exactly $\alpha t + ct^2 = c$. This means that $x := tx_0, y := ty_0$ would have the desired properties. To put it more formally, on P_1 we could define

$$\begin{aligned}\phi_{p,q,r;1}(a, b, c) &:= \phi(ac^{r/p} + bc^{r/q}, c)c^{r/p},\\ \phi_{p,q,r;2}(a, b, c) &:= \phi(ac^{r/p} + bc^{r/q}, c)c^{r/q}.\end{aligned}$$

These maps are Borel-measurable since ϕ has this property. (This can easily be verified.)

On the other sets of the partition we proceed similarly; as a further example we consider P_3.

We start with $x_0 := |c|^{r/p}, y_0 := -|c|^{r/q}$. Then $ax_0 + by_0 + x_0y_0 = \alpha - |c| = \alpha + c$, where $\alpha := (a|c|^{r/p} - b|c|^{r/q}) \geq 0$. Therefore, $x := \phi(\alpha, c)x_0, y := \phi(\alpha, c)y_0$ will behave as desired. Here is the formal definition:

$$\begin{aligned}\phi_{p,q,r;1}(a, b, c) &:= \phi\big(a|c|^{r/p} - b|c|^{r/q}, c\big)|c|^{r/p},\\ \phi_{p,q,r;2}(a, b, c) &:= -\phi\big(a|c|^{r/p} + b|c|^{r/q}, c\big)|c|^{r/q}.\end{aligned}$$

The case $\mathbb{K} = \mathbb{R}$, assertion (ii). The strategy is similar. This time $\mathbb{R}^3$ is partitioned as the disjoint union of Q_1, Q_2, Q_3, Q_4 where

$$\begin{aligned}Q_1 &:= \big\{(a, b, c) \mid c \geq 0, a\varepsilon + cb/\varepsilon \geq 0\big\};\\ Q_2 &:= \big\{(a, b, c) \mid c \geq 0, a\varepsilon + cb/\varepsilon < 0\big\};\\ Q_3 &:= \big\{(a, b, c) \mid c < 0, a\varepsilon - |c|b/\varepsilon \geq 0\big\};\\ Q_4 &:= \big\{(a, b, c) \mid c < 0, a\varepsilon - |c|b/\varepsilon < 0\big\}.\end{aligned}$$

What has to be done on Q_1? First one tries it with $x_0 := \varepsilon$ and $y_0 := c/\varepsilon$, but $ax_0 + by_0 + x_0y_0 = a\varepsilon + cb/\varepsilon + c$ might be too large. Therefore, one passes to $x := tx_0, y := ty_0$ with $t := \phi(a\varepsilon + cb/\varepsilon, c)$. It should be clear how one has to argue on Q_2, Q_3, and Q_4.

The case $\mathbb{K} = \mathbb{C}$, assertion (i). In the complex case we have to argue much more subtly. For complex a, b, c we have to find $x, y \in \mathbb{C}$ such that $ax + by + xy = c$ and also $|x| \leq |c|^{r/p}, |y| \leq |c|^{r/q}$ hold. It will suffice to assume that $c \neq 0$ since for $c = 0$ one can simply put $x = y = 0$.

In order to copy the strategy for the real case, we will first have to rotate a, b such that $ax + by$ are "in the same direction" as c. Here is our *road map.*

Claim 1. *For all a, b there exist z_1, z_2 with $|z_1| = |z_2| = z_1z_2 = 1$ such that $az_1 + bz_2 + z_1z_2 = 1 + r$ for some $r \geq 0$. To state it otherwise, there exists z with $|z| = 1$ such that $az + b\overline{z} \in [0, \infty[$. This will have to be done in such a way that $(a, b) \mapsto (z, r)$ is measurable.*

Claim 2. *For a, b, and c there exist z_1', z_2', and $r \geq 0$ such that*

- *$z_1'z_2' = c$, $|z_1'| \leq |c|^{r/p}, |z_2'| \leq |c|^{r/q}$;*
- *$az_1' + bz_2' + z_1'z_2' = (1 + r)c$.*

Claim 3. *Whenever complex numbers A, B "point to the same direction" (i.e., if there is an $r \geq 0$ such that $A = rB$), then there exists a $t \in [0,1]$ with $At + Bt^2 = B$. (This corresponds to the ϕ-function defined in the real case.)*

It will remain to find t for the special case $A = az_1' + bz_2'$ and $B = c$. Then $x := tz_1'$ and $y := tz_2'$ will have the desired properties.

