CHAPTER III

Banach spaces which are M-ideals in their biduals

III.1 Examples and stability properties

In this section we begin our study of Banach spaces X which are M-ideals in their biduals. The reason for the rich theory of these spaces is mainly their nice stability behaviour (Theorem 1.6) and the *natural* L-decomposition of X^{***} (Proposition 1.2).

We write i_X for the isometric embedding of a Banach space X into its bidual X^{**}. However, in most of the following we will simply regard X as a subspace of X^{**}.

Definition 1.1 *Let X be a Banach space.*

(a) *X is called an M-*embedded space *if X is an M-ideal in X^{**}.*

(b) *X is called an L-*embedded space *if X is an L-summand in X^{**}.*

REMARKS: (a) These names are chosen to reflect the M-(L-)type embedding of X in X^{**}.

(b) Since by Theorem I.1.9 M-summands in dual spaces are w^*-closed, no nonreflexive Banach space X can be an M-summand in X^{**}, that is nonreflexive M-embedded spaces are proper M-ideals (cf. Definition II.3.1).

(c) Because of the following we can (should it be necessary) restrict ourselves to real spaces.

> *If X is a complex Banach space, then X is M- (L-)embedded if and only if $X_{\mathbb{R}}$, i.e. X considered as a real Banach space, is M- (L-)embedded.*

[PROOF: The mapping $X^* \longrightarrow (X_{\mathbb{R}})^*$, $x^* \longmapsto \operatorname{Re} x^*$ is an $\mathbb{R}$-linear isometry, hence $(X^*)_{\mathbb{R}} \cong (X_{\mathbb{R}})^*$. Applying this also to X^{**} yields

$$(X^{**})_{\mathbb{R}} \cong ((X^*)_{\mathbb{R}})^* \cong (X_{\mathbb{R}})^{**}.$$

Since the isomorphism between $(X^{**})_{\mathbb{R}}$ and $(X_{\mathbb{R}})^{**}$ maps $X_{\mathbb{R}}$ onto $i_{X_{\mathbb{R}}}(X_{\mathbb{R}})$ we get the claim by Proposition I.1.23.]

Recall for the following that the natural projection from the third dual X^{***} of a Banach space X onto $i_{X^*}(X^*)$ is the mapping $\pi_{X^*} := i_{X^*} \circ i_X{}^*$.

Proposition 1.2 *For a Banach space X the following are equivalent:*

(i) *X is an M-ideal in X^{**}.*

(ii) *The natural projection π_{X^*} from X^{***} onto $i_{X^*}(X^*)$ is an L-projection.*

PROOF: (ii) $\Rightarrow$ (i): Since the kernel of π_{X^*} is $X^\perp$, we have $X^{***} = X^* \oplus_1 X^\perp$.

(i) $\Rightarrow$ (ii): By assumption $X^\perp$ is the kernel of an L-projection P in X^{***}. Since π_{X^*} is a contractive projection with kernel $X^\perp$ we get by Proposition I.1.2 that $P = \pi_{X^*}$. □

For easy reference we state:

Corollary 1.3 *If X is an M-embedded space, then X^* is an L-embedded space.*

The converse of this statement is not true (consider e.g. $X = c$; cf. also Proposition 2.7). In Proposition IV.1.9 we will characterise those L-embedded spaces which are duals of M-embedded spaces.

Banach spaces which are M-ideals in their biduals appear naturally among classical spaces. In the following we will give examples of sequence spaces, function spaces, operator spaces, and spaces of analytic functions which are M-embedded. The underlying idea in most of the examples is that there is a "$o(\cdot)$-$O(\cdot)$" relation between the elements of X and those of X^{**}. (o and O are to denote the usual Landau symbols.) In this case the proofs use the 3-ball property with the following idea: Suppose X is a space of functions on some set S which we would like to prove to be M-embedded. To find $y \in X$ for given $x^{**} \in B_{X^{**}}$, $x_i \in B_X$, $(i = 1, 2, 3)$ and $\varepsilon > 0$ such that $\|x^{**} + x_i - y\| \le 1 + \varepsilon$ choose a subset $K \subset S$ with $|x_i| < \varepsilon$ off K (the o-condition) and try to define $y = x^{**}$ on K and $y = 0$ elsewhere. This idea works perfectly well for $c_0(S)$ which will turn out to be the archetype of an M-embedded space, but it needs some work to adapt it to other examples. Also, it is sufficient to assume that the x_i lie in some dense subset of B_X, since we have an $\varepsilon > 0$ at our disposal.
Since not all of the following spaces are equally well-known let us recall some definitions first.

THE ORLICZ SPACES h_M AND ℓ_M, $H_M(I)$ AND $L_M(I)$. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous convex function such that $M(0) = 0$. We shall also assume that M is nondegenerate, i.e. $M(t) > 0$ if $t > 0$. The Orlicz sequence space ℓ_M consists of all sequences (x_n) for which

$$\sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) < \infty \quad \text{for some } \rho > 0 ,$$

h_M denotes the subspace of ℓ_M where

$$\sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) < \infty \quad \text{for all } \rho > 0.$$

ℓ_M is a Banach space under the (Luxemburg) norm

$$\|x\|_M = \inf\left\{\rho > 0 \;\middle|\; \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) \leq 1\right\}.$$

(Actually, the infimum is attained.) Under suitable conditions $(h_M)^{**}$ is canonically isometric with ℓ_M, namely if the conjugate function M^* satisfies the Δ_2-condition at zero, which is $\limsup_{t\to 0} M^*(2t)/M^*(t) < \infty$. The conjugate function is defined by $M^*(u) = \max\{tu - M(t) \mid 0 < t < \infty\}$. We refer to Chapter 4 of [421] for these notions, basic facts (to be used below without further comment) and nontrivial results on Orlicz sequence spaces.
In the case of function spaces the investigations can be reduced to the intervals $I = (0,1)$ resp. $I = (0,\infty)$ equipped with Lebesgue measure. For an Orlicz function M as above, the Orlicz spaces $L_M(I)$ and $H_M(I)$ are defined by

$$\begin{aligned} L_M(I) &= \left\{f \;\middle|\; \int_I M\left(\frac{|f(s)|}{\rho}\right) ds < \infty \quad \text{for some } \rho > 0\right\}, \\ H_M(I) &= \left\{f \;\middle|\; \int_I M\left(\frac{|f(s)|}{\rho}\right) ds < \infty \quad \text{for all } \rho > 0\right\}. \end{aligned}$$

The norm in L_M is of course

$$\|f\|_M = \inf\left\{\rho > 0 \;\middle|\; \int_I M\left(\frac{|f(s)|}{\rho}\right) ds \leq 1\right\}.$$

The relevant Δ_2-conditions have to be checked at ∞ if $I = (0,1)$ and at both zero and ∞ if $I = (0,\infty)$. If M^* satisfies the Δ_2-condition, then $(H_M)^{**} = L_M$, and $H_M = L_M$ if (and only if) M satisfies the Δ_2-condition (cf. [422, p. 120]).

The Lorentz spaces $d(w,1)$ and $L^{p,1}$. Let $w = (w_n)$ be a decreasing sequence of real numbers such that $w_1 = 1$, $w_n \to 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. The Lorentz sequence space $d(w,1)$ consists of all sequences (u_n) such that

$$\|(u_n)\|_{w,1} = \sup_{\pi} \sum_{n=1}^{\infty} |u_{\pi(n)}| w_n < \infty$$

where π ranges over all the permutations of $\mathbb{N}$. In the case $w_n = n^{\frac{1}{p}-1}$ $(1 < p \leq \infty)$ this space is known as $\ell^{p,1}$. Under the canonical duality the dual of $d(w,1)$ can be represented as the space of sequences (x_n) for which $\sum x_n u_n$ is absolutely convergent for all $(u_n) \in d(w,1)$. This implies $(x_n) \in c_0$. If (x_n^*)[1] denotes the nonincreasing

[1] We adopt here the symbol x_n^* and f^* for the decreasing rearrangement of a sequence or a function since it is commonly used in the literature. No confusion should arise with the notation x^* used in this book to denote an element of the dual space X^*.

rearrangement of the sequence of moduli $(|x_n|)$, the dual norm can be calculated as follows:

$$\|(x_n)\| = \sup_n \frac{\sum_{j=1}^n x_j^*}{\sum_{j=1}^n w_j}$$

It is also known that the subspace $d(w,1)_*$ of $d(w,1)^*$ consisting of all sequences for which

$$\lim_n \frac{\sum_{j=1}^n x_j^*}{\sum_{j=1}^n w_j} = 0$$

is a predual of $d(w,1)$, which we call the canonical predual. Moreover, $d(w,1)_*$ coincides with the closed linear span of the unit vectors e_n in $d(w,1)^*$. (All the results mentioned above can be found e.g. in [241].)

For Lorentz function spaces let I be one of the intervals $(0,1)$ or $(0,\infty)$ equipped with Lebesgue measure. Although the following could be done for the spaces $L(W,1)$ considered in [422, p. 120], we confine ourselves to $L^{p,1}$.

For a measurable function f on I let f^* be its nonincreasing rearrangement and f^{**} its maximal function

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

If

$$\|f\|_{p,1} = \frac{1}{p}\int_I s^{\frac{1}{p}} f^*(s)\, \frac{ds}{s}\,,$$

then $L^{p,1}(I) = \{f \mid \|f\|_{p,1} < \infty\}$, and $\|\,.\,\|_{p,1}$ is a complete norm on $L^{p,1}$. If $1 < p < \infty$ and $1/p + 1/q = 1$, then the dual of $L^{p,1}$ coincides with the "weak L^q-space" $L^{q,\infty}$, and the dual norm is

$$\|f\|_{q,\infty} = \sup_t t^{\frac{1}{q}} f^{**}(t).$$

(See Lorentz' paper [425] where $L^{p,1}$ is denoted by $\Lambda(\frac{1}{p})$ and $L^{q,\infty}$ by $M(\frac{1}{p})$, or compare Chapter IV.4 of the monograph [70].) It is known that the closure of the set of bounded functions whose support has finite measure in $(L^{p,1})^* = L^{q,\infty}$ is a predual of $L^{p,1}$; this follows for instance from Theorem I.4.1 in [70]. We denote this closure by $(L^{p,1})_*$ and call it the canonical predual of $L^{p,1}$.

The Banach algebra H^∞ and the Hardy spaces H^p. If F is a bounded analytic function on the open unit disk $\mathbb{D}$, then a well-known result in complex variables asserts that

$$\lim_{r\to 1} F(rz) =: f(z)$$

exists for almost every $z \in \mathbb{T}$. The mapping which assigns to each such F its boundary function $f \in L^\infty(\mathbb{T})$ then identifies the space $H^\infty(\mathbb{D})$ isometrically with a closed subalgebra of $L^\infty(\mathbb{T})$ which is denoted by H^∞. (Often we shall not explicitly distinguish between $H^\infty(\mathbb{D})$ and H^∞.) Also, H^∞ is the weak* closed linear span of the functions $1, z, z^2, \ldots$ in $L^\infty(\mathbb{T})$. Likewise the boundary function f is known to exist almost everywhere if F is an analytic function on $\mathbb{D}$ subject to the growth condition

$$\sup_{r<1}\left(\frac{1}{2\pi}\int_0^{2\pi} |F(re^{it})|^p dt\right)^{1/p} =: \|F\|_p < \infty,$$

and $\|F\|_p$ coincides with the norm of f in $L^p(\mathbb{T})$. These boundary functions make up a closed subspace of $L^p(\mathbb{T})$ denoted by H^p, which is called a Hardy space.
Under the duality $\langle f, g\rangle = \int f\overline{g}\, dt/2\pi$ the dual of $H^1_0 = \{f \in H^1 |\ f(0) = 0\}$ is isometric with the quotient space $L^\infty(\mathbb{T})/(H^1_0)^\perp = L^\infty(\mathbb{T})/H^\infty$. On the other hand, H^1_0 is isometric with the dual of the quotient space $C(\mathbb{T})/A$ by the classical F. and M. Riesz theorem. (Here A denotes the disk algebra.) Consequently $L^\infty(\mathbb{T})/H^\infty$ "is" the bidual of $C(\mathbb{T})/A$, and the canonical copy of $C(\mathbb{T})/A$ in its bidual is the quotient space $(H^\infty + C(\mathbb{T}))/H^\infty$. To see this one has to take into account that the distance of a continuous function to A is the same as its distance to H^∞. (We remark in passing that this distance formula and hence the identification of the canonical copy of $C(\mathbb{T})/A$ in $L^\infty(\mathbb{T})/H^\infty$ is the essence of Sarason's proof that $H^\infty + C(\mathbb{T})$ is closed in $L^\infty(\mathbb{T})$, cf. [169, p. 163] or [243, p. 376].) For a detailed discussion of the above facts and more information we refer to the monographs [169], [180] or [243]. We also point out that it is convenient in this setting to abbreviate $C(\mathbb{T})$ by C and $L^\infty(\mathbb{T})$ by L^∞.

The spaces of analytic functions with weighted supremum norm $A_0(\phi)$ and $A_\infty(\phi)$. We denote by $\mathbb{D}$ the open unit disk and let $\phi : [0, 1] \to \mathbb{R}$ be a positive continuous decreasing function such that $\phi(0) = 1$ and $\phi(1) = 0$. We define

$$\begin{aligned} A_\infty(\phi) &= \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ analytic}, \|f\|_\phi = \sup_{z\in\mathbb{D}} |f(z)| \cdot \phi(|z|) < \infty\}, \\ A_0(\phi) &= \{f \in A_\infty(\phi) \mid \lim_{|z|\to 1} |f(z)| \cdot \phi(|z|) = 0\}. \end{aligned}$$

These are Banach spaces under the norm $\|\,.\,\|_\phi$, and $A_\infty(\phi)$ is canonically isometric with $A_0(\phi)^{**}$ ([539] or [78]).

The Bloch spaces B_0 and B. These are the spaces

$$B = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ analytic},\ f(0) = 0,\ \|f\| = \sup_{z\in\mathbb{D}} |f'(z)| \cdot (1 - |z|^2) < \infty\},$$

and

$$B_0 = \{f \in B \mid \lim_{|z|\to 1} |f'(z)| \cdot (1 - |z|^2) = 0\}$$

(the "little Bloch space"). That B is the bidual of B_0 and more information on Bloch functions and the Bloch spaces can be found in the surveys [37] and [131].

Examples 1.4 *The following spaces are M-embedded spaces:*

(a) c_0 *(more generally $c_0(\Gamma)$).*
(b) *The Orlicz sequence space h_M if M^* satisfies the Δ_2-condition at zero while M fails it.*
(c) *The canonical predual $d(w, 1)_*$ of the Lorentz sequence space $d(w, 1)$.*
(d) *The Orlicz function space $H_M(I)$ if M^* satisfies the appropriate Δ_2-condition while M fails it.*
(e) *The canonical predual $(L^{p,1})_*$ of the Lorentz function space $L^{p,1}$.*
(f) *$K(H)$ if H a Hilbert space.*

(g) $K(\ell^p, \ell^q)$ *if* $1 < p \le q < \infty$.
(h) $C(\mathbb{T})/A$ *where* $\mathbb{T}$ *denotes the unit circle and* A *the disk algebra.*
(i) *The weighted spaces of analytic functions* $A_0(\phi)$.
(j) *The little Bloch space* B_0.

PROOF: All arguments except that for (h) use the restricted 3-ball property (Theorem I.2.2) with the idea outlined on page 102.

(a) This is immediate; see also Example I.1.4(a).