Proof of Claim 1. Let a, b be given. We have to find z with $|z| = 1$ such that $az + b\overline{z} \in [0, \infty[$. This is simple if $a = 0 = b$ (choose an arbitrary z), or $a \neq 0 = b$ (put $z := \overline{a}/|a|$), or $a = 0 \neq b$ (put $z := \overline{b}/|b|$), or $0 < |a| = |b|$ (choose z such that $z^2 = -b/a$). Thus, we may concentrate on a situation where $0 < |b| < |a|$. (The case $|a| < |b|$ can be treated similarly.)

What are the $r \geq 0$ where one finds a z with $|z| = 1$ such that $az + b\overline{z} = r$? One must find a z with $|z| = 1$ that solves the quadratic equation $az^2 + b = rz$. The solutions are

$$z_{1,2} = \frac{r \pm \sqrt{r^2 - 4ab}}{2a}$$

so that it would suffice to find a nonnegative r with the property

$$|r + \sqrt{r^2 - 4ab}| = |2a|,$$

where the square root is defined suitably.

But precisely this is prepared in the following lemma, where it is also shown that the map $(a, b) \mapsto (z, r)$ can be chosen to be Borel-measurable. $\square$

Lemma 2.2. *Let C_2^- be the set $\{(a, b) \mid a, b \in \mathbb{C}, |b| < |a|, \operatorname{Im}(ab) \leq 0\}$ ($\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and the imaginary parts of a complex number). Denote by $W^+ : \{z \mid \operatorname{Im} z \geq 0\} \to \{z \mid \operatorname{Im} z, \operatorname{Re} z \geq 0\}$ the natural square root: if z is written as $re^{i\phi}$ with $\phi \in [0, \pi]$, then $W^+(z) := \sqrt{r}e^{i\phi/2}$. There is a Borel-measurable $\psi^- : C_2^- \to [0, \infty[$ with the following property: if one puts $r := \psi^-(a, b)$, then $|r + W^+(r^2 - 4ab)| = |2a|$.*

Similarly, we define C_2^+ as $\{(a, b) \mid a, b \in \mathbb{C}, |b| < |a|, \operatorname{Im}(ab) > 0\}$ and $W^- : \{z \mid \operatorname{Im} z < 0\} \to \{z \mid \operatorname{Im} z < 0, \operatorname{Re} z \geq 0\}$ as the function that maps $re^{i\phi}$ (with $\phi \in [-\pi, 0]$) to $\sqrt{r}e^{i\phi/2}$. There is a Borel-measurable $\psi^+ : C_2^+ \to [0, \infty[$ with the following property: if one puts $r := \psi^+(a, b)$, then $|r + W^-(r^2 - 4ab)| = |2a|$.

Proof. Let $(a, b) \in C_2^-$ be given. Consider the map $h : t \mapsto |t + W^+(t^2 - 4ab)|$ for $t \geq 0$. For $t = 0$ the absolute value of h is $2\sqrt{|a||b|} < 2|a|$, whereas for $t > 2|a|$ one has $|h(t)| > 2|a|$. (Here one uses the fact that $\operatorname{Re} t\overline{W^+(t - 4ab)} \geq 0$ and that $|z + w|^2 \geq |z|^2 + |w|^2$ if $\operatorname{Re} z\overline{w} \geq 0$.)

Therefore, the following definition is reasonable:

$$\psi^-(a, b) := \sup\big\{t \bigm| 0 \leq t, \big|t + W^+(t^2 - 4ab)\big| < 2|a|\big\}.$$

Then ψ^- is Borel-measurable since, as a consequence of the continuity of W^+, the sets $\{(a, b) \mid \psi^+(a, b) > \eta\}$ are open for every real η.

The second assertion is proved in a similar way. $\square$

Proof of Claim 2. Choose any w_1, w_2 such that $w_1 w_2 = c$ and $|w_1| \leq |c|^{r/p}$ and $|w_2| \leq |c|^{r/q}$. We note in passing that this choice is possible in such a way that $c \mapsto (w_1, w_2)$ is Borel-measurable. We have assumed that $c \neq 0$, and so $w_1 \neq 0 \neq w_2$. Thus, it is reasonable to apply the assertion of Claim 1 as follows: there are z_1, z_2 of absolute value one with $z_1 z_2 = 1$ such that $(a/w_2)z_1 + (b/w_1)z_2 + z_1 z_2 = 1 + r$ for a suitable $r \geq 0$. We multiply this equation with c and put $z_1' := w_1 z_1$, $z_2' := w_2 z_2$. These numbers have the claimed properties. □

Proof of Claim 3. Let A, B be codirectional: $A = rB$ with an $r \geq 0$. If we multiply this equation with $\overline{B}/|B|$ and put $\alpha := A\overline{B}/|B|, \beta := \overline{B}/|B|$, then we arrive at a situation where we can apply our mapping ϕ from the beginning of the proof of the real case. Thus, with $t := \phi(\alpha, \beta) \in [0, 1]$ we know that $t\alpha + t^2\beta = \beta$; that is, $tA + t^2 B = B$. □

As already noted, this completes the proof of part (i) of the proposition. All constructions were explicit so that the definition

$$\phi_{p,q,r;1}(a, b, c) := x, \qquad \phi_{p,q,r;2}(a, b, c) := y$$

(with x, y as above) gives rise to two Borel-measurable maps. We omit to make this precise with some clumsy formulas.

The case $\mathbb{K} = \mathbb{C}$*, assertion (ii).* With an easy modification of the preceding proof we can also treat part (ii) of the proposition. This time, in Claim 2, we choose w_1, w_2 simply as $w_1 := \varepsilon$ and $w_2 := c/\varepsilon$. All other steps are completely similar. □

We now turn to the *proof of Theorem 1.1*.

The case $p, q, r < \infty$. Let $\varepsilon_0 > 0$ and $f \in L^p$, $g \in L^q$ be given. We have to find $\delta_0 > 0$ such that for $h \in L^r$ with $\|h\|_r \leq \delta_0$ one can select $d_1 \in L^p$ with $\|d_1\|_p \leq \varepsilon_0$ and $d_2 \in L^q$ with $\|d_2\|_q \leq \varepsilon_0$ such that $(f + d_1)(g + d_2) = fg + h$; that is, $gd_1 + fd_2 + d_1 d_2 = h$.

Let such f, g be given. We define d_1, d_2 as follows:

$$d_1(\omega) := \phi_{p,q,r;1}\big(g(\omega), f(\omega), h(\omega)\big), \qquad d_2(\omega) := \phi_{p,q,r;2}\big(g(\omega), f(\omega), h(\omega)\big).$$

Then d_1, d_2 are measurable as compositions of $\mathcal{A}$-measurable maps with Borel-measurable ones. As a consequence of $|d_1(\omega)| \leq |h(\omega)|^{r/p}$ and $|d_2(\omega)| \leq |h(\omega)|^{r/q}$ we have the following estimation:

$$\|d_1\|_p^p = \int_\Omega \big|d_1(\omega)\big|^p \, d\omega \leq \int_\Omega \big|h(\omega)\big|^r \, d\omega = \|h\|_r^r;$$

that is, $\|d_1\|_p \leq \|h\|_r^{r/p}$, and, similarly, one has $\|d_2\|_q \leq \|h\|_r^{r/q}$. Thus, it suffices to choose δ_0 so small that $\delta_0 \leq \min\{\varepsilon_0^{p/r}, \varepsilon_0^{q/r}\}$. For example, the special case $p = q = 2$ and $r = 1$ leads to the condition $\delta_0 \leq \varepsilon_0^2$.

The case $q = r < \infty = p$. Here the mappings $\phi_{\varepsilon_0;1}, \phi_{\varepsilon_0;2}$ come into play. If we use them in the definition of d_1, d_2, then we know that $|d_1(\omega)| \leq \varepsilon_0$ and $|d_2(\omega)| \leq |h(\omega)|/\varepsilon_0$ for all ω. Consequently, $\|d_1\|_\infty \leq \varepsilon_0$ and $\|d_2\|_q \leq \|h\|_q/\varepsilon_0$. Hence, it suffices to choose $\delta_0 > 0$ so small that $\delta_0/\varepsilon_0 \leq \varepsilon_0$; that is, $\delta_0 \leq \varepsilon_0^2$.

The case $p = r < \infty = q$. Here one simply has to reverse the roles of f and g.