(b) By what was said above the bidual of h_M is ℓ_M in this situation. (The assumption on M is made only to exclude the reflexive case.) Suppose $x^{**} \in B_{\ell_M}$, i.e. $\sum M(|x^{**}(n)|) \le 1$, $x_i \in B_{h_M}$ $(i = 1, 2, 3)$ and $\varepsilon > 0$ are given. There is no loss of generality in assuming $\|x_i\| < 1$ and $x_i(n) = 0$ for $n > n_0$, since such sequences are dense in the unit ball of h_M. Then $\sum_{n=1}^{n_0} M(|x_i(n)|) < 1$. Choose $m_0 > n_0$ such that

$$\sum_{n=1}^{n_0} M(|x_i(n)|) + \sum_{n=m_0}^{\infty} M(|x^{**}(n)|) \le 1.$$

Put $y(n) = x^{**}(n)$ for $n < m_0$ and $y(n) = 0$ otherwise. Then $y \in h_M$ and

$$\sum M(|x^{**}(n) + x_i(n) - y(n)|) = \sum_{n=1}^{n_0} M(|x_i(n)|) + \sum_{n=m_0}^{\infty} M(|x^{**}(n)|) \le 1,$$

hence $\|x^{**} + x_i - y\| \le 1$.

(c) We have already remarked that the bidual of $d(w,1)_*$ can canonically be identified with $d(w,1)^*$ and that $\operatorname{lin}(e_n)$ is dense in $d(w,1)_*$. Given $x^{**} = (a(n)) \in B_{d(w,1)^*}$, $x_i = (x_i(n)) \in B_{d(w,1)_*}$ with finite support, and $\varepsilon > 0$ we first show the existence of $y \in d(w,1)_*$ with $\|x^{**} + x_i - y\| \le 1 + \varepsilon$ under the additional assumption

$$\max\{j \mid x_i^*(j) \ne 0\} =: k_i = k \qquad \text{for all } i$$

and

$$\sum_{j=1}^{k} x_i^*(j) \le \sum_{j=1}^{k} a^*(j) \qquad \text{for all } i.$$

Pick N such that $x_i(n) = 0$ for $n > N$ and

$$|a(n)| \le \min\{\delta, a^*(k)\} \qquad \text{for } n > N,$$

where $\delta = \min_i x_i^*(k)$. Then define the sequence $y = (y(n))$ by $y(n) = a(n)$ if $n \le N$ and $y(n) = 0$ otherwise. If $z_i(n) = a(n) + x_i(n) - y(n)$, then

$$\begin{aligned} z_i^*(j) &= x_i^*(j) \qquad \text{for } j \le k, \\ z_i^*(j) &\le a^*(j) \qquad \text{for } j > k \end{aligned}$$

by construction. Hence

$$\frac{\sum_{j=1}^{n} z_i^*(j)}{\sum_{j=1}^{n} w(j)} \le 1 \qquad \text{if } n \le k,$$

since $\|x_i\| \le 1$, and for $n > k$ we have by the second part of our assumption

$$\frac{\sum_{j=1}^{n} z_i^*(j)}{\sum_{j=1}^{n} w(j)} \;\le\; \frac{\sum_{j=1}^{n} a^*(j)}{\sum_{j=1}^{n} w(j)} \;\le\; \|x^{**}\| \;\le\; 1,$$

hence $\|x^{**} + x_i - y\| \le 1$.
To dispose of the extra assumption we may clearly demand $x^{**} \notin d(w,1)_*$ (because otherwise one could trivially choose $y = x^{**}$). In this case we cannot have $x^{**} \in \ell^1$ either. Hence we find $l \ge k_i$ such that

$$\sum_{j=1}^{k_i} x_i^*(j) < \sum_{j=1}^{l} a^*(j). \tag{$*$}$$

We now modify x_i as follows: If $x_i(n) \neq 0$, then let $\xi_i(n) = x_i(n)$. At $l - k_i$ indices where $x_i(n) = 0$ let $\xi_i(n) = \eta$ (here $\eta > 0$ is a small number to be specified in a moment); otherwise let $\xi_i(n) = 0$. The number η should be chosen so small that $\|x_i - \xi_i\| \le \varepsilon$ and

$$\sum_{j=1}^{l} \xi_i^*(j) \le \sum_{j=1}^{l} a^*(j)$$

hold; this is possible by $(*)$. By the first part of the proof, some $y \in d(w,1)_*$ such that

$$\left\| x^{**} + \frac{\xi_i}{1+\varepsilon} - y \right\| \le 1$$

exists, hence $\|x^{**} + x_i - y\| \le 1 + 2\varepsilon$.

(d) The idea of the proof is the same as in part (e) below, while the details, which we leave to the reader, follow – mutatis mutandis – the proof of part (b).

(e) We have already pointed out that the bidual is the space

$$L^{q,\infty}(I) = \{ f \mid \|f\|_{q,\infty} < \infty \}$$

where $1/p + 1/q = 1$. We first deal with the case $I = (0,1)$. Suppose $f \in L^{q,\infty}$, $\|f\|_{q,\infty} \le 1$, bounded functions g_i with $\|g_i\|_{q,\infty} \le 1$ and some $\varepsilon > 0$ are given. (The boundedness assumption may be imposed on the g_i since the bounded functions form a dense subspace of $(L^{p,1})_*$.) The idea is to define the function g by $\chi_{\{|f| \le r\}} \cdot f$ for sufficiently large r.
Since the g_i and hence the g_i^{**} are bounded, there exists some $t_1 > 0$ such that

$$t^{\frac{1}{q}} \cdot g_i^{**}(t) \le \varepsilon \;\; \text{for } t \le t_1,$$

whereas

$$t^{\frac{1}{q}} \cdot g_i^{**}(t) \le 1 \;\; \text{for } t > t_1$$

anyway. Next we choose $t_0 \le t_1$ such that

$$\int_0^{t_0} f^*(s)\ ds \le \varepsilon\, t_1^{\frac{1}{p}},$$

and finally let $r = f^*(t_0)$. Then $g = \chi_{\{|f|\le r\}} \cdot f$ is bounded and hence in $(L^{p,1}(0,1))_*$. By the subadditivity of the maximal operator $(\phi+\psi)^{**} \le \phi^{**} + \psi^{**}$ [70, Theorem II.3.4] we have

$$\|f + g_i - g\|_{q,\infty} \le \sup_t \Big(t^{\frac{1}{q}} \cdot g_i^{**}(t) + t^{\frac{1}{q}} \cdot (f-g)^{**}(t) \Big)$$

and for $t > t_1$ one computes

$$\begin{aligned} t^{\frac{1}{q}} \cdot (f-g)^{**}(t) &= t^{-\frac{1}{p}} \int_0^t (f-g)^*(s)\ ds \\ &\le t_1^{-\frac{1}{p}} \int_0^{t_0} (f-g)^*(s)\ ds \\ &\le \varepsilon \end{aligned}$$

by the choice of t_0. Since for $t \le t_1$

$$t^{\frac{1}{q}} \cdot (f-g)^{**}(t) \le \|f-g\|_{q,\infty} \le \|f\|_{q,\infty} \le 1,$$

these estimates, together with the ones for g_i^{**}, prove the inequality

$$\|f + g_i - g\|_{q,\infty} \le 1 + \varepsilon.$$

In the case of the infinite interval $(0,\infty)$ one has to combine the above approach (at 0) with the method of part (c) (at ∞) to accomplish the proof. We omit the details. (Note that $L^{p,1}(0,1)$ and $L^{p,1}(0,\infty)$ are not even isomorphic for $1 < p < \infty$ [111].)

(f) It is well known that the bidual of $K(H)$ is $L(H)$ (see for example [592, p. 64] or [158, p. 247]). Hence the result follows from the coincidence of M-ideals and closed two-sided ideals in C^*-algebras: Theorem V.4.4.

(g) In Example VI.4.1 we will prove that $K(\ell^p, \ell^q)$ in an M-ideal in $L(\ell^p, \ell^q)$. This, however, is our assertion since the latter space is the bidual of $K(\ell^p, \ell^q)$ [158, p. 247]. (The condition $p \le q$ in (g) is imposed only to assure nontrivial, meaning nonreflexive examples; see [158, p. 248].)

(h) As mentioned above we will abbreviate $C(\mathbb{T})$ by C and $L^\infty(\mathbb{T})$ by L^∞. To show that $X = C/A = (H^\infty + C)/H^\infty$ is an M-embedded space we will prove that $X^\perp$ is an L-summand in X^{***}. Let K denote the maximal ideal space of L^∞. In the following we identify, using the Gelfand transform $f \mapsto \widehat{f}$, H^∞ and C with proper closed subalgebras of $C(K)$ ($\cong L^\infty$). Then our problem is to find an L-projection from from $X^{***} \cong (H^\infty)^\perp$ ($\subset M(K)$) onto $X^\perp \cong (H^\infty + C)^\perp$.

To exhibit such a projection, the basic idea is this. Let us denote by m the normalized Lebesgue measure on $\mathbb{T}$. Then there is, by the Riesz representation theorem, a uniquely determined probability measure $\widehat{m} \in M(K)$ satisfying

$$\int_{\mathbb{T}} f\, dm = \int_K \widehat{f}\, d\widehat{m} \qquad \forall f \in L^\infty.$$

The Lebesgue decomposition $\mu = \mu_a + \mu_s$ with respect to $\widehat{m}$ gives rise to an L-projection $P : \mu \mapsto \mu_s$ acting on $M(K)$ (Example I.1.6(b)). We intend to show that $(H^\infty)^\perp$ is an invariant subspace of P and that P maps $(H^\infty)^\perp$ onto $(H^\infty + C)^\perp$ which clearly settles our problem. For this the use of an abstract F. and M. Riesz theorem will turn out to be instrumental.
We will now give the details of this argument. The F. and M. Riesz theorem we have in mind reads as follows [95, Th. 4.2.2, Cor. 4.2.3] (a more general version can be found in [239, Th. II.7.6, Cor. II.7.9]):

> Theorem (Abstract F. and M. Riesz theorem) *Let B be a function algebra on the compact space K and ϕ be a multiplicative linear functional which admits a unique representing measure σ on K. Writing the Lebesgue decomposition of a measure $\mu \in M(K)$ with respect to σ as $\mu = \mu_a + \mu_s$ one obtains:*
>
> (a) *If $\mu \in B^\perp$, then $\mu_a, \mu_s \in B^\perp$.*
> (b) *If $\mu \in B_\phi^\perp := (\ker \phi)^\perp$, then $\mu_s \in B^\perp$ and $\mu_a \in B_\phi^\perp$.*

We shall apply this theorem to the algebra H^∞ (strictly speaking to its image under the Gelfand transform) with the above K and the functional

$$\phi(f) = \int_{\mathbb{T}} f \, dm.$$

Since $\phi(f) = F(0)$ if f is the boundary function of the bounded analytic function F on $\mathbb{D}$, ϕ is clearly multiplicative. Moreover, $\widehat{m}$ represents ϕ by definition, and $\widehat{m}$ is unique with respect to this property as shown for instance in [95, Lemma 4.1.1], [239, p. 38] or [243, p. 201]. (The crucial property to be used there is that H^∞ is a so-called logmodular algebra, see [239, p. 37] or [243, p. 66].) Part (a) of the above theorem now assures that $P((H^\infty)^\perp) \subset (H^\infty)^\perp$.
It is left to show that $P((H^\infty)^\perp) = (H^\infty + C)^\perp$. Since $P((H^\infty)^\perp) = M_{\text{sing}}(K) \cap (H^\infty)^\perp$ (compare Lemma I.1.15) this amounts to showing that

$$(H^\infty)^\perp \cap M_{\text{sing}}(K) = (H^\infty)^\perp \cap C^\perp.$$

To see "$\subset$" let $\mu \in (H^\infty)^\perp$ be $\widehat{m}$-singular. It will be convenient to denote the function $z \mapsto z^k$ $(z \in \mathbb{T},\ k \in \mathbb{Z})$ simply by z^k. In the notation of part (b) of the abstract F. and M. Riesz theorem the measure $\widehat{z^{-1}}\mu$ belongs to $(H^\infty)_\phi^\perp$ (since $\mu \in (H^\infty)^\perp$) so that by $\widehat{m}$-singularity $(\widehat{z^{-1}}\mu)_s = \widehat{z^{-1}}\mu \in (H^\infty)^\perp$. Continuing inductively we find that $\widehat{z^{-k}}\mu \in (H^\infty)^\perp$ for all $k \in \mathbb{N}$, hence in particular $\int \widehat{z^{-k}}\, d\mu = 0$. Since $\int \widehat{z^n}\, d\mu = 0$ for $n \in \mathbb{N}_0$ anyway, the density of $\operatorname{lin}\{z^k \mid k \in \mathbb{Z}\}$ (= the trigonometric polynomials) in $C = C(\mathbb{T})$ shows the desired inclusion.
For the proof of "$\supset$" take $\mu \in (H^\infty)^\perp$ with $\int \widehat{g}\, d\mu = 0$ for all $g \in C$. We want to show that its absolutely continuous part μ_a vanishes. By P-invariance of $(H^\infty)^\perp$ we have $\mu_s \in (H^\infty)^\perp$, so the inclusion already proved shows that $\int \widehat{g}\ d\mu_s = 0$ for all $g \in C$. Then by assumption this holds also for $\mu_a = \mu - \mu_s$. Thus, writing $\mu_a = f\widehat{m}$ for some $f \in L^1(\widehat{m})$, we obtain

$$\int \widehat{g} f \, d\widehat{m} = 0 \qquad \text{for all } g \in C.$$

Since the Gelfand transform of L^∞ extends uniquely to an isometric isomorphism between $L^1(\mathbb{T}, m)$ and $L^1(K, \widehat{m})$ (see [243, p. 202]) this allows us to conclude $f = 0$.

(i) Let $f \in A_\infty(\phi)$, $g_i \in A_0(\phi)$ with norm ≤ 1, and $\varepsilon > 0$ be given. For $r < 1$ let $f_r(z) = f(rz)$. Then $f_r \in A_0(\phi)$, $\|f_r\|_\phi \le \|f\|_\phi$ (since ϕ is decreasing) and $\lim_{r\to 1} f_r = f$ uniformly on compact subsets of $\mathbb{D}$.
First choose $\varrho_1 < 1$ such that

$$|g_i(z)| \cdot \phi(|z|) \le \varepsilon \qquad \text{for } |z| \ge \varrho_1$$

and then $r_1 < 1$ with

$$\sup_{|z|\le\varrho_1} |f(z) - f_{r_1}(z)| \le \varepsilon.$$

Since $f_{r_1} \in A_0(\phi)$, we have for some $\varrho_2 > \varrho_1$

$$|f_{r_1}(z)| \cdot \phi(|z|) \le \varepsilon \qquad \text{for } |z| \ge \varrho_2$$

so that

$$|f(z) - f_{r_1}(z)| \cdot \phi(|z|) \le 1 + \varepsilon \qquad \text{for } |z| \ge \varrho_2.$$

In the next step we pick $r_2 \ge r_1$ such that

$$\sup_{|z|\le\varrho_2} |f(z) - f_{r_2}(z)| \le \varepsilon.$$

In this way we find increasing sequences (ϱ_k) and (r_k) such that $|f(z) - f_{r_k}(z)| \cdot \phi(|z|) \le 1 + \varepsilon$ for all z except in the annulus $\varrho_k < |z| < \varrho_{k+1}$. Now fix an integer $m \ge 2/\varepsilon$ and let $h = \frac{1}{m}\sum_{k=1}^m f_{r_k}$. Then $h \in A_0(\phi)$, and we claim

$$|f(z) + g_i(z) - h(z)| \cdot \phi(|z|) \le 1 + 3\varepsilon \qquad \text{for all } z \in \mathbb{D}.$$

If $|z| \le \varrho_1$, this is clear since $\|g_i\|_\phi \le 1$ and $|f(z) - f_{r_k}(z)| \le \varepsilon$ for all k. Otherwise, we have $|g_i(z)| \cdot \phi(|z|) \le \varepsilon$ and $|f(z) - f_{r_k}(z)| \cdot \phi(|z|) \le 1 + \varepsilon$ for all but possibly one k. However, by what was noted above $\|f - f_{r_k}\|_\phi \le 2$ holds anyway so that

$$|f(z) - h(z)| \cdot \phi(|z|) \;\le\; \frac{m-1}{m}(1+\varepsilon) + \frac{1}{m} \cdot 2 \;\le\; 1 + 2\varepsilon.$$

(j) This is a special case of (i) since B_0 is isometric (by $f \mapsto f'$) to $A_0(\phi)$ with the weight function $\phi(r) = 1 - r^2$. □

Remarks: (a) The induction argument used in the proof of the part (h) above is similar to the one employed in [239, Th. 7.10, p. 45] to deduce the classical F. and M. Riesz theorem from the general one.

(b) We shall later obtain several proofs of the fact that $C(\mathbb{T})/A$ is M-embedded as consequences of more general results; see Remark 1.8 and Example IV.4.11. Although the above proof is not the shortest possible, it is very natural if one adopts the idea (compare I.1.15 and I.1.16) that the desired L-projection should be the restriction of an L-projection of the superspace $M(K)$ leaving the subspace $(H^\infty)^\perp$ invariant. The abstract F. and M. Riesz theorem is in fact a statement about invariant subspaces of L-projections so that its use above seems quite appropriate.

(c) We will give more examples of Banach spaces which are M-ideals in their biduals in the Notes and Remarks section.

Some consequences of general results on M-embedded spaces for concrete examples will be explicitly noted in the following sections. We start with the answer to a question of Sarason and Adamjan, Arov and Krein; cf. [38, p. 608].

Corollary 1.5 *$H^\infty + C$ is proximinal in L^∞.*

Although $H^\infty + C$ is not an M-ideal in L^∞ (because it contains the constant functions), so that Proposition II.1.1 is not applicable right away, we will derive this by M-ideal methods.

Proof: It was noted above that the canonical copy of C/A in its bidual is $(H^\infty + C)/H^\infty$. So this space is, as an M-ideal, proximinal in L^∞/H^∞ by Proposition II.1.1. Combined with the following elementary observation

> *If X is a Banach space and E and F are subspaces with $E \subset F \subset X$ such that F/E is proximinal in X/E and E is proximinal in X, then F is proximinal in X.*

and the fact that H^∞ is proximinal in L^∞ (as a w^*-closed subspace), this yields the claim. ☐

The next theorem describes the hereditary properties of M-embedded spaces.

Theorem 1.6 *The class of M-embedded Banach spaces is stable by taking*

(a) *subspaces,*
(b) *quotients,*
(c) *c_0-sums.*

Proof: (a) For a subspace Y of X we have to show by Proposition 1.2 that π_{Y^*} is an L-projection. Denote by $i : Y \longrightarrow X$ the inclusion mapping and recall that $\pi_{Y^*} i^{***} = i^{***}\pi_{X^*}$. Moreover, given $y^{***} \in Y^{***}$ one can find $x^{***} \in X^{***}$ with $i^{***}x^{***} = y^{***}$ and $\|x^{***}\| \le \|y^{***}\|$. For y^{***} and x^{***} as above we have (using that π_{X^*} is an L-projection)

$$\begin{aligned}
\|y^{***}\| &\le \|\pi_{Y^*} y^{***}\| + \|y^{***} - \pi_{Y^*} y^{***}\| \\
&= \|\pi_{Y^*} i^{***} x^{***}\| + \|i^{***} x^{***} - \pi_{Y^*} i^{***} x^{***}\| \\
&= \|i^{***} \pi_{X^*} x^{***}\| + \|i^{***} x^{***} - i^{***} \pi_{X^*} x^{***}\| \\
&\le \|i^{***}\| \left(\|\pi_{X^*} x^{***}\| + \|x^{***} - \pi_{X^*} x^{***}\|\right) \\
&= \|x^{***}\| \\
&\le \|y^{***}\|.
\end{aligned}$$

This shows that π_{Y^*} is an L-projection.

(b) Replace i in the above proof by the quotient map $q : X \longrightarrow X/Y$ and note that q^{***} is isometric.

(c) If X_i $(i \in I)$ are M-embedded spaces and X is the c_0-sum $(\oplus \sum X_i)_{c_0(I)}$ then $X^{**} = (\oplus \sum X_i^{**})_{\ell^\infty(I)}$. Direct verification of the 3-ball property shows the claim. □

Remarks 1.7 (a) Although subspaces and quotients of M-embedded spaces are again M-embedded, this property is *not* a 3-space property, i.e.

Y a subspace of X, Y and X/Y M-embedded $\not\Longrightarrow$ X M-embedded.

[Consider e.g. $X = c_0 \oplus_1 c_0$ and $Y = c_0 \times \{0\}$. Since X^{**} has a nontrivial L-summand, it can't contain a nontrivial M-ideal by Theorem I.1.8.]

(b) The property of being an M-embedded space is clearly not isomorphically invariant, however it is invariant under almost isometric isomorphism. Recall that the Banach-Mazur distance of two Banach spaces X and Y is defined as $d(X,Y) = \inf\{\|T\|\|T^{-1}\| \mid T : X \to Y \text{ isomorphism}\}$ and that X and Y are said to be almost isometric if $d(X,Y) = 1$.

If X is an M-embedded space and Y a Banach space with $d(X,Y) = 1$, then Y is M-embedded.

[This is straightforward using the 3-ball property.]

Remark 1.8 The stability property of M-embedded spaces combined with the theorems of Nehari and Hartman (see below) immediately gives a second proof of Example 1.4(h): Recall that an operator $T \in L(H^2)$ is called a *Hankel operator* on the Hardy space H^2 if there is a sequence $(a_n)_{n\in\mathbb{N}_0}$ such that $t_{ij} := \langle T\gamma_j, \gamma_i \rangle = a_{i+j}$, where $i, j \in \mathbb{N}_0$ and $\gamma_k(z) = z^k$, $k \in \mathbb{Z}$. [The harmonic analysis type notation γ_k for the natural basis vectors is convenient also here; cf. Section IV.4.] The condition on T is easily seen to be equivalent to $S^*T = TS$, where S denotes the unilateral shift on H^2, i.e. $S\gamma_i = \gamma_{i+1}$, and S^* is its Hilbert space adjoint.
To give the precise statements of the quoted theorems we need some more notation: For $f \in L^\infty(\mathbb{T})$ we write M_f for the corresponding multiplication operator $M_f g := fg$ on $L^2(\mathbb{T})$. Further, $J \in L(L^2(\mathbb{T}))$ is the complex conjugation $Jf = \overline{f}$, P denotes the orthogonal projection from $L^2(\mathbb{T})$ onto H^2 and $i : H^2 \longrightarrow L^2(\mathbb{T})$ the inclusion map. Finally, we let $H_0^\infty = \{zf \mid f \in H^\infty\} \cong \{f \in H^\infty(\mathbb{D}) \mid f(0) = 0\}$.

Theorem (Nehari and Hartman)

(a) *The mapping*

$$\begin{array}{rccl} \mathfrak{H} : & L^\infty/H_0^\infty & \longrightarrow & L(H^2) \\ & f + H_0^\infty & \longmapsto & H_f := PJM_f i \end{array}$$

is an isometric isomorphism onto the space of Hankel operators on H^2.

(b)
$$d(H_f, K(H^2)) = d(f, H_0^\infty + C)$$

In particular: H_f is a compact operator iff $f \in H_0^\infty + C$.

By the above the restriction of $\mathfrak{H}$ to $C/A_0 \cong (H_0^\infty + C)/H_0^\infty$ is an isometry into $K(H^2)$, so C/A_0 is an M-embedded space by Theorem 1.6(a) and Example 1.4(f). Observing that the isometrical isomorphism $M_{\gamma_1} : C(\mathbb{T}) \longrightarrow C(\mathbb{T})$ maps A onto A_0, we obtain Example 1.4(h). (We refer to the survey article [505] and the books [474] and [506] for a detailed exposition of the theory of Hankel operators. We explicitly mention the remarkably simple geometric proof of the above theorem due to Parrot which is outlined in [505, p. 430–432].)

We will complete our discussion of the stability properties of M-embedded spaces by showing that this is in fact a separably determined property of a Banach space. To show this we need a proposition which is of independent interest.

Proposition 1.9 *For a Banach space X, the following assertions are equivalent:*

(i) *X is an M-ideal in its bidual.*

(ii) *For all $x \in B_X$, all sequences (x_n) in B_X, all weak* cluster points x^{**} of (x_n) and all $\varepsilon > 0$ there is some $u \in \text{co}\,\{x_1, x_2, \ldots\}$ such that*

$$\|x + x^{**} - u\| \leq 1 + \varepsilon.$$

(iii) *For all $x \in B_X$, all sequences (x_n) in B_X and all $\varepsilon > 0$ there is some $n \in \mathbb{N}$ and there are $u \in \text{co}\,\{x_1, \ldots, x_n\}$, $t \in \text{co}\,\{x_{n+1}, x_{n+2}, \ldots\}$ such that*

$$\|x + t - u\| \leq 1 + \varepsilon.$$

(iv) *For all $x \in B_X$ and all $x^{**} \in B_{X^{**}}$ there is a net (x_α) in B_X weak* converging to x^{**} such that*

$$\limsup \|x + x^{**} - x_\alpha\| \leq 1.$$

PROOF: (i) $\Rightarrow$ (ii): If this were false, then the inequality $\|x + x^{**} - u\| > 1 + \varepsilon$ would hold for all $u \in A := \text{co}\,\{x_1, x_2, \ldots\}$. Consequently, A and the ball $B_{X^{**}}(x + x^{**}, 1 + \varepsilon/2)$ could strictly be separated by some $x^{***} \in X^{***}$, with $\|x^{***}\| = 1$ say. By assumption we have a decomposition

$$x^{***} = x^* + x_s^{***} \in X^* \oplus X^\perp, \quad \|x^{***}\| = \|x^*\| + \|x_s^{***}\|.$$

Hence

$$\begin{aligned} 1 + \frac{\varepsilon}{2} &\leq \text{Re}\, x^{***}(u - (x + x^{**})) \\ &\leq |x^*(x)| + |x^*(x^{**} - u)| + |x_s^{***}(x^{**})| \\ &\leq \max\{\|x\|, \|x^{**}\|\}(\|x^*\| + \|x_s^{***}\|) + |x^*(x^{**} - u)| \\ &\leq 1 + |x^*(x^{**} - u)| \end{aligned}$$

for all $u \in A$; however $|x^*(x^{**} - x_n)| < \varepsilon/2$ for some n since x^{**} is a weak* cluster point of the sequence (x_n). This leads to a contradiction.

(ii) $\Rightarrow$ (iii): Let x^{**} be a weak* cluster point of the sequence (x_n). We apply (ii) to obtain $\|x + x^{**} - u\| \leq 1 + \varepsilon/3$ for some $u \in \text{co}\,\{x_1, x_2, \ldots\}$, say $u \in \text{co}\,\{x_1, \ldots, x_n\}$.

Suppose that $\|x+t-u\| > 1+\varepsilon$ holds for each $t \in \operatorname{co}\{x_{n+1}, x_{n+2}, \ldots\} =: A$. Again this implies that A and $B_X(u-x, 1+\varepsilon/2)$ can strictly be separated. Thus, for some $x^* \in X^*$ with $\|x^*\| = 1$

$$1+\varepsilon/2 \le \operatorname{Re} x^*(t-(u-x)) \qquad \forall t \in A.$$

But since $x^{**} \in \overline{A}^{w*}$, this yields

$$\begin{aligned} 1+\frac{\varepsilon}{2} &\le \operatorname{Re} x^*(x+x^{**}-u) \\ &\le \|x+x^{**}-u\| \\ &\le 1+\frac{\varepsilon}{3}, \end{aligned}$$

a contradiction.

(iii) $\Rightarrow$ (iv): Again we argue by contradiction. Suppose that for some $x \in B_X$, $x^{**} \in B_{X^{**}}$ there is no such net. Consequently there is, for some $\varepsilon > 0$, a convex weak* neighbourhood V of x^{**} such that

$$\|x+x^{**}-v\| > 1+\varepsilon \qquad \forall v \in V. \tag{1}$$

Pick $x_1 \in V \cap B_X$ and put $B_1 = -x + x_1 + (1+\varepsilon)B_{X^{**}}$. This is a weak* compact set not containing x^{**} (by (1)), so there is a convex weak* neighbourhood $W_1 \subset V$ of x^{**} such that $W_1 \cap B_1 = \emptyset$. This means

$$\|x+w-x_1\| > 1+\varepsilon \qquad \forall w \in W_1.$$

Next choose $x_2 \in W_1 \cap B_X$, put $B_2 = -x + \operatorname{co}\{x_1, x_2\} + (1+\varepsilon)B_{X^{**}}$ and find a convex weak* neighbourhood $W_2 \subset W_1$ of x^{**} satisfying $W_2 \cap B_2 = \emptyset$, i.e.

$$\|x+w-u\| > 1+\varepsilon \qquad \forall w \in W_2,\ u \in \operatorname{co}\{x_1, x_2\}.$$

Continuing in this manner, we inductively define a sequence of points (x_n) in B_X and a sequence of convex weak* neighbourhoods $V \supset W_1 \supset W_2 \supset \ldots$ such that $x_{n+1} \in W_n$ and

$$\|x+w-u\| > 1+\varepsilon \qquad \forall w \in W_n,\ u \in \operatorname{co}\{x_1, \ldots, x_n\}.$$

for all $n \in \mathbb{N}$. This is a contradiction to (iii), since $\operatorname{co}\{x_{n+1}, x_{n+2}, \ldots\} \subset W_n$.

(iv) $\Rightarrow$ (i): We shall verify that the canonical projection from X^{***} onto X^* is an L-projection. To this end decompose a given $x^{***} \in B_{X^{***}}$ into $x^{***} = x^* + x_s^{***} \in X^* \oplus X^\perp$. For an arbitrary $\varepsilon > 0$ pick $x \in B_X$, $x^{**} \in B_{X^{**}}$ such that $\operatorname{Re} x^*(x) \ge \|x^*\| - \varepsilon$ and $\operatorname{Re} x_s^{***}(x^{**}) \ge \|x_s^{***}\| - \varepsilon$. By (iv) there is a net (x_α) such that, for sufficiently large α,

$$|x^*(x^{**}-x_\alpha)| \le \varepsilon \ \text{ and } \ \|x+x^{**}-x_\alpha\| \le 1+\varepsilon.$$

This yields

$$\begin{aligned} \|x\| + \|x_s^{***}\| - 2\varepsilon &\le \operatorname{Re}\left(x^*(x) + x_s^{***}(x^{**})\right) \\ &\le \operatorname{Re}\langle x^* + x_s^{***}, x+x^{**}-x_\alpha\rangle + \varepsilon \\ &\le \|x^{***}\| \cdot (1+\varepsilon) + \varepsilon \end{aligned}$$

so that in fact $\|x^*\| + \|x_s^{***}\| = \|x^{***}\|$. □

We remark that condition (iv) has implicitly appeared in the discussion of our examples at the beginning of this section. From a conceptual point of view, condition (iii) is the most interesting one. First of all, it is formulated in terms of X alone; no a priori knowledge of the bidual space is needed. (However, applying (iii) in concrete instances seems to be technically unpleasant.) As part (iii) involves just sequences so that only separable subspaces have to be checked, we immediately get:

Corollary 1.10 *A Banach space is an M-ideal in its bidual if and only if every separable subspace is.*

In the remainder of this chapter we will present several properties of M-embedded spaces. These results can be thought of as giving necessary conditions on a Banach space to be M-embedded. Let us conclude this first section with the reference to Proposition VI.4.4 where we will show that X is necessarily M-embedded if $K(X)$ is an M-ideal in $L(X)$.