The case $p = q = r = \infty$. The mappings $\phi_{2,2,1;1}$ and $\phi_{2,2,1;2}$ lead to the desired result. They provide d_1, d_2 with $(f + d_1)(g + d_2) = fg + h$ such that pointwise

$$\left|d_1(\omega)\right|, \left|d_2(\omega)\right| \leq \sqrt{\left|h(\omega)\right|}.$$

Therefore, it suffices to put $\delta_0 := \varepsilon_0^2$, and this completes the proof of Theorem 1.1. □

It should be noted that one could treat other multiplication maps with the same approach. Here are two examples.

Example 1. We consider the pointwise multiplication from $l^\infty \times c_0$ to c_0. We claim that this mapping is also uniformly open. Let $\varepsilon_0 > 0$, $(f_n) \in l^\infty$, and $(g_n) \in c_0$ be given. For $(h_n) \in c_0$ we find with the help of $\phi_{\varepsilon_0;1}$ and $\phi_{\varepsilon_0;2}$ sequences (d_n) and (d'_n) with $|d_n| \leq \varepsilon_0$ and $|d'_n| \leq |h_n|/\varepsilon_0$ that satisfy $(f + d)(g + d') = fg + h$. Then $(d_n) \in l^\infty$ and $(d'_n) \in c_0$ hold, and $\|(d_n)\| \leq \varepsilon_0$ and $\|(d'_n)\| \leq \|h\|/\varepsilon_0$. Thus, $\delta_0 := \varepsilon_0^2$ would be an admissible choice to prove the uniform openness of this multiplication.

Example 2. Next we investigate pointwise multiplication from $l^1 \times c_0$ to l^1. This map is also uniformly open.

To prove this claim, we consider any $(x_n) \in l^1$, $(y_n) \in c_0$, and a positive ε. Let δ be any number with $0 < \delta < \varepsilon^2$ and $(d_n) \in l^1$ such that $\|(d_n)\|_1 \leq \delta$. We will find $(x'_n) \in l^1$, $(y'_n) \in c_0$ with $\|(x'_n)\|_1, \|(y'_n)\|_\infty \leq \varepsilon$ such that

$$\big((x_n) + (x'_n)\big)\big((y_n) + (y'_n)\big) = (x_n)(y_n) + (d_n);$$

that is,

$$x_n y'_n + y_n x'_n + x'_n y'_n = d_n \quad (n = 1, 2, \ldots),$$

and this would prove the assertion.

In a first step we choose positive $c_1, c_2, \ldots$ such that $(c_n) \in c_0$ and $\|(c_n)\|_\infty = 1$ such that $\|(d_n/\sqrt{c_n})\|_1 \leq \varepsilon^2$. (It is an easy exercise to show that such c_n's exist.) Then we find, with the help of Proposition 3(ii), numbers x''_n, y''_n with

$$|x''_n| \leq \varepsilon, \qquad |y''_n| \leq \frac{d_n}{c_n} \cdot \frac{1}{\varepsilon}$$

such that

$$\frac{x_n}{\sqrt{c_n}} y''_n + \frac{y_n}{\sqrt{c_n}} x''_n + x''_n y''_n = \frac{d_n}{c_n}.$$

Then the numbers $x'_n := x''_n \sqrt{c_n}$, $y'_n := y''_n \sqrt{c_n}$ will have the claimed properties.

Similarly, one can treat a much more general situation. Let K be σ-compact topological space provided with a Borel measure μ. We denote by $L_0^\infty(K, \mu)$ the uniform closure of the collection of functions in $L^\infty(K, \mu)$ that are supported by a compact subset of K. (This is the natural generalization of c_0.) Then pointwise multiplication from $L^p(K, \mu) \times L_0^\infty(K, \mu)$ to $L^p(K, \mu)$ is uniformly open for every $p \in [1, \infty[$ (for $p = 1$ this result is proved in [10]). The crucial ingredient is again part (ii) of Proposition 3. As a preparation one has to find, for given $d \in L^p$ with $\|d\|_p < \varepsilon^2$, a positive $c \in L_0^\infty$ with $\|c\|_\infty \leq 1$ and $\|d/c\|_p \leq \varepsilon^2$. The proposition provides the desired functions as in the previous discrete example. (The measurability can be guaranteed by σ-compactness. Due to this property, it suffices to glue together measurable functions that are defined on a countable partition of K.)