III.2 Isometric properties

The first group of results in the following section is devoted to the study of surjective isometries and contractive projections in M-embedded spaces and their duals. After this we collect what is known about representation and characterisation of these spaces. At the end various other isometric properties – such as unique preduals and renormings – are discussed.

Proposition 2.1 *If X is an M-ideal in its bidual, then $\pi_{X^{**}}$ is the only contractive projection from $X^{(4)}$ onto X^{**}.*

Proof: We will show the following stronger statement:

$$\|x^{**}\| \le \|x^{(4)} + x^{**}\| \quad \forall\, x^{**} \in X^{**} \quad \text{implies} \quad x^{(4)} \in X^{*\perp}. \tag{$*$}$$

This shows the claim, because, for a contractive projection P from $X^{(4)}$ onto X^{**} and $x^{(4)} \in \ker P$, $(*)$ yields $x^{(4)} \in X^{*\perp}$, therefore $\ker P \subset X^{*\perp}$, hence $P = \pi_{X^{**}}$.
To prove $(*)$ observe first that by assumption we have $X^{***} = X^* \oplus_1 X^\perp$, hence $X^{(4)} = X^{*\perp} \oplus_\infty X^{\perp\perp}$. Since $X \subset X^{\perp\perp}$ we have in particular

$$\|x^{*\perp} + x\| = \max\{\|x^{*\perp}\|, \|x\|\} \quad \text{for} \quad x^{*\perp} \in X^{*\perp},\ x \in X.$$

Under the hypothesis of $(*)$ we now decompose $x^{(4)} = x^{*\perp} - y^{**} \in X^{*\perp} \oplus X^{**}$ and easily get

$$\|y^{**} + x^{**}\| \le \|x^{*\perp} + x^{**}\| \quad \forall x^{**} \in X^{**}. \tag{$\dagger$}$$

Assuming $y^{**} \neq 0$ we find $n \in \mathbb{N}$ such that $n\,\|y^{**}\| > \|x^{*\perp}\|$. By Goldstine's theorem and w^*-lower semicontinuity of the norm, there is a net (y_α) in X such that $y_\alpha \longrightarrow ny^{**}$ with respect to $\sigma(X^{**}, X^*)$ and $\|x^{*\perp}\| \le \|y_\alpha\| \le n\,\|y^{**}\|$. But then by $(\dagger)$

$$\|y^{**} + y_\alpha\| \le \|x^{*\perp} + y_\alpha\| = \max\{\|x^{*\perp}\|, \|y_\alpha\|\} = \|y_\alpha\| \le n\|y^{**}\|,$$

hence (passing to the limit)

$$\|y^{**} + ny^{**}\| \leq n\|y^{**}\|,$$

a contradiction. □

We remark that Proposition 2.1 finds its proper place among Godefroy's general results on uniqueness of preduals (in particular it is a consequence of Theorem 3.1 below and [258, Th. II.1 and Ex. II.2.3.b]). However the above direct proof using the geometric assumption is also interesting.

Proposition 2.2 *Let X be a Banach space which is an M-ideal in its bidual. Then every surjective isometry I of X^{**} is the bitranspose of a surjective isometry of X.*

PROOF: Let such an $I : X^{**} \longrightarrow X^{**}$ be given. As easily seen $P := I^{**-1}\pi_{X^{**}}I^{**}$ is a contractive projection of $X^{(4)}$ onto X^{**}, hence $P = \pi_{X^{**}}$ by Proposition 2.1. But then $\pi_{X^{**}}I^{**} = I^{**}\pi_{X^{**}}$, which is known to imply the w^*-continuity of I. Consequently $I = J^*$ for some surjective isometry $J : X^* \longrightarrow X^*$. To show the claim it is enough to prove that $I(X) = X$, which by the Hahn-Banach theorem means $I^*(X^\perp) = X^\perp$. But $I^* = J^{**}$ is a surjective isometry of X^{***} which maps the L-summand X^* onto itself; therefore it also maps the complementary L-summand $X^\perp$ onto itself. □

In our next theorem we collect some results on the M-structure of M-embedded spaces. Recall from Definition I.3.7 that we denote by $Z(X)$ the centralizer of a Banach space X.

Theorem 2.3 *Suppose X is an M-ideal in X^{**}.*

(a) *Every M-ideal J in X is an M-summand.*

(b) *If X has no nontrivial M-summands, then every nontrivial M-ideal of X^{**} contains X.*

(c) *Every M-projection in X^{**} is the bitranspose of an M-projection in X (in particular, every L-projection in X^* is w^*-continuous).*

(d) $Z(X) \cong Z(X^{**})$.

PROOF: (a) Let P be the M-projection in X^{**} onto $J^{\perp\perp}$. Since M-ideals are left invariant under M-projections (Lemma I.3.5) $P_{|X}$ is an M-projection in X onto $J^{\perp\perp} \cap X = J$ (see Lemma I.1.15).

(b) Let J be a nontrivial M-ideal in X^{**}. Then $J \cap X$ is an M-ideal in X (see Propositions I.1.11 and I.1.17), hence an M-summand by part (a). The assumption gives that $J \cap X$ equals $\{0\}$ or X. We will show that the first case cannot arise. In fact, then J and X are complementary M-summands in $J \oplus X$ by Proposition I.1.11(c). If now $J \neq \{0\}$ there is an $x^{**} \in J \cap (X^{**} \setminus X)$, so $P_X(x^{**})$ is a closed ball in X with radius $d(x^{**}, X)$ – see the introduction to Section II.1. But

> *For every Banach space X and $x^{**} \in X^{**} \setminus X$ the set $P_X(x^{**})$ has no interior points relative to X.*

[Proof: Assuming that this is not the case, we may suppose after a suitable translation that 0 is an interior point of $P_X(x^{**})$, i.e. there is a $\delta > 0$ such that $y \in X$ and $\|y\| < \delta$ imply $\|x^{**}-y\| = \|x^{**}\|$ $(= d(x^{**}, X))$. Take $x^* \in S_{X^*}$ with $\operatorname{Re} x^{**}(x^*) > \|x^{**}\| - \delta/4$ and $x \in S_X$ with $\operatorname{Re} x^*(x) > 1/2$. For $x_\delta := (\delta/2)x$ we have $\|x_\delta\| = \delta/2$ and $\operatorname{Re} x^*(x_\delta) > \delta/4$. So

$$\|x^{**} + x_\delta\| \geq |(x^{**} + x_\delta)(x^*)| \geq \operatorname{Re} x^{**}(x^*) + \operatorname{Re} x^*(x_\delta) > \|x^{**}\|$$

gives a contradiction with $y = -x_\delta$.]
This shows that $J = \{0\}$ if $J \cap X = \{0\}$, so that $J \cap X = X$ as claimed.

(c) If P is an M-projection in X^{**}, then $I := 2P - Id_{X^{**}}$ is a surjective isometry of X^{**}, hence $I = {I_0}^{**}$ for some surjective isometry I_0 of X by Proposition 2.2. It easily follows that P is the bitranspose of an M-projection in X.

(d) Recall from Theorem I.1.10 that $\mathbb{P}_M(E)$ denotes the set of M-projections of a Banach space E. By part (c) the isometric injection $\phi : L(X) \longrightarrow L(X^{**})$, $T \longmapsto T^{**}$ maps $\mathbb{P}_M(X)$ onto $\mathbb{P}_M(X^{**})$, i.e.

$$\mathbb{P}_M(X^{**}) = \phi\left(\mathbb{P}_M(X)\right) \subset \phi\left(\overline{\operatorname{lin}}\, \mathbb{P}_M(X)\right).$$

Hence by Theorem I.3.14(c)

$$Z(X^{**}) = \overline{\operatorname{lin}}\, \mathbb{P}_M(X^{**}) \subset \phi\left(\overline{\operatorname{lin}}\, \mathbb{P}_M(X)\right) \subset \phi\left(Z(X)\right).$$

But the converse inclusion $\phi(Z(X)) \subset Z(X^{**})$ is true in arbitrary Banach spaces X by Corollary I.3.15(a). ☐

Remarks: (a) Part (c) of the above theorem shows that in the situation of (b) X^{**} has no nontrivial M-summands. However, there may be proper M-ideals between X and X^{**}. Take for example $X = K(H)$, H a nonseparable Hilbert space. Then X has no nontrivial M-ideals (Corollary VI.3.7) but $J := \{T \in L(H) \mid \operatorname{ran} T \text{ separable}\}$ is a closed two-sided ideal, hence an M-ideal (Theorem V.4.4) in $L(H) = K(H)^{**}$ and $X \subsetneqq J \subsetneqq X^{**}$. In Corollary 2.12 we will see that also for separable M-embedded spaces X there may exist M-ideals between X and X^{**}. We remark that $L(\ell^p), 1 < p < \infty$, contains no other M-ideals than $K(\ell^p)$ as will be shown in Corollary V.6.6.

(b) The above-mentioned examples also serve to show that there are Banach spaces with a trivial centralizer containing nontrivial M-ideals. (For the centralizer of $L(H)$ cf. Theorem V.4.7 or Theorem VI.1.2.)

(c) We refer to Lemma 4.1 for another result on the continuity of contractive projections in the dual of an M-embedded space.

Lemma 2.4 *Let X be a Banach space and recall that π_{X^*} denotes the natural projection from X^{***} onto X^*, i.e. $\pi_{X^*} = i_{X^*} \circ {i_X}^*$. The following are equivalent:*

(i) *π_{X^*} is the only contractive projection P in X^{***} with $\ker P = X^\perp$.*
(ii) *The only operator $T \in L(X^{**})$ such that $\|T\| \leq 1$ and $T_{|X} = Id_X$ is $T = Id_{X^{**}}$.*
(iii) *For every surjective isometry U of X, the only $T \in L(X^{**})$ such that $\|T\| \leq 1$ and $T_{|X} = U$ is $T = U^{**}$.*

Banach spaces which are M-ideals in their biduals have these properties.

Because of (iii) a space X fulfilling one (hence all) of the above statements is said to have the *unique extension property.*

PROOF: The last statement follows from Proposition I.1.2 and Proposition 1.2.

(i) $\Rightarrow$ (ii): With T as in (ii) define $P := T^*\pi_{X^*}$. Since $Tx = x$ for $x \in X$ one gets $T^*x^* - x^* \in X^\perp$ for $x^* \in X^*$. It is then easy to show that P is a projection with $\ker P = X^\perp$; of course $\|P\| \le 1$. By assumption $x^* = \pi_{X^*}x^* = Px^* = T^*\pi_{X^*}x^* = T^*x^*$, which shows $Tx^{**} = x^{**}$.

(ii) $\Rightarrow$ (i): With P as in (i) define $T := \left(P|_X\right)^* i_{X^{**}}$. Since $\ker P = X^\perp$ one gets $Px^*|_X = x^*|_X$ and this easily yields $T|_X = Id_X$; of course $T \in L(X^{**})$ and $\|T\| \le 1$. The assumption then shows $Px^* = x^*$. Therefore P and π_{X^*} have the same ranges and kernels, so they coincide.

(iii) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): With U and T as in (iii) define $V := (U^{**})^{-1}T$. Then $\|V\| \le 1$ and $V|_X = Id_X$, hence $V = Id_{X^{**}}$ by assumption. □

We refer to [269] for more information on the unique extension property. It is custom made for proving

Proposition 2.5 *If X is a Banach space with the unique extension property and X has the MAP [MCAP], then X^* has the MAP [MCAP] with adjoint operators.*
In particular: If X is an M-embedded space with the MAP [MCAP], then X^* has the MAP [MCAP].

PROOF: In the proof we use the symbols s_{op} and w_{op} for the strong and the weak operator topology.
Let (K_α) be a net in the unit ball of $F(X)\,[K(X)]$ which converges to Id_X in the strong operator topology, in particular $K_\alpha \xrightarrow{w_{op}} Id_X$. We claim

$$K_\alpha^* \xrightarrow{w_{op}} Id_{X^*}, \quad \text{i.e. } x^{**}(K_\alpha^* x^*) \longrightarrow x^{**}(x^*) \quad \text{for all } x^* \in X^*,\ x^{**} \in X^{**}.$$

We will show that every subnet (K_β^*) has a subnet (K_γ^*) such that $K_\gamma^* \xrightarrow{w_{op}} Id_{X^*}$. Indeed, since $(X^*\widehat{\otimes}_\pi X^{**})^* \cong L(X^{**})$, there is by w^*-compactness of $B_{L(X^{**})}$ some $T \in L(X^{**})$ with $\|T\| \le 1$ and a subnet such that $K_\gamma^{**} \xrightarrow{w^*} T$, in particular

$$\left(K_\gamma^{**}x^{**}\right)(x^*) \longrightarrow \left(Tx^{**}\right)(x^*) \quad \text{for } x^* \in X^*,\ x^{**} \in X^{**}. \qquad (*)$$

Using this for $x^{**} = x \in X$ we find by the w_{op}-convergence of (K_α) that $Tx = x$. By the unique extension property we have $T = Id_{X^{**}}$, hence $(*)$ shows our claim.
To finish the proof recall that the topologies w_{op} and s_{op} yield the same dual space (e.g. [178, Theorem VI.1.4]), so

$$Id_{X^*} \in \overline{\text{co}}^{\,w_{op}}(K_\alpha^*) = \overline{\text{co}}^{\,s_{op}}(K_\alpha^*) \subset \overline{B_{F(X^*)}}^{\,s_{op}} \left[\overline{B_{K(X^*)}}^{\,s_{op}}\right]. \qquad □$$

We remark that the MAP passes from X^* to X [158, Cor. 9, p. 244], but the corresponding statement for the MCAP does not hold, as was recently shown by P. Casazza.

Except Proposition 1.9 there is no internal characterisation known – neither isometric, nor isomorphic – of Banach spaces which are M-ideals in their biduals (in contrast to what has been shown in Theorem II.3.10 and Theorem II.4.9 for proper M-ideals). However, the following proposition presents a general decomposition procedure for M-embedded spaces. We are also able to characterise the M-embedded spaces in some classes of Banach spaces completely (see Theorem 3.11 for an isomorphic result in this vein).

Proposition 2.6 *Let X be a Banach space which is an M-ideal in its bidual. Then there exists a set I and Banach spaces X_i ($i \in I$) which are M-ideals in their biduals and contain no nontrivial M-ideals such that $X \cong (\oplus \sum X_i)_{c_0(I)}$.*

PROOF: For $p \in \operatorname{ex} B_{X^*}$ denote by N_p the intersection of all w^*-closed L-summands of X^* containing p. By Proposition I.1.11 N_p is the smallest w^*-closed L-summand containing p. Note that every L-summand of X^* is w^*-closed by Theorem 2.3(c). Then

(1) $N_p = N_q$ or $N_p \cap N_q = \{0\}$ for $p, q \in \operatorname{ex} B_{X^*}$.

(2) M_p, the (w^*-closed) L-summand complementary to N_p is a maximal proper L-summand in X^*.