3. Multilinear maps

We start with a *combinatorial result.* Fix $n \in \mathbb{N}$ with $n \geq 2$, and denote by I_n the set of nonempty subsets α of $\{1, \ldots, n\}$. As before, $\mathbb{K}$ denotes the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers. For a given family $\mathbf{a} = (a_\alpha)_{\alpha \in I_n}$ of elements of $\mathbb{K}$, we consider the polynomial $P_{\mathbf{a}} : \mathbb{K}^n \to \mathbb{K}$ that is defined by

$$P_{\mathbf{a}}(z_1, \ldots, z_n) := \sum_{\alpha \in I_n} a_\alpha \prod_{i \in \alpha} z_i.$$

For example, if $n = 3$, then $P_{\mathbf{a}}$ has the form

$$a_1 z_1 + a_2 z_2 + a_3 z_3 + a_{12} z_1 z_2 + a_{13} z_1 z_3 + a_{23} z_2 z_3 + a_{123} z_1 z_2 z_3;$$

here and in the sequel we will use the notation a_1 for $a_{\{1\}}$, and so on. (Such polynomials, where there are no powers larger than one, are called 0–1-*polynomials.*)

It will be convenient to write $\vec{z}$ for the elements of $\mathbb{K}^n$, and we will abbreviate the expression $\prod_{i \in \alpha} z_i$ by $\vec{z}^{\alpha}$.

Our main result on the range of such polynomials reads as follows.

Proposition 3.1. *Let $a = (a_\alpha)_{\alpha \in I_n}$ be given such that $a_{12 \cdots n} = 1$. Then the set*

$$\bigl\{P_a(\vec{z}) \bigm| |z_1|, \ldots, |z_n| \leq 1\bigr\}$$

contains all $w \in \mathbb{K}$ with $|w| \leq 1$.

Proof. First we consider the case $\mathbb{K} = \mathbb{R}$.

Let $(a_\alpha)_{\alpha \in I_n}$ be a family of real numbers as in the proposition. Then the following combinatorial result holds.

Claim.

(i) *There exists $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{-1, 1\}^n$ with $\sigma_1 \cdots \sigma_n = 1$ such that*

$$\sum_{\alpha \in I_n, \alpha \neq \{1, \ldots, n\}} a_\alpha \vec{\sigma}^{\alpha} \geq 0.$$

(ii) *Also, one can find $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{-1, 1\}^n$ with $\sigma_1 \cdots \sigma_n = -1$ such that*

$$\sum_{\alpha \in I_n, \alpha \neq \{1, \ldots, n\}} a_\alpha \vec{\sigma}^{\alpha} \leq 0.$$

As an illustration we consider an example in the case $n = 3$. We choose the six real numbers

$$a_1 = 2, \qquad a_2 = -10, \qquad a_3 = 0, \qquad a_{12} = 6, \qquad a_{13} = 3, \qquad a_{23} = -6,$$

and it is claimed that it is possible to find $\sigma_1, \sigma_2, \sigma_3 \in \{-1, +1\}$ with $\sigma_1 \sigma_2 \sigma_3 = 1$ such that

$$\sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3 + \sigma_1 \sigma_2 a_{12} + \sigma_1 \sigma_2 a_{12} + \sigma_2 \sigma_3 a_{23} \geq 0.$$

And really, $\sigma_1 = 1, \sigma_2 = -1$, and $\sigma_3 = -1$ have this property. Similarly, one can provide $\sigma_1, \sigma_2, \sigma_3 \in \{-1, +1\}$ with $\sigma_1\sigma_2\sigma_3 = -1$ (e.g., $\sigma_1 = 1, \sigma_2 = 1, \sigma_3 = -1$) in this case such that

$$\sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3 + \sigma_1\sigma_2 a_{12} + \sigma_1\sigma_2 a_{12} + \sigma_2\sigma_3 a_{23} \leq 0.$$

Proof of the Claim. Choose a probability space $(\Omega, \mathcal{E}, \mathbb{P})$ and identically distributed independent random variables $X_1, \dots, X_n : \Omega \to \{-1, +1\}$ such that $\mathbb{P}(X_i = \pm 1) = 0.5$ for all i.

For a proper subset α of $\{1, \dots, n\}$ we put $X_\alpha := \prod_{i\in\alpha} X_i$, and the random variable X is defined by $\prod_{i=1}^n X_i$. Then $\int_{\{X=1\}} X_\alpha \, d\mathbb{P} = 0$.