[To see (1) use Lemma I.1.5 and the Krein-Milman theorem. Further, assuming $X^* = N_p \oplus_1 M_p = N \oplus_1 M$ with $M_p \subset M$ one gets $N_p \supset N$ by Theorem I.1.10. Part (1) quickly shows that $N = \{0\}$ or $N = N_p$.]
By (1) an equivalence relation is defined on $\operatorname{ex} B_{X^*}$ by $p \sim q$ if $N_p = N_q$. Put $I = \operatorname{ex} B_{X^*}/\!\sim$ and $X_i = (M_p)_\perp$ for $p \in i$. By the above $(X_i)_{i \in I}$ is a family of minimal nontrivial M-summands in X with pairwise trivial intersection. Thus we have an isometric embedding of $c_{00}(I, X_i)$ into X (where $c_{00}(I, X_i)$ denotes the space of "sequences" of finite support), hence of $c_0(I, X_i)$ into X. But $c_0(I, X_i) \cong \overline{\operatorname{lin}} \bigcup X_i$ is an M-ideal in X (Proposition I.1.11(a)), hence an M-summand by Theorem 2.3(a). The assumption of a proper inclusion of this embedding would give

$$\{0\} \subsetneqq \Big(\overline{\operatorname{lin}} \bigcup X_i\Big)^\perp = \Big(\bigcup X_i\Big)^\perp = \bigcap {X_i}^\perp = \bigcap M_p$$

so that $\bigcap M_p$ is a nontrivial w^*-closed L-summand which contains no extreme point of B_{X^*}: a contradiction. By Theorems 2.3(a) and 1.6(a), the X_i are as desired. □

Proposition 2.7 *For an L^1-predual space X the following assertions are equivalent:*
(i) *X is an M-ideal in X^{**}.*
(ii) *$X \cong c_0(I)$ for some set I.*

PROOF: (ii) ⇒ (i): Example 1.4(a).
(i) ⇒ (ii): Since X^* is isometric to $L^1(\mu)$ we have $X^{**} \cong C(K)$ for some compact space K, so X, being an M-ideal in X^{**}, is of the form $J_D \cong C_0(K \setminus D)$ for some closed subset D of K (Example I.1.4(a)). For every closed subset A of $K \setminus D$ the M-ideal J_A – considered in $C_0(K \setminus D)$ – is an M-summand by Theorem 2.3(a), hence A is open by Example I.1.4(a). So $I := K \setminus D$ is discrete. □

As an application we get the following Banach space result:

Corollary 2.8 *Let X be a Banach space. Then $d(X, c_0) = 1$ implies $X \cong c_0$.*

PROOF: The assumption implies $d(X^*, \ell^1) = 1$ and, as easily seen, this yields that X^* is an $\mathcal{L}^{1,1+\varepsilon}$-space for all $\varepsilon > 0$. Corollary 5 to Theorem 7.1 in [418] entails $X^* \cong L^1(\mu)$ for some measure space (S, Σ, μ). But X is an M-embedded space by Remark 1.7(b), so Proposition 2.7 implies $X \cong c_0(I)$ for some set I. Clearly, I is countable by separability. □

We remark that it is possible to find two equivalent norms $|\,.\,|_1$ and $|\,.\,|_2$ on c_0 such that

$$d\left((c_0, |\,.\,|_1), (c_0, |\,.\,|_2)\right) = 1 \quad \text{but} \quad (c_0, |\,.\,|_1) \not\cong (c_0, |\,.\,|_2)$$

(see [483, p. 230] or Proposition 2.13 below). The above corollary shows that this cannot be achieved with the natural norm $\|\,.\,\|_\infty$ on c_0.

We now turn to a "noncommutative" analogue of Proposition 2.7.

Proposition 2.9 *Let A be a C^*-algebra. Then the following assertions are equivalent:*

(i) *A is an M-ideal in A^{**}.*

(ii) *A is isometrically $*$-isomorphic to the c_0-sum of algebras of compact operators on some Hilbert spaces.*

Note that by Theorem V.1.10 and Theorem V.4.4 below, (i) admits the following reformulation in purely algebraic terms:

(i*) *A is a two-sided ideal in its enveloping von Neumann algebra.*

Another algebraic characterisation of M-embedded C^*-algebras, proved in [76, Theorem 5.5] (and in part below as well) is:

(iii) *Every maximal commutative C^*-subalgebra of A is generated by its minimal projections.*

PROOF: (ii) $\Rightarrow$ (i): This implication follows from Theorem 1.6(c) and Example 1.4(f).

(i) $\Rightarrow$ (ii): By Proposition 2.6 there is no loss of generality in assuming that A contains no nontrivial M-ideals; that is we may suppose, anticipating Theorem V.4.4, that A contains no nontrivial closed two-sided ideals. In this case we shall show that A is isometrically and algebraically $*$-isomorphic to the C^*-algebra $K(H)$ for some Hilbert space, thus settling our claim. (To show the isomorphism in (ii) to be algebraic in the general case, too, we again invoke Theorem V.4.4.)

In what follows we call a self-adjoint idempotent element p of a C^*-subalgebra $B \subset A$ minimal with respect to B if the only idempotent elements $q \in B$ satisfying $0 \le q \le p$ are $q = 0$ or $q = p$.

Let M be a maximal abelian C^*-subalgebra of A. Thus $M \cong C_0(\Omega)$ for some locally compact Hausdorff space Ω by the Gelfand-Naimark theorem. Then we assert:

$$\begin{aligned} M &= \overline{\text{lin}}\,\{p \in M \mid p \text{ is minimal with respect to } M\} \;=:\; M_1 \\ &= \overline{\text{lin}}\,\{p \in M \mid p \text{ is minimal with respect to } A\} \;=:\; M_2. \end{aligned}$$

Indeed, since M is an M-ideal in M^{**} (by assumption and Theorem 1.6(a)) we conclude from Proposition 2.7 that $M \cong c_0(I)$ for some set I. Hence we have $M = M_1$, and clearly $M_2 \subset M_1$ holds. On the other hand, if an idempotent element $p \in M$ is minimal with respect to M and $0 \le q \le p$ for some idempotent $q \in A$, then p and q commute, and by maximality of M we infer that $q \in M$. Hence p is minimal with respect to A as well, and the inclusion $M_1 \subset M_2$ follows.
As a result, A is the closed linear span of its minimal idempotents, since every self-adjoint element is contained in some M as above.
Now let $p \in A$, $p \neq 0$ be a minimal idempotent of A. Then we claim:

$$pAp = \mathbb{C}p \tag{1}$$

To see this, note that $B := pAp$ is a unital C^*-subalgebra with unit p, and p is clearly minimal with respect to B. Now B is an M-ideal in B^{**} (Theorem 1.6), but on the other hand B has the IP by Proposition II.4.2(b). Hence B is reflexive as a result of Theorem II.4.4, consequently finite dimensional (cf. [360, p. 288]). The minimality of the unit implies that B must be one-dimensional. [If not, B would contain an abelian C^*-subalgebra M with $p \in M$ and $d := \dim(M) \ge 2$. Since M can be identified with $\mathbb{C}^d$, the product being pointwise multiplication, and p with $(1, \dots, 1)$, p cannot be minimal with respect to M, let alone B, if $d \ge 2$.]
Let $H = Ap$. We introduce a scalar product on H as follows. If xp, $yp \in H$, then $py^*xp \in pAp$ is a scalar multiple of p (by (1)) which we denote by $\langle xp, yp \rangle$:

$$\langle xp, yp \rangle p = (yp)^*(xp) = py^*xp \tag{2}$$

Then $\langle xp, xp \rangle = \|\langle xp, xp \rangle p\| = \|(xp)^*(xp)\| = \|xp\|^2$ so that the norm induced by $\langle \cdot, \cdot \rangle$ coincides with the C^*-algebra norm inherited from A. Hence H is a Hilbert space. Finally we define

$$T : A \to L(H), \qquad (Ta)(xp) = axp.$$

Clearly, this is a $*$-homomorphism; and T is injective since otherwise $\ker(T)$ would be a nontrivial closed two-sided ideal of A.
To complete the proof we wish to show that $\operatorname{ran}(T) = K(H)$. First of all, if $u : zp \mapsto \langle zp, yp \rangle xp$ is an operator on H with rank 1, then an easy computation reveals $T(xpy^*) = u$; and it follows that $K(H) \subset \operatorname{ran}(T)$. For the converse inclusion it is enough to show that $Tq \in K(H)$ if q is a minimal idempotent of A. (Recall from the first part of the proof that these q generate A.) Now observe $qAq = \mathbb{C}q$ (by (1)) and consequently

$$Tq\,K(H)\,Tq \subset Tq\,T(A)\,Tq = \mathbb{C}Tq.$$

Thus either $Tq\,K(H)\,Tq = 0$ in which case $Tq = 0$ or $\mathbb{C}Tq = Tq\,K(H)\,Tq \subset K(H)$. Hence $Tq \in K(H)$ in either case, and the proof is finished. □

In the remainder of this section we will collect various other isometric properties of M-embedded spaces. To start with, we recall the following definition: A Banach space X is said to be a *strongly unique predual* (of X^*), if, for every Banach space Y, every isometric isomorphism from X^* onto Y^* is w^*-continuous, hence is the adjoint of an isometric isomorphism from Y onto X. It is well-known (and routine to verify) that

X has this property iff π_{X^*} is the only contractive projection from X^{***} onto X^* with w^*-closed kernel.
Clearly a strongly unique predual is a *unique predual*, which of course means that $X^* \cong Y^*$ implies $X \cong Y$. This latter property is not easy to handle; in particular no example is known of a space which is a unique predual, but not a strongly unique predual. We refer to Godefroy's recent survey [258] for detailed information.

Proposition 2.10 *Let X be a nonreflexive Banach space which is an M-ideal in its bidual. Then*

(a) *X^* is a strongly unique predual of X^{**},*
(b) *X is not a strongly unique predual of X^*.*

PROOF: (a) By what was noted above this is an immediate consequence of Proposition 2.1.

(b) By assumption and Proposition 1.2 we have

$$X^{***} = X^* \oplus_1 X^{\perp}.$$

Consider $X^{\perp}$ as the dual space of X^{**}/X and choose $y^{***} \in X^{\perp}$, $\|y^{***}\| = 1$, and a w^*- (i.e. $\sigma\left((X^{**}/X)^*, (X^{**}/X)\right)$-) closed hyperplane H in $X^{\perp}$ such that

$$1 \le \|y^{***} + h^{***}\| \quad \text{for all} \quad h^{***} \in H.$$

[Take for example a norm-attaining $y^{***} \in (X^{**}/X)^*$, $\|y^{***}\| = 1$, and put $H = \ker i_{x^{**}}$, where $y^{***}(x^{**}) = 1 = \|x^{**}\|$.] Because the w^*-topology on $(X^{**}/X)^*$ coincides with the relative w^*-topology of X^{***} on $X^{\perp}$, the hyperplane H is $\sigma(X^{***}, X^{**})$-closed in X^{***}. Hence for $0 < s \le 1$ and $p^* \in X^*$, $\|p^*\| = 1$,

$$Z := H \oplus \mathbb{K}(y^{***} + sp^*)$$

is w^*-closed in X^{***}. Let P be the projection onto X^* associated with the decomposition

$$X^{***} = X^* \oplus Z.$$

For $x^* + z^{***} \in X^* \oplus Z$ we have

$$x^* + z^{***} = x^* + [h^{***} + \lambda(y^{***} + sp^*)] = (x^* + \lambda sp^*) + (h^{***} + \lambda y^{***}) \in X^* \oplus_1 X^{\perp}.$$

Since $\|\lambda sp^*\| = |\lambda| s \le |\lambda| \le \|h^{***} + \lambda y^{***}\|$ for all $\lambda \in \mathbb{K}$, $h^{***} \in H$ we get

$$\begin{aligned} \|P(x^* + z^{***})\| &= \|x^*\| \\ &\le \|x^* + \lambda sp^*\| + \|\lambda sp^*\| \\ &\le \|x^* + \lambda sp^*\| + \|h^{***} + \lambda y^{***}\| \\ &= \|x^* + z^{***}\|. \end{aligned}$$

So P is a contractive projection onto X^* with w^*-closed kernel Z and is different from π_{X^*}, since $Z \ne X^{\perp}$. □

REMARKS: (a) It is easy to verify that for the projection P in the above proof we have $\|Id - P\| = 1 + s$. For the predual Y of X^* obtained by $Y^{\perp} = Z$ we thus get by some formal calculations $\|Id_{Y^{***}} - \pi_{Y^*}\| = 1 + s$. This shows in particular that there are Banach spaces Y for which $\|Id_{Y^{***}} - \pi_{Y^*}\|$ can be any number between 1 and 2. (The first example of this kind was given in [352] using a renorming of c_0.)

(b) If X is an M-embedded space then we have by Corollary 1.3 and Proposition 2.10(a) that the L-embedded space X^* is a strongly unique predual. However the following is open:

PROBLEM: *Is every L-embedded space X a strongly unique predual of X^*?*

The answer is yes for the "classical" Examples IV.1.1 (see [258, section V]), but in general this problem seems to be difficult. Note that a counterexample would solve the long-standing question whether a Banach space which contains no isomorphic copy of c_0 is a strongly unique predual. (By Theorem IV.2.2 L-embedded spaces are weakly sequentially complete; a fortiori, they do not contain copies of c_0.)

(c) The authors of [27] give an isometric representation of the dual of the little Bloch space B_0 as the minimal Möbius invariant space $\mathcal{M}$, and they ask in [26] whether B_0 is the only isometric predual of $\mathcal{M}$ and whether $\mathcal{M}$ is the only isometric predual of B. An explicit solution of the case of $\mathcal{M}$ is contained in [449], whereas the first question is left open there. Proposition 2.10 and Example 1.4(j) answer both questions.
Incidentally, M-ideal arguments can also be used to reprove Theorem 2 of [133] which asserts that the onto isometries of $B = B_0^{**}$ coincide with the second adjoints of the onto isometries of B_0: Proposition 2.2. We also obtain from Proposition II.4.2 and Theorem II.4.4 that the unit ball of B_0 cannot contain any strongly extreme points, a fact first pointed out in [132] by ad-hoc methods.

In the next proposition we describe a renorming technique for M- (and L-) embedded spaces. In order not to obscure the main idea we don't give the most general form.

Proposition 2.11 *Let X be an M-embedded space, Y a Banach space and $T : Y \longrightarrow X$ a weakly compact operator. Then*

$$|x^*|^* := \|x^*\| + \|T^*x^*\|$$

defines an equivalent dual norm on X^* for which $(X, |\,.\,|)$ is an M-embedded space.

PROOF: Since $x^* \longmapsto \|T^*x^*\|$ is w^*-lower semicontinuous, $|\,.\,|^*$ is an equivalent dual norm (see e.g. [154, p. 106]). To calculate $|\,.\,|^{***}$ on X^{***} without using $|\,.\,|^{**}$ we employ the following argument:
By definition the operator

$$\begin{array}{rcl} S \;:\; (X^*, |\,.\,|^*) & \longrightarrow & X^* \oplus_1 Y^* \\ x^* & \longmapsto & (x^*, T^*x^*) \end{array}$$

is isometric, hence $S^{**} : (X^{***}, |\,.\,|^{***}) \longrightarrow X^{***} \oplus_1 Y^{***}$ is isometric, too. As easily seen $S^{***}x^{***} = (x^{***}, T^{***}x^{***})$, so $|x^{***}|^{***} = \|x^{***}\| + \|T^{***}x^{***}\|$. By w-compactness and w^*-continuity of T^* we get $T^{***} = \pi_{Y^*}T^{***} = T^{***}\pi_{X^*}$, therefore $T^{***}x^{***} = T^*x^*$

where $x^{***} = x^* + x^\perp \in X^* \oplus_1 X^\perp$. Hence

$$\begin{aligned} |x^{***}|^{***} &= |x^* + x^\perp|^{***} \\ &= \|x^* + x^\perp\| + \|T^{***}(x^* + x^\perp)\| \\ &= \|x^*\| + \|x^\perp\| + \|T^* x^*\| \\ &= |x^*|^* + \|x^\perp\| \end{aligned}$$

$x^* = 0$ yields $|x^\perp|^{***} = \|x^\perp\|$ (so that the two norms agree on $X^\perp$), and we can continue

$$= \quad |x^*|^{***} + |x^\perp|^{***}$$

Hence $X^\perp$ is, with respect to the new norm, still an L-summand in X^{***}. □

Applying the above proposition to the formal identity $I : \ell^2 \to c_0$, we find:

Corollary 2.12 *There is an equivalent norm $|\,.\,|$ on c_0 such that $(c_0, |\,.\,|)^*$ is strictly convex, hence $(c_0, |\,.\,|)$ is smooth, and $(c_0, |\,.\,|)$ is an M-ideal in its bidual. Moreover, there are uncountably many M-ideals in $(c_0, |\,.\,|)^{**}$.*

PROOF: The injectivity of $I^* : \ell^1 \to \ell^2$ yields the strict convexity of $|\,.\,|^*$, and the smoothness of $|\,.\,|$ follows [154, p. 23, p. 100]. Every ideal in ℓ^∞ containing c_0 yields an M-ideal in ℓ^∞/c_0 (Prop. I.1.17) and $(c_0, |\,.\,|)^{**}/(c_0, |\,.\,|)$ is isometric to ℓ^∞/c_0 by what was noted in the above proof. □

In Theorem 4.6(e) we will see that a similar renorming is possible for all M-embedded spaces. Also, in Remark IV.1.17 we will point out an example of a strictly convex M-embedded space with a smooth dual, namely $C(\mathbb{T})/A$.