To prove this, let Y be the product of the spaces X_i with $i \notin \alpha$. Then $X = X_\alpha Y$, and X_α, Y are independent and uniformly distributed on $\{-1, 1\}$. It follows that

$$\int_{\{X=1\}} X_\alpha \, d\mathbb{P} = \int_{\{X_\alpha = Y = 1\}} X_\alpha \, d\mathbb{P} + \int_{\{X_\alpha = Y = -1\}} X_\alpha \, d\mathbb{P} = 1/4 - 1/4 = 0.$$

Let Z be the random variable

$$\sum_{\alpha \in I_n, \alpha \neq \{1,\dots,n\}} a_\alpha \prod_{i\in\alpha} X_i.$$

By our claim, the integral $\int_{\{X=1\}} Z \, d\mathbb{P}$ vanishes so that there must be $\omega \in \Omega$ with $X(\omega) = 1$ and $Z(\omega) \geq 0$. With $\sigma_i := X_i(\omega)$ we thus have found $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$ with $\sigma_1 \cdots \sigma_n = 1$ and $\sum_{\alpha\in I_n, \alpha\neq\{1,\dots,n\}} a_\alpha \prod_{i\in\alpha} \sigma_i \geq 0$.

(ii) The proof is similar; this time one works with the integral over $\{X = -1\}$. □

Now the proposition can easily be proved: by the claim there are numbers not smaller than 1 and not larger than -1 in $\{P_{\mathbf{a}}(\vec{z}) \mid |z_1|, \dots, |z_n| \leq 1\}$, and, since this set is obviously connected, it will also contain $[-1, 1]$.

It remains to treat the case $\mathbb{K} = \mathbb{C}$. There, a much stronger variant of the proposition holds: one can even consider tuples $(z_1, \dots, z_n)$ so that $z_1 = \cdots = z_n$ and $|z_i| \leq 1$.

This follows at once from the following observation.

Let $b_1, \dots, b_{n-1}$ be complex numbers and P the polynomial

$$z \mapsto b_1 z + b_2 z^2 + \cdots + b_{n-1} z^{n-1} + z^n.$$

Then for every w_0 with $|w_0| \leq 1$ there exists a $z_0 \in \mathbb{C}$ such that $|z_0| \leq 1$ and $P(z_0) := w_0$.

To prove this, fix an arbitrary $w_0 \in \mathbb{C}$. We consider the polynomial $z \mapsto -w_0 + P(z)$. It can be written as $(z - z_1) \cdots (z - z_n)$ with suitable $z_1, \dots, z_n$. We have $P(z_j) = w_0$ for every j and $-w_0 = (-1)^n z_1 \cdots z_n$. Thus, there must be a j with $|z_j| \leq \sqrt[n]{|w_0|} \leq 1$, and this implies our claim. □

After these preparations we are ready for the proof of Theorem 1.2.

Choose any $(\hat{y}_1, \dots, \hat{y}_n) \in X_1 \times \cdots \times X_n$ such that $T(\hat{y}_1, \dots, \hat{y}_n) = 1$. Fix an $\varepsilon > 0$ and find $\eta > 0$ so small that $\|\eta \hat{y}_i\| \leq \varepsilon$ for all i. (For simplicity the norm will be denoted by the same symbol $\|\cdot\|$ for all spaces under consideration.) Then, for arbitrary $(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$ and $z_1, \dots, z_n$ with $|z_1|, \dots, |z_n| \le 1$, the number

$$T(x_1 + \eta z_1 \hat{y}_1, \dots, x_n + \eta z_n \hat{y}_n) - T(x_1, \dots, x_n)$$

is of the form $P_{\mathbf{a}}(z_1, \dots, z_n)$ (with $a_{12\cdots n} = \eta^n$) for suitable a_α as a consequence of the fact that T is multilinear. Thus, by Proposition 3.1, all $w \in B(0, \eta^n)$ can be written in this form with suitable $z \in \mathbb{K}^n$ that satisfy $|z_1|, \dots, |z_n| \le 1$. This means that $\delta := \eta^n$ has the claimed properties.

4. Quantitative results and invitations to further study

We will check whether our choice of δ, witnessing the uniform openness in Theorems 1.1 and 1.2, is optimal. Let $f\colon X \to Y$ be a uniformly open map between metric spaces X and Y. Then the function $M_f\colon]0,\infty] \to]0,\infty]$ given by

$$M_f(\varepsilon) := \sup\bigl\{\delta > 0 \bigm| B\bigl(f(x), \delta\bigr) \subset f\bigl[B(x, \varepsilon)\bigr] \text{ for all } x \in X\bigr\}$$

is called the *modulus of uniform openness* of f.

First, consider pointwise multiplication $\Phi\colon L^p \times L^q \to L^r$ in Theorem 1.1. From the final part of the proof of this theorem it follows that we could choose $\delta := \min\{\varepsilon^{p/r}, \varepsilon^{q/r}\}$ if $p, q \in]1, \infty[$, and $\delta = \varepsilon^2$ otherwise; hence, $M_\Phi(\varepsilon) \ge \min\{\varepsilon^{p/r}, \varepsilon^{q/r}\}$ if $p, q \in]1, \infty[$, and $M_\Phi(\varepsilon) \ge \varepsilon^2$ otherwise.

On the other hand, if $f \in L^p$ and $g \in L^q$ are such that $\|fg\|_r \ge \varepsilon^2$, then, by the Hölder inequality,

$$\varepsilon^2 \le \|fg\|_r \le \|f\|_p \cdot \|g\|_q,$$

and so $\|f\|_p \ge \varepsilon$ or $\|g\|_q \ge \varepsilon$; hence, $M_\Phi(\varepsilon) \le \varepsilon^2$ and thus we have $\min\{\varepsilon^{p/r}, \varepsilon^{q/r}\} \le M_\Phi(\varepsilon) \le \varepsilon^2$. In particular, in each of the following cases, (i) $p = q = 2$ and $r = 1$; (ii) $p = \infty$; (iii) $q = \infty$, we get the equality.

It would be interesting to find the exact value of $M_\Phi(\varepsilon)$ for the remaining cases.

Now, consider n-linear functionals as in Theorem 1.2. If $T : X_1 \times \cdots \times X_n \to \mathbb{K}$ is n-linear, then we set $\|T\| := \sup\{\|T(x_1, \dots, x_n)\| : x_i \in X_i, \|x_i\| \le 1\}$. It is well known that $\|T\| < \infty$ if and only if T is continuous, and that $\|\cdot\|$ is a norm on the space of all continuous n-linear functionals.

Proposition 4.1. *Let $X_1, \dots, X_n$ be normed spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let T be a nontrivial n-linear functional from $X_1 \times \cdots \times X_n$ to $\mathbb{K}$. Then $M_T(\varepsilon) = \|T\|\varepsilon^n$ for all $\varepsilon > 0$. In particular, $M_T(\varepsilon) = \infty$ for every $\varepsilon > 0$ when T is not continuous.*

Proof. Fix $\varepsilon > 0$ and $\sigma > 0$ with $\sigma < \|T\|$. Pick any $(y_1, \dots, y_n) \in X_1 \times \cdots \times X_n$ such that $\|y_i\| = 1$ for all $i = 1, \dots, n$, and $\sigma < T(y_1, \dots, y_n) \le \|T\|$. Set $\lambda := (T(y_1, \dots, y_n))^{1/n}$, and set $(\hat{y}_1, \dots, \hat{y}_n) := (y_1/\lambda, \dots, y_n/\lambda)$. Then $T(\hat{y}_1, \dots, \hat{y}_n) = 1$. By the final part of the proof of Theorem 1.2, we obtain

$$M_T(\varepsilon) \ge (\lambda\varepsilon)^n = T(y_1, \dots, y_n)\varepsilon^n \ge \sigma\varepsilon^n,$$

and, since $\sigma < \|T\|$ is arbitrary, we have $M_T(\varepsilon) \ge \|T\|\varepsilon^n$. In particular, if T is not continuous, then we obtain $M_T(\varepsilon) = \infty$.

Now we show the opposite inequality in the case when T is continuous. Let $z := \|T\|\varepsilon^n$ and assume that $T(x_1, \ldots, x_n) = z$. Then

$$\begin{aligned}\|T\|\varepsilon^n &= T(x_1, \ldots, x_n)\\ &= \|x_1\| \cdots \|x_n\| T\Big(\frac{x_1}{\|x_1\|}, \ldots, \frac{x_n}{\|x_n\|}\Big) \leq \|x_1\| \cdots \|x_n\| \|T\|.\end{aligned}$$

Hence, $\|x_1\| \cdots \|x_n\| \geq \varepsilon^n$, and so $\|x_i\| \geq \varepsilon$ for some $i = 1, \ldots, n$. Therefore, $z \notin T[B(0, \varepsilon)]$ and $M_T(\varepsilon) \leq \|T\|\varepsilon^n$. □

Now, let us calculate the modulus of openness for pointwise multiplication in the finite-dimensional case.