In the last renorming result we show that the existence of nontrivial M-ideals or M-summands is not preserved by almost isometries.

Proposition 2.13 *There is an M-embedded space X isomorphic to c_0 without nontrivial M-ideals such that $d(X, X \oplus_\infty X) = 1$.*

PROOF: Let $E_1 := \mathbb{K}$ and define inductively $E_{n+1} := E_n \oplus_{2^n} E_n$. Put $X_n := c_0(E_n)$ and denote the norm of X_n by $\|\ \|_n$. For $x \in c_0$ we obtain $\|x\|_n \le \|x\|_{n+1} \le 2^{1/2^n}\|x\|_n$, hence

$$\|x\|_n \le 2^{1/2^{n-1}} 2^{1/2^{n-2}} \dots 2^{1/2}\|x\|_1 \le 2^{\sum_{k=1}^\infty 1/2^k}\|x\|_1 = 2\|x\|_1.$$

Thus $\|x\| := \sup_n \|x\|_n$ defines a norm equivalent to $\|\ \|_1$, the usual sup-norm of c_0. Since

$$\|x\|_n \le \|x\| \le 2^{\sum_{k=n}^\infty 1/2^k}\|x\|_n = 2^{2^{-n+1}}\|x\|_n \qquad (*)$$

we get $d(X_n, X) \to 1$ where $X = (c_0, \|\,.\,\|)$.
Since X_n is an M-embedded space by Theorem 1.6, the uniform estimate $(*)$ and the 3-ball property yield that X is M-embedded, too. From $X \overset{1+\varepsilon}{\simeq} X_n \cong X_n \oplus_\infty X_n \overset{1+\varepsilon}{\simeq} X \oplus_\infty X$ we deduce $d(X, X \oplus_\infty X) = 1$. Finally every $x \in c_{00}$ with unit norm is an extreme point of B_X: if $\operatorname{supp}(x) \subset \{1, \dots, 2^n\}$ we get $\|x\|_{n+1} = \|x\|_{n+2} = \dots = 1$ and

the strict convexity of E_{n+k} implies $x \in \operatorname{ex} B_X$. So $\overline{\operatorname{ex}}^{\|\ \|} B_X = S_X$, hence X can't have nontrivial M-summands and by Theorem 2.3 no nontrivial M-ideals either. □

We now give a characterisation of the Hahn-Banach smoothness of a Banach space X, considered as a subspace of X^{**}, and then go on to apply it to M-embedded spaces.

Lemma 2.14 *For a Banach space X and $x^* \in S_{X^*}$ the following are equivalent:*

(i) *x^* has a unique norm preserving extension to a functional on X^{**}.*

(ii) *The relative w- and w^*-topologies on B_{X^*} agree at x^*, meaning that the function $Id_{B_{X^*}} : (B_{X^*}, w^*) \to (B_{X^*}, w)$ is continuous at x^*.*

Proof: (i) ⇒ (ii): Let (x^*_α) be a net in B_{X^*} with $x^*_\alpha \xrightarrow{w^*} x^*$. By $\sigma(X^{***}, X^{**})$-compactness of $B_{X^{***}}$ we find for every subnet (x^*_β) of (x^*_α) an $x^{***} \in B_{X^{***}}$ and a subnet (x^*_γ) such that $x^*_\gamma \longrightarrow x^{***}$ with respect to $\sigma(X^{***}, X^{**})$. Since $x^*_\gamma \longrightarrow x^*$ with respect to $\sigma(X^*, X)$ we infer that x^{***} is a norm preserving extension of x^*, hence $x^{***} = x^*$. So the $\sigma(X^{***}, X^{**})$-convergence of (x^*_γ) to x^* is the desired $\sigma(X^*, X^{**})$-convergence of (x^*_γ) to x^*.

(ii) ⇒ (i): Let $x^{***} = x^* + x^\perp$ be a norm preserving extension of x^*. By Goldstine's theorem there is a net (x^*_α) in B_{X^*} such that $x^*_\alpha \longrightarrow x^{***}$ with respect to $\sigma(X^{***}, X^{**})$. In particular $x^*_\alpha \xrightarrow{w*} x^*$ on B_{X^*}, hence by assumption $x^*_\alpha \xrightarrow{w} x^*$. This means $x^*_\alpha \longrightarrow x^*$ with respect to $\sigma(X^{***}, X^{**})$ so that $x^\perp = 0$. □

Corollary 2.15 *If X is an M-ideal in its bidual, then the relative w- and w^*-topologies on B_{X^*} agree on S_{X^*}, meaning that the function $Id_{B_{X^*}} : (B_{X^*}, w^*) \to (B_{X^*}, w)$ is continuous at all $x^* \in S_{X^*}$.*

Proof: This is an immediate consequence of Proposition I.1.12 and Lemma 2.14. □

Remarks: (a) It is an amusing exercise to give a direct proof of Corollary 2.15 using the 3-ball property.

(b) If the assumption in Corollary 2.15 is strengthened to "$K(X)$ is an M-ideal in $L(X)$" (see Proposition VI.4.4), then the conclusion that the relative $\|\,.\,\|$- and w^*-topologies agree on S_{X^*} obtains (Proposition VI.4.6). However, for this stronger conclusion it does not suffice to assume only that X is an M-ideal in X^{**}: renorm ℓ^2 by $\ell^2 \simeq \mathbb{K} \oplus_\infty \ell^2$; then $\|(1, e_n)\| = 1 = \|(1, 0)\|$, $(1, e_n) \xrightarrow{w} (1, 0)$, but $(1, e_n) \overset{\|\ \|}{\not\longrightarrow} (1, 0)$.

Corollary 2.16 *If X is an M-ideal in its bidual, then X^* contains no proper norming subspace.*

Proof: If $V \subset X^*$ is norming, which means that $\|x\| = \sup_{x^* \in B_V} |x^*(x)|$ for all $x \in X$, then B_V is w^*-dense in B_{X^*} by the Hahn-Banach theorem. So every point $x^* \in S_{X^*}$ where the relative w- and w^*-topologies on B_{X^*} agree belongs to V. Hence the statement follows from Corollary 2.15. □

In Proposition 3.9 we will prove a stronger statement.

III.3 Isomorphic properties

By what was said in Remark (b) following Definition 1.1, nonreflexive M-embedded spaces are proper M-ideals, so they contain isomorphic copies of c_0 by Theorem II.4.7. This, and Theorem 3.1 below (M-embedded spaces are Asplund spaces, equivalently duals of M-embedded space have the Radon-Nikodým property), were the first results of isomorphic nature in M-structure theory.
The main part of the following section is devoted to the properties (V) and (u) of Pełczyński, two properties which imply the containment of c_0 in nonreflexive M-embedded spaces. Although in our situation property (u) is the stronger one, we will also give an independent proof of property (V) since it uses a completely different – and in our opinion, a very natural – approach to the problem. We will conclude this section with some more results of isomorphic nature, among them a characterisation of M-embedded spaces among separable $\mathcal{L}^\infty$-spaces.

First we deal with the Radon-Nikodým property (RNP). This property of a Banach space is defined and studied in detail in the monographs [93] and [158]. The RNP, originally defined in measure theoretic terms, can be equivalently characterised in a number of ways. The case of a dual Banach space is especially pleasing, since it is a theorem that X^* has the RNP if and only if every continuous convex function from an open convex subset O of X into $\mathbb{R}$ is Fréchet differentiable on a dense G_δ-subset of O. Moreover, either of these two properties is equivalent to the requirement that separable subspaces of X have separable duals. Banach spaces fulfilling the above differentiability hypothesis are called Asplund spaces, so X is an Asplund space if and only if X^* has the RNP. References for these results are [93, p. 91, p. 132], [158, p. 82, p. 195, p. 213] and [496, p. 34, p. 75].
Depending on the consequences one wants to draw from Theorem 3.1 below, it is sometimes more advantageous to use the Asplund property, sometimes one is better off using the RNP of the dual in order to give effective proofs. In particular, the result quoted in Corollary 3.2 is more easily derived from the Asplund property of X, whereas we shall have occasion to employ the (measure theoretic) RNP of the dual in Chapter VI.
The following theorem will frequently be used in the following. Its proof relies ultimately on Corollary 2.15.
For a Banach space X we denote the least cardinal $\mathfrak{m}$ for which X has a dense subset of cardinality $\mathfrak{m}$, the so-called density character of X, by $\operatorname{dens} X$. Thus $X \neq \{0\}$ is separable if and only if $\operatorname{dens} X = \aleph_0$.

Theorem 3.1 *If X is an M-embedded space and Y a subspace of X, then* $\operatorname{dens} Y = \operatorname{dens} Y^*$. *In particular, separable subspaces of X have separable duals. Consequently M-embedded spaces are Asplund spaces, and duals of M-embedded spaces have the RNP.*

PROOF: By Theorem 1.6 we may assume that $Y = X$. For a dense subset $\{x_i \mid i \in I\}$ of X we choose $x_i^* \in S_{X^*}$ such that $x_i^*(x_i) = \|x_i\|$. Then $\overline{\operatorname{lin}}\,\{x_i^* \mid i \in I\}$ is norming, hence equal to X^* by Corollary 2.16. For the final statement recall the above remarks. (Actually, here we only use the comparatively easy implications from [158, p. 82] or [496, p. 34].) □

A point x^* of a closed bounded convex subset K of a dual space X^* is called w^*-strongly

exposed (written $x^* \in w^*\text{-sexp}\,(K)$) if there is an $x \in X$ which strongly exposes x^*, meaning $\operatorname{Re} y^*(x) < \operatorname{Re} x^*(x)$ for $y^* \in K \setminus \{x^*\}$ and the sets $\{y^* \in K \mid \operatorname{Re} y^*(x) > \operatorname{Re} x^*(x) - \varepsilon\}$ ($\varepsilon > 0$) form a neighbourhood base of x^* in K for the relative norm topology (see [93, Definition 3.2.1(w^*)] or [496, Definition 5.8]).
The following consequence of the above is occasionally useful.

Corollary 3.2 *If X is an M-embedded Banach space, then*

$$B_{X^*} = \overline{\text{co}}^{\|\ \|} w^*\text{-sexp}\, B_{X^*}.$$

PROOF: Since X is an Asplund space by Theorem 3.1 one knows from [496, Theorem 5.12] (or [93, Theorem 4.4.1]) that

$$B_{X^*} = \overline{\text{co}}^{w*} w^*\text{-sexp}\, B_{X^*}.$$

Now, Corollary 2.15 yields for $C := \text{co}\, w^*\text{-sexp}\, B_{X^*}$ the inclusion $\overline{C}^{w} \supset S_{X^*}$, thus also

$$\overline{C}^{\|\ \|} = \overline{C}^{w} \supset \overline{S_{X^*}}^{w} = B_{X^*}$$

for infinite dimensional X. (The finite dimensional case is trivial.) □

Because the next property we want to prove for M-embedded spaces is possibly not so well-known, we wish to not only give the definition but also to list its most important consequences for easy reference.

3.3 THE PROPERTIES (V) AND (V^*)

These properties were introduced by A. Pełczyński in [485], where also most of the following can be found. For further information see e.g. [511], [548], [267], [268] and [156]. Recall that a series $\sum x_n$ in a Banach space X is called *weakly unconditionally Cauchy (wuC)*, if for every $x^* \in X^*$ we have $\sum |x^*(x_n)| < \infty$. Equivalently, $\sum x_n$ is a wuC-series if there is some $M \geq 0$ such that $\|\sum \alpha_n x_n\| \leq M \max |\alpha_n|$ for all finitely supported scalar sequences (α_n) [157, p. 44]. We refer to [157] and [572] for basic properties of wuC-series.

DEFINITION. (a) A Banach space X is said to have *property* (V) if every subset K of X^* satisfying

$$\lim_n \sup_{x^* \in K} |x^*(x_n)| = 0$$

for every wuC-series $\sum x_n$ in X is relatively weakly compact.
(b) A Banach space X is said to have *property* (V^*) if every subset K of X satisfying

$$\lim_n \sup_{x \in K} |x_n^*(x)| = 0$$

for every wuC-series $\sum x_n^*$ in X^* is relatively weakly compact.

Note that (by considering suitably defined operators into ℓ^1) the converse implications in the above definition are always true, e.g. for a relatively weakly compact subset K of X^* and $\sum x_n$ wuC in X one has $\lim_n \sup_{x^* \in K} |x^*(x_n)| = 0$.
These properties were introduced in order to understand why operators from $C(K)$-spaces to spaces not containing c_0 are always weakly compact, and they are closely related to unconditionally converging operators which we define next.

DEFINITION. Let X and Y be Banach spaces. An operator $T \in L(X,Y)$ is called *unconditionally converging* if for every wuC-series $\sum x_n$ in X the series $\sum Tx_n$ is unconditionally convergent in Y. We write $U(X,Y)$ for the set of all unconditionally converging operators from X to Y. This makes U a closed Banach operator ideal (see [497, p. 48]).

These operators are characterised in the following lemma.

Lemma 3.3.A *Let X and Y be Banach spaces.*

(a) *For $T \in L(X,Y)$ one has*

$T \notin U(X,Y) \iff$ *there is a subspace X_0 of X which is isomorphic to c_0 such that $T|_{X_0}$ is an isomorphism.*

(b) *For $T \in L(Y,X)$ one has*

$T^* \notin U(X^*,Y^*) \iff$ *there is a subspace Y_0 of Y which is isomorphic to ℓ^1 such that $T|_{Y_0}$ is an isomorphism and $T(Y_0)$ is complemented in X.*

For a proof see [511, p. 270 and p. 272]. Maybe some readers will find it misleading to call this result a lemma, after all it contains two beautiful theorems of Bessaga and Pełczyński in the special case $X = Y$ and $T = Id_X$, namely "A Banach space X contains no copy of c_0 iff every wuC-series $\sum x_n$ in X is unconditionally converging" and "X contains a complemented copy of ℓ^1 iff X^* contains c_0". In our context, however, it serves only as a preparation to the following characterisation:

Theorem 3.3.B *Let X be a Banach space.*

(a) *The following are equivalent:*

(i) *X has property (V).*

(ii) $U(X,\,.\,) \subset W(X,\,.\,)$

(iii) *For all Banach spaces Y and for all operators $T : X \longrightarrow Y$ which are not weakly compact there is a subspace X_0 of X isomorphic to c_0 such that $T|_{X_0}$ is an isomorphism.*

(b) *The following are equivalent:*

(i*) *X has property (V^*).*

(ii*) $U^{dual}(\,.\,,X) \subset W(\,.\,,X)$

(iii*) *For all Banach spaces Y and all operators $T : Y \longrightarrow X$ which are not weakly compact there is a subspace Y_0 of Y isomorphic to ℓ^1 such that $T|_{Y_0}$ is an isomorphism and $T(Y_0)$ is complemented in X.*

[The statement (ii*) reads: for every Banach space Y every operator $T \in L(Y,X)$ is weakly compact if T^* is unconditionally converging – see [497, p. 67] for the dual of an operator ideal.] The equivalence of the second and the third statements is immediate from Lemma 3.3.A, "(i) $\Longleftrightarrow$ (ii)" is proved in [485], the corresponding proof of part (b) is similar. [Note that $W(E,F) \subset U(E,F)$ for all Banach spaces E and F; see e.g. [497, p. 51].]