Example. For every integer $n \geq 1$, consider the multiplication $\Phi_n : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathbb{K}^n$ is provided with the Euclidean norm. In this case, $M_{\Phi_n}(\varepsilon) = \varepsilon^2/\sqrt{n}$, and so it depends strongly on n; indeed, by Proposition 4.1 we obtain $M_{\Phi_1}(\varepsilon) = \varepsilon^2$. Now fix any $n > 1$, $\varepsilon > 0$ and any $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{K}^n$. Choose $z = (z_1, \ldots, z_n) \in \mathbb{K}^n$ such that $\|z - xy\| < \varepsilon^2/\sqrt{n}$. Then there are $\delta_1, \ldots, \delta_n > 0$ such that $|x_i y_i - z_i| < \delta_i$ for $i = 1, \ldots, n$ and

$$\sqrt{\sum_{i=1}^{n} \delta_i^2} < \frac{\varepsilon^2}{\sqrt{n}}.$$

Now, using M_{Φ_1}, for every $i = 1, \ldots, n$ pick x_i', y_i' such that $|x_i - x_i'| < \sqrt{\delta_i}$, $|y_i - y_i'| < \sqrt{\delta_i}$, and $z_i = x_i' y_i'$. Hence, by the Hölder inequality,

$$\|x - x'\| < \sqrt{\sum_{i=1}^{n} \delta_i} \leq \sqrt{\sqrt{n}\sqrt{\sum_{i=1}^{n} \delta_i^2}} \leq \sqrt{\sqrt{n}\frac{\varepsilon^2}{\sqrt{n}}} = \varepsilon,$$

and, in the same way, $\|y - y'\| < \varepsilon$; hence, $M_{\Phi_n}(\varepsilon) \geq \varepsilon^2/\sqrt{n}$. Now let $\delta := \varepsilon^2/\sqrt{n}$ and take $z := (\delta/\sqrt{n}, \ldots, \delta/\sqrt{n})$. Then $\|z\| = \delta$, and if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are such that $z = xy$, then, by the Hölder inequality, we have

$$\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2} \geq \sum_{i=1}^{n} |x_i y_i| = \sum_{i=1}^{n} z_i = \sqrt{n}\delta = \varepsilon^2.$$

Thus, $M_{\Phi_n}(\varepsilon) \leq \varepsilon^2/\sqrt{n}$.

Finally, let us state some open problems concerning the openness of bilinear maps.

1. It is interesting to establish whether the notion of uniform openness is essentially stronger than the notion of openness for the class of continuous bilinear surjective operators between Banach spaces.

2. Let $BV[0,1]$ stand for the Banach algebra of real-valued functions on $[0,1]$ of bounded variation with the norm $\|f\| := |f(0)| + V(f, [0,1])$, where $V(f, [0,1])$ denotes the total variation of f on $[0,1]$. We do not know whether pointwise multiplication in $BV[0,1]$ is open.

3. Consider the Cauchy product of series as a bilinear continuous map from $l^1 \times l^1$ onto l^1. We do not know whether it is open. Note that the Cauchy product is a particular case of the operation of convolution. In general, consider the Banach algebra $L^1(G)$ of μ-integrable functions with respect to the Haar measure where G is a locally compact Hausdorff topological group. One can ask about the openness of the operator of convolution between $L^1(G) \times L^1(G)$ and $L^1(G)$.

4. Let $A(X)$ denote the Banach algebra of bounded linear operators from a Banach space X (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) onto itself. In [3], an example of X is given where the operator of composition $\circ$ in $A(X)$ is not open and the same argument works in both of the cases $\mathbb{R}$ and $\mathbb{C}$. We would like to obtain sufficient and/or necessary conditions on X for the openness of $\circ$, or to decide whether $\circ$ is open for some concrete algebras $A(X)$. For instance, consider the algebra of 2×2 matrices with terms in $\mathbb{K}$ (they are associated with linear maps from $\mathbb{K}^2$ onto $\mathbb{K}^2$). Is the map, which associates with two such matrices their product, an open map?

Acknowledgment. Filip Strobin's research was supported by NCN grant no. 2012/07/D/ST1/02087.

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