Corollary 3.3.C

(a) *A nonreflexive Banach space with property* (V) *contains a subspace isomorphic to* c_0.

(b) *A nonreflexive Banach space with property* (V^*) *contains a complemented subspace isomorphic to* ℓ^1.

The remaining statements are from [485]:

Proposition 3.3.D *Let* X *be a Banach space.*

(a) *If* X *has property* (V), *then* X^* *has property* (V^*).

(b) *If* X^* *has property* (V), *then* X *has property* (V^*).

Proposition 3.3.E *Let* X *be a Banach space and* Y *a subspace of* X.

(a) *If* X *has property* (V), *then* X/Y *has property* (V).

(b) *If* X *has property* (V^*), *then* Y *has property* (V^*).

Proposition 3.3.F *A Banach space* X *with property* (V^*) *is weakly sequentially complete.*

Clearly reflexive Banach spaces have the properties (V) and (V^*). To give at least one nonreflexive example we mention that $C(K)$-spaces have property (V). This is essentially a consequence of Grothendieck's characterisation of relatively weakly compact subsets of $M(K)$ [485, Theorem 1].

Now back to M-ideals.

Theorem 3.4 *Every* M*-embedded Banach space* X *has property* (V).

PROOF: We use Theorem 3.3.B, part (a), and show that for every Banach space Y every nonweakly compact operator $T \in L(X,Y)$ is not unconditionally converging. To achieve this we will construct a wuC-series $\sum x_n$ in X such that (Tx_n) does not converge to zero. Since T is not weakly compact there is $x^{**} \in X^{**}$ with $T^{**}x^{**} \in Y^{**} \setminus Y$. We may assume without restriction $\|x^{**}\| = 1$. Put $\alpha := d(T^{**}x^{**}, Y) > 0$ and take $\varepsilon_n > 0$ with $\prod(1+\varepsilon_n) < \infty$. We will inductively construct $x_n \in X$ such that

$$\|\pm x_1 \pm x_2 \cdots \pm x_n\| \leq \prod_{k=1}^{2n}(1+\varepsilon_k), \qquad n \in \mathbb{N} \tag{1}$$

$$\|Tx_n\| > \frac{\alpha}{2} \qquad \text{for } n \geq 2 \tag{2}$$

Since (1) says that $\sum x_n$ is weakly unconditionally Cauchy this will finish the proof. The case $n = 1$ is trivial. So assume $x_1, \dots, x_n$ are given satisfying (1) and (2). Put $\eta := \prod_{k=1}^{2n}(1+\varepsilon_k)$. Then $\|\frac{1}{\eta}(\pm x_1 \pm x_2 \cdots \pm x_n)\| \le 1$, $\|x^{**}\| = 1$ and, since X is an M-ideal in X^{**}, we find (Theorem I.2.2) $z \in X$ such that

$$\left\| x^{**} + \frac{1}{\eta}(\pm x_1 \pm x_2 \cdots \pm x_n) - z \right\| \le 1 + \varepsilon_{2n+1},$$

i.e.

$$\|\eta\,(x^{**} - z) + (\pm x_1 \pm x_2 \cdots \pm x_n)\| \le \eta\,(1 + \varepsilon_{2n+1}).$$

Put

$$E := \operatorname{lin}(x^{**}, z, x_1, \dots, x_n) \subset X^{**}.$$

Because $d(T^{**}x^{**}, Y) = \alpha$ we have $\alpha \le \|T^{**}x^{**} - Tz\| = \|T^{**}(x^{**} - z)\|$. So we find $y^* \in B_{Y^*}$ such that

$$|T^{**}(x^{**} - z)(y^*)| > \frac{\alpha}{2}.$$

Put $G := \operatorname{lin}\{Ty^*\} \subset X^*$. By the principle of local reflexivity there is an operator $A : E \longrightarrow X$ such that

- $Ax = x$ for $x \in E \cap X$,
- $\|A\| \le 1 + \varepsilon_{2n+2}$,
- $g^*(Ae^{**}) = e^{**}(g^*)$ for $e^{**} \in E,\ g^* \in G$.

Put $x_{n+1} := A(\eta\,(x^{**} - z))$. Then

$$\begin{aligned} \|x_{n+1} \pm x_1 \cdots \pm x_n\| &= \|A[\eta\,(x^{**} - z) + (\pm x_1 \cdots \pm x_n)]\| \\ &\le \|A\|\eta\,(1 + \varepsilon_{2n+1}) \\ &\le \prod_{k=1}^{2(n+1)} (1 + \varepsilon_k) \end{aligned}$$

and

$$\begin{aligned} \|Tx_{n+1}\| &\ge |y^*(Tx_{n+1})| \\ &= |(T^*y^*)(x_{n+1})| \\ &= \eta\,|(T^*y^*)(A(x^{**} - z))| \\ &= \eta\,|(x^{**} - z)(T^*y^*)| \\ &= \eta\,|T^{**}(x^{**} - z)(y^*)| \\ &> \eta\,\frac{\alpha}{2} \;>\; \frac{\alpha}{2}. \end{aligned}$$

Hence $x_1, \dots, x_n, x_{n+1}$ satisfy (1) and (2). □

We shall prove in Theorem IV.2.7 the companion result that L-embedded spaces enjoy property (V^*).

As a first application of the above theorem let us give an example which answers the question of Pełczyński [485, p. 646] whether the converse of Proposition 3.3.D holds:

Example 3.5 *The space $X = (\oplus \sum \ell^\infty(n))_{\ell^1}$ has property (V^*), but X^* fails property (V).*

PROOF: X is the dual of $X_* = \big(\oplus \sum \ell^1(n)\big)_{c_0}$, which is an M-embedded space by Theorem 1.6. So X has property (V^*) by the above Theorem 3.4 and Proposition 3.3.D. But $X^* = \big(\oplus \sum \ell^1(n)\big)_{\ell^\infty}$ contains a 1-complemented subspace isometric to ℓ^1 (see the proof of Example IV.1.7(b)). Since by Proposition 3.3.E property (V) passes to quotients (in particular to complemented subspaces) and ℓ^1 fails property (V) by Corollary 3.3.C, we get the desired conclusion. □

The first example to answer Pełczyński's question about the converse of Proposition 3.3.D was given in [548]. Let us mention in passing that the Bourgain-Delbaen spaces [91] may serve as examples of spaces X failing property (V), but whose duals have property (V^*).

By Propositions 3.3.D and 3.3.F the dual of a Banach space with property (V) is weakly sequentially complete. The following result gives a slightly stronger statement, which may also be viewed as an abstract version of Phillips's lemma.

Proposition 3.6 *If a Banach space X has property (V), then the natural projection π_{X^*} from X^{***} onto X^* is w^*-w-sequentially continuous.*

PROOF: It is enough to show $\pi_{X^*}(x_n^{***}) = x_n^{***}|_X \xrightarrow{w} 0$ for a w^*-null sequence (x_n^{***}) in X^{***}. The sequence (x_n^{***}) induces an operator $T \in L(X^{**}, c_0)$ by means of $Tx^{**} = (x_n^{***}(x^{**}))$, and $(x_n^{***}|_X)$ corresponds to $T|_X$. If $(x_n^{***}|_X)$ were not weakly null, then $T|_X$ would not be weakly compact, hence, by Theorem 3.3.B, there would exist a subspace X_0 of X isomorphic to c_0 such that the restriction of $T|_X$ to X_0 is an isomorphism. We would therefore obtain a diagram

$$\begin{array}{ccc} X^{**} & \xrightarrow{\quad T \quad} & c_0 \\ \uparrow & & \downarrow P \\ X_0 & \xleftarrow[\;{T|_{X_0}}^{-1}\;]{} & T(X_0) \simeq c_0 \end{array}$$

where P is a projection existing by Sobczyk's theorem (Corollary II.2.9). ${T|_{X_0}}^{-1} \circ P \circ T$ is then a projection from the dual space X^{**} onto a subspace isomorphic to c_0. But it is quickly checked that a complemented subspace V of a dual space is complemented in V^{**}, a property not shared by c_0. This contradiction completes the proof. □

It seems appropriate to gather some consequences of what has been shown so far.

Corollary 3.7 *Let X be a nonreflexive Banach space which is an M-ideal in its bidual. Then:*

(a) *Every subspace of X has property (V). In particular: X contains a copy of c_0, X is not weakly sequentially complete, and X fails the Radon-Nikodým property.*
(b) *X^* has property (V^*). In particular: X^* is weakly sequentially complete and contains a complemented copy of ℓ^1.*
(c) *X is not complemented in X^{**}.*
(d) *X^{**}/X is not separable.*
(e) *Every subspace or quotient of X which is isomorphic to a dual space is reflexive.*
(f) *Every operator from X to a space not containing c_0 (in particular, every operator from X to X^*) is weakly compact.*

PROOF: (a) This follows from Theorem 1.6 and Theorem 3.4.

(b) Theorem 3.4, Proposition 3.3.D, Proposition 3.3.F and Corollary 3.3.C.

(c) Assuming X to be complemented in X^{**} we infer from (a) and [528, Theorem 1.3] that X contains a subspace isomorphic to ℓ^∞. Hence X contains ℓ^1 – a separable space with nonseparable dual – contradicting Theorem 3.1.

(d) By (a) c_0 embeds into X, hence ℓ^∞/c_0 embeds into X^{**}/X.

(e) A subspace or quotient Y of X is again M-ideal in its bidual (Theorem 1.6). If Y is nonreflexive then Y is not complemented in Y^{**} by part (c); in particular Y is not isomorphic to a dual space.

(f) Theorem 3.4 and Theorem 3.3.B. □

That nonreflexive M-embedded spaces contain (arbitrarily good) copies of c_0 can be deduced from Theorem II.4.7, too. Let us also mention that we shall present an improvement of part (b) in Theorems IV.2.2 and IV.2.7.

The following property of a Banach space was introduced by Pełczyński in [484].

DEFINITION. A Banach space X is said to have *property* (u) if for every weak Cauchy sequence (x_n) in X there is a wuC-series $\sum y_k$ in X such that $(x_n - \sum_{k=1}^n y_k)$ converges to zero weakly.

A reformulation of this is: X has property (u) if and only if every x^{**} in the w^*-sequential closure of X in X^{**} is also the w^*-limit of a wuC-series $\sum y_k$ in X; we write $x^{**} = \sum^* y_k$ in this case.

REMARKS: (a) It is clear that weakly sequentially complete Banach spaces have property (u). To see other examples we mention that order continuous Banach lattices have property (u); cf. e.g. [422, Prop. 1.c.2].
(b) In the Notes and Remarks section of Chapter IV we will explain how property (u) may be viewed as a statement about the continuity of elements of the bidual considered as functions on (B_{X^*}, w^*).

Theorem 3.8 *Every M-embedded space X has property (u).*

PROOF: For a sequence (x_n) in X which is w^*-convergent to some $x^{**} \in X^{**}$ consider the separable M-embedded space $Y = \overline{\text{lin}}\,(x_n)$ and note that the σ-topology from Remark I.1.13 is just the w^*-topology in our situation. Hence the assertion is a special case of Theorem I.2.10. □

While property (u) always passes to subspaces (see Lemma I.2.9) it is generally not inherited by quotients. It seems noteworthy that quotients of M-embedded spaces do enjoy property (u) by Theorems 1.6 and 3.8.

Let us briefly comment on the relation between the properties (u) and (V). Pełczyński has shown in [485, Prop. 2] that a Banach space with property (u) in which every bounded sequence contains a weak Cauchy subsequence has property (V). The additional assumption is clearly fulfilled for M-embedded spaces, since $B_{Y^{**}}$ is w^*-metrizable if Y is a separable M-embedded space as a result of Theorem 3.1. By the way, the celebrated ℓ^1-theorem of Rosenthal (see e.g. [421, Theorem 2.e.5]) states that every bounded sequence in X contains a weak Cauchy subsequence if and only if ℓ^1 does not embed into X. Therefore a restatement of Pełczyński's above proposition is:

> *A Banach space with property (u) which doesn't contain an isomorphic copy of ℓ^1 has property (V).*

We refer to the Notes and Remarks section for consequences of Theorem 3.8 on property (X) of Godefroy and Talagrand.

The next result is a quantitative version of Corollary 2.16. Recall from p. 67 that the characteristic $r(V, X^*)$ of a subspace V of a dual space X^* is given by

$$r(V, X^*) = \max\{r \geq 0 \mid rB_{X^*} \subset \overline{B_V}^{w*}\} = \inf_{x \in S_X} \sup_{x^* \in B_V} |x^*(x)|.$$

Proposition 3.9 *If X is an M-embedded space, then for every proper subspace V of X^* one has $r(V, X^*) \leq 1/2$.*

PROOF: Since $r(V, X^*) \leq r(W, X^*)$ if $V \subset W$, it is enough to show the claim for hyperplanes. Let $V = \ker x_0^{**}$, $\|x_0^{**}\| = 1$. Take $x^* \in X^*$ such that $\|x^*\| = 1/2 + \varepsilon$ and $x_0^{**}(x^*) > 1/2$. Assuming $(1/2 + \varepsilon)B_{X^*} \subset \overline{B_V}^{w*}$ we could find $v_\alpha \in B_V$ with $v_\alpha \overset{w*}{\longrightarrow} x^*$. For a $\sigma(X^{***}, X^{**})$-accumulation point $x^{***} \in B_{X^{***}}$ of (v_α) we obtain $x^{***}|_X = x^*$. Therefore $x^{***} = x^* + x^\perp$ with $x^\perp \in X^\perp$ and $\|x^\perp\| = \|x^{***}\| - \|x^*\| \leq 1/2 - \varepsilon$. Also $x^{***}(x_0^{**}) = 0$ since $x_0^{**}(v_\alpha) = 0$. But then

$$|x^\perp(x_0^{**})| = |(x^{***} - x^*)(x_0^{**})| = |x_0^{**}(x^*)| > 1/2 > \|x^\perp\|\,,$$

which contradicts $\|x_0^{**}\| = 1$ and hence refutes the inclusion $(1/2 + \varepsilon)B_{X^*} \subset \overline{B_V}^{w*}$. □

The example $X = c_0$ and $V = \{(\lambda_n) \mid \sum \lambda_n = 0\}$ shows that $1/2$ is optimal in the above proposition.

For unexplained notation in the following corollary we refer to [421] and [174].

Corollary 3.10 *Every basic sequence (e_n) in an M-embedded space with basis constant $B < 2$ is shrinking.*

Proof: Since $\overline{\text{lin}}\,\{e_1, e_2, \ldots\}$ is M-embedded we may assume that (e_n) is a basis. The coefficient functionals (e_n^*) are a basis of $V := \overline{\text{lin}}\,\{e_1^*, e_2^*, \ldots\}$ and $r(V, X^*) \geq 1/B > 1/2$ (see [174, Propositions 6.1 and 6.3]). Hence $V = X^*$ by Proposition 3.9 which means that (e_n) is shrinking. □

To elucidate the connection with the unique extension property from p. 118 and Proposition 2.5 let us remark that according to the above corollary $P_n^* x^* \longrightarrow x^*$ $(x^* \in X^*)$ for the projections P_n associated to a basis (e_n) if $\sup \|P_n\| < 2$. (Note that $P_n x \longrightarrow x$ for all $x \in X$ anyway.)

We conclude this section with an isomorphic characterisation of M-embedded spaces within the class of separable $\mathcal{L}^\infty$-spaces. For the definition and basic properties of $\mathcal{L}^p$-spaces we refer to [418], [419], and [420].

Theorem 3.11 *Let X be a separable $\mathcal{L}^\infty$-space which is an M-ideal in its bidual. Then X is isomorphic to c_0.*

Proof: We will show that X is a complemented subspace of any separable superspace Y containing X. Zippin's famous characterisation of separably injective spaces [660] then gives the claim.
As every separable Banach space embeds into $C[0,1]$ it is sufficient to prove that every subspace of $C[0,1]$ isometric to X is the range of a continuous linear projection. Since X is an $\mathcal{L}^\infty$-space, its bidual is injective [419]. Note that $X^{**} \cong X^{\perp\perp}$ and X is an M-ideal in $X^{\perp\perp}$. Now $X^{\perp\perp}$ is injective, and it is therefore complemented in $C[0,1]^{**}$. Let P denote a projection from $C[0,1]^{**}$ onto $X^{\perp\perp}$; hence

$$C[0,1]^{**} \simeq X^{\perp\perp} \oplus \ker P.$$

We now renorm $C[0,1]^{**}$ (and thus its subspace $C[0,1]$) to the effect that

$$(C[0,1]^{**}, |\,.\,|) \cong X^{\perp\perp} \oplus_\infty \ker P.$$

Observe that the norms $\|\,.\,\|$ and $|\,.\,|$ coincide on $X^{\perp\perp}$. We conclude that $X^{\perp\perp}$ is an M-summand in $(C[0,1]^{**}, |\,.\,|)$, hence X (being an M-ideal in $X^{\perp\perp}$) is an M-ideal in $(C[0,1]^{**}, |\,.\,|)$, a fortiori X is an M-ideal in $(C[0,1], |\,.\,|)$.
We remark that $(C[0,1]/X)^{**}$ is isomorphic to $\ker P$, which has the BAP (it is a complemented subspace of the space $C[0,1]^{**}$ which has the MAP in its original norm). Therefore $C[0,1]/X$ is a separable space with the BAP, hence, by Theorem II.2.1 and Remark II.2.2(b), X is a complemented subspace of $C[0,1]$. □

We do not know if the above theorem holds also in the nonseparable case. Its proof essentially used results which are limited to separable spaces; however the decomposition of M-embedded spaces given in Theorem 4.5 gives the impression that it might be valid in general.

Lewis and Stegall have shown that X^* is isomorphic to ℓ^1 if X is an $\mathcal{L}^\infty$-space with a separable dual [393, p. 182]. So Theorem 3.11 is equivalent to: If X is an M-embedded space such that $X^* \simeq \ell^1$, then $X \simeq c_0$. Here is yet another way to look at Theorem 3.11 which should be compared with the isometric result in Proposition 2.7.

Corollary 3.12 *Let X be a separable M-embedded space such that X^* is isomorphic to a complemented subspace of $L^1(\mu)$ for some measure μ. Then X is isomorphic to c_0.*

This corollary follows from Theorem 3.11, since the condition on X^* is equivalent to requiring that X is an $\mathcal{L}^\infty$-space (see [420, Theorem II.5.7]).
Applying this to $X = C(\mathbb{T})/A$, one recovers that H^1 is not complemented in $L^1(\mathbb{T})$ [542, Example 5.19].

III.4 Projectional resolutions of the identity in M-embedded spaces

Our next aim is to establish a set of isomorphic properties (Theorem 4.6) for M-embedded spaces which show that in some weak sense these spaces behave like subspaces of $c_0(\Gamma)$ and thus support the intuition that M-embedded spaces in a way "vanish at infinity". These properties can all be deduced from the (isometric) result in Theorem 4.5: M-embedded spaces admit a shrinking projectional resolution of the identity. The proof of this fact uses a variation of the Lindenstrauss compactness argument and is prepared in the next four lemmata.
We remark here that for a proof of the existence of a "nice" projectional resolution of the identity some degree of smoothness of the space, i.e. some degree of continuity of the support mapping, is always needed (see the Notes and Remarks section). It was the achievement of Fabian and Godefroy in [212] to show that the assumption "X is an Asplund space" via the Jayne-Rogers selection theorem gives a $\|\,.\,\|$-Baire-1 selector of the support mapping and that this is enough to prove the existence of a projectional resolution of the identity in X^*. However, when dealing with M-embedded spaces, we have much more geometrical structure at our disposal, which allows us – using the ideas of Sims and Yost from [570] and [571] – to give the following "elementary" proof avoiding the selection theorem.
We will work with a special kind of projection on the dual of a Banach space. So let us recall:

- If M is a subspace of the Banach space X, then a linear operator $T : M^* \to X^*$ such that Tm^* is a norm-preserving extension of m^* for all $m^* \in M^*$ is called a *Hahn-Banach extension operator.*
- There exists a Hahn-Banach extension operator $T : M^* \to X^*$ if and only if $M^\perp$ is the kernel of a contractive linear projection Q in X^* (in which case $Q = Ti^*$, where $i : M \to X$ is the inclusion mapping).

In the next lemma the geometric structure comes into play.

Lemma 4.1 *If M is a subspace of an M-embedded space X and $T : M^* \longrightarrow X^*$ is a Hahn-Banach extension operator, then T is w^*-continuous. The projection $Q = Ti^*$ is the adjoint of a contractive projection P in X with range M.*

PROOF: (We use two results which will be proved in the next chapter.) By Proposition IV.1.5 the range of Q is an L-embedded space, hence Proposition IV.1.10 shows that $\operatorname{ran} Q$ is w^*-closed in X^*. A projection in a dual space with w^*-closed kernel and range is w^*- continuous, i.e. $Q = P^*$ for a projection P in X. Since $M^\perp = \ker Q$ we get $\operatorname{ran} P = M$ by standard duality. It is easily deduced from this that also T is w^*-continuous. □

Lemma 4.2 *Let X be a Banach space, B a finite dimensional subspace of X, k a positive integer, ε a positive real number, and G a finite subset of X^*. Then there is a finite dimensional subspace Z containing B such that for every subspace E of X containing B and satisfying $\dim E/B \le k$, we find an operator $T : E \longrightarrow Z$ such that $Tx = x$ for $x \in B$, $\|T\| \le 1 + \varepsilon$ and $|f(Tx) - f(x)| \le \varepsilon\|x\|$ for $x \in E$, $f \in G$.*

PROOF: If there were only finitely many subspaces $E_i = B \oplus N_i$ we would put $Z := B \oplus \bigcup N_i$. The idea is now to reduce our problem to this "case": Find $(N_i)_{i \le i_0}$ such that for all $E = B \oplus N$ as in the statement of the lemma there is an N_i with $\|b+n\| \sim \|b+n_i\|$. The details are easier than expected:
Let $G = \{f_1, \dots, f_m\}$ and let P be a projection on X with range B. Put $U := \ker P$. So $X = B \oplus U$. Choose M so large that

$$M > \frac{5k\|Id - P\|}{\varepsilon} \qquad \text{and} \qquad \frac{M+1}{M-1} < 1 + \varepsilon.$$

Let

$$\begin{array}{ll} (b_\rho)_{\rho \le r} & \text{be a finite } 1/M\text{-net for } \{b \in B \mid \|b\| \le M\}, \\ (\lambda^\sigma)_{\sigma \le s} & \text{be a finite } 1/M\text{-net for } S_{\ell^1(k)}. \end{array}$$

Define

$$\begin{array}{rccl} \Phi : & (B_U)^k & \longrightarrow & \mathbb{K}^{rs} \times \mathbb{K}^{mk} = \mathbb{K}^{rs+mk} \\ & (u_1, \dots, u_k) & \longmapsto & \left(\left(\|b_\rho + \sum_{\kappa=1}^k \lambda_\kappa^\sigma u_\kappa\| \right)_{\substack{\rho \le r \\ \sigma \le s}} , \ (f_\mu(u_\kappa))_{\substack{\mu \le m \\ \kappa \le k}} \right) \end{array}$$

Since $\Phi({B_U}^k)$ is totally bounded we find $u_1, \dots, u_n$ in ${B_U}^k$ such that

$$(\Phi u^\nu)_{\nu \le n} \quad \text{is a finite } 1/M\text{-net for } \Phi({B_U}^k)$$

where we may take any norm on $\mathbb{K}^{rs+mk}$ for which the coefficient functionals have norm ≤ 1. Put

$$Z := B \oplus \operatorname{lin}\{u_\kappa^\nu \mid 1 \le \kappa \le k,\ 1 \le \nu \le n\}.$$

Now given $E \supset B$ with $\dim E/B = k$ there are $u_1, \dots, u_k \in U$ such that $E = B \oplus \operatorname{lin}(u_1, \dots, u_k)$. By Auerbach's lemma (see e.g. [421, Prop. 1.c.3]) $u = (u_1, \dots, u_k)$ may be chosen so that

$$\|u_\kappa\| = 1\ , \ \ 1 \le \kappa \le k \quad \text{and} \quad \left\| \sum_{\kappa=1}^k \lambda_\kappa u_\kappa \right\| \ge \frac{1}{k} \sum_{\kappa=1}^k |\lambda_\kappa| \quad \text{for all } (\lambda_\kappa) \in \mathbb{K}^k. \tag{1}$$

So there is an $\nu \le n$ such that

$$\|\Phi u - \Phi u^\nu\| < \frac{1}{M},$$

i.e. with $\lambda u := \sum_{\kappa=1}^k \lambda_\kappa u_\kappa$ for $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{K}^k$ we have

$$\Big| \|b_\rho + \lambda^\sigma u\| - \|b_\rho + \lambda^\sigma u^\nu\| \Big| < \frac{1}{M} \quad \text{for all } \rho \le r,\ \sigma \le s, \tag{2}$$

$$\Big| f_\mu(u_\kappa) - f_\mu(u_\kappa^\nu) \Big| < \frac{1}{M} \quad \text{for all } \mu \le m,\ \kappa \le k. \tag{3}$$

Define

$$\begin{array}{rcccl} T & : & E & \longrightarrow & Z \\ & & b + \lambda u & \longmapsto & b + \lambda u^\nu. \end{array}$$

Obviously T fixes B. To show $\|T\| \le 1 + \varepsilon$ it is sufficient to prove

$$\|b + \lambda u^\nu\| \le (1+\varepsilon)\|b + \lambda u\| \qquad \text{for} \quad \|\lambda\|_{\ell^1(k)} = 1.$$

Assume first $\|b\| \le M$. Then $\|b - b_\rho\| < 1/M$ for some ρ and $\|\lambda - \lambda^\sigma\| < 1/M$ for some σ. Consequently

$$\begin{aligned} \|b + \lambda u^\nu\| &\le \|b_\rho + \lambda^\sigma u^\nu\| + \frac{1}{M} + \frac{1}{M} \\ &\overset{(2)}{\le} \|b_\rho + \lambda^\sigma u\| + \frac{3}{M} \\ &\le \|b + \lambda u\| + \frac{5}{M}. \end{aligned}$$

Also

$$\|b + \lambda u\| \ge \frac{1}{\|Id - P\|} \Big\| \sum \lambda_\kappa u_\kappa \Big\| \overset{(1)}{\ge} \frac{1}{k\|Id - P\|} \sum |\lambda_\kappa| = \frac{1}{k\|Id - P\|} > \frac{5}{\varepsilon M},$$

i.e. $5/M \le \varepsilon \|b + \lambda u\|$.
For $\|b\| > M$ we have

$$\|b + \lambda u^\nu\| \le \|b\| + 1 \qquad \text{and} \qquad \|b + \lambda u\| \ge \|b\| - 1,$$

so

$$\frac{\|b + \lambda u^\nu\|}{\|b + \lambda u\|} \le \frac{\|b\| + 1}{\|b\| - 1} < \frac{M+1}{M-1} < 1 + \varepsilon.$$

Finally for any $x = b + \sum \lambda_\kappa u_\kappa \in E$

$$\begin{aligned} |f_\mu(x) - f_\mu(Tx)| &= \left| f_\mu\Big(\sum_\kappa \lambda_\kappa (u_\kappa - u_\kappa^\nu) \Big) \right| \\ &\le \sum |\lambda_\kappa| \, |f_\mu(u_\kappa - u_\kappa^\nu)| \end{aligned}$$

$$\begin{aligned}
&\overset{(3)}{\leq} \frac{1}{M}\sum|\lambda_\kappa| \\
&\overset{(1)}{\leq} \frac{k}{M}\left\|\sum\lambda_\kappa u_\kappa\right\| \\
&= \frac{k}{M}\|(Id-P)x\| \\
&\leq \frac{k\|Id-P\|}{M}\|x\| \\
&< \frac{\varepsilon}{5}\|x\|.
\end{aligned}$$

□

We will find a shrinking projectional resolution of the identity (for the definition see Theorem 4.5) by constructing an increasing family of subspaces M_α which admit Hahn-Banach extension operators. To achieve this we employ a transfinite induction argument whose first step is the next lemma. Let us remark that Lindenstrauss has proved the above Lemma 4.2 with $G = \emptyset$. Our modification, suggested by D. Yost, proves useful when we want to show the ranges of the adjoint projections in the dual to be increasing.

Lemma 4.3 *If X is a Banach space, L a separable subspace of X, and F a separable subspace of X^*, then X has a separable subspace M containing L which admits a Hahn-Banach extension operator $T : M^* \longrightarrow X^*$ satisfying $TM^* \supset F$.*

PROOF: Let (x_n) be a sequence dense in L and (f_n) a sequence dense in F. Starting with $M_1 = \{0\}$ we inductively define subspaces M_n as follows: Putting $B_n := \operatorname{lin}(M_n, x_n)$, $G_n := \{f_1, \dots, f_n\}$ we let M_{n+1} be the subspace Z given by Lemma 4.2 when $B = B_n$, $k = n$, $\varepsilon = \frac{1}{n}$, and $G = G_n$. We may assume $\dim M_{n+1}/B_n \geq n+1$. Clearly $M := \overline{\bigcup M_n}$ is separable and contains L.
For $n \in \mathbb{N}$ define

$$I_n := \{E \subset X \mid B_n \subset E,\ \dim E/B_n \leq n\}$$

and put

$$I := \bigcup I_n.$$

Since $E \in I_n,\ F \in I_m$ implies $E + F + B_{\dim E + \dim F} \in I_{\dim E + \dim F}$, we have that I is a directed set. The condition $\dim M_{n+1}/B_n \geq n+1$ implies $\dim B_{n+1}/B_n \geq n+1$, and this easily gives that for each $E \in I$ there is a unique $n \in \mathbb{N}$ such that $E \in I_n$. So by Lemma 4.2 there exists $T_E : E \longrightarrow M_{n+1} \subset M$ such that $\|T_E\| \leq 1 + \frac{1}{n}$, $T_E|_{B_n} = Id_{B_n}$ and $|f_i(T_E x) - f_i(x)| \leq \frac{1}{n}\|x\|$ for all $x \in E$ and $1 \leq i \leq n$.
Extend T_E (nonlinearly) to X by setting

$$\overline{T_E}(x) := \begin{cases} T_E x & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

Since $\|\overline{T_E}x\| \leq 2\|x\|$ and regarding $\overline{T_E}x \in M \subset M^{**}$ as an element of M^{**} we have

$$(\overline{T_E}x)_x \in \prod_{x \in X} B_{M^{**}}(0, 2\|x\|).$$

Hence by compactness $\big((\overline{T_E}x)_x\big)_E$ has a convergent subnet $\big((\overline{T_{E'}}x)_x\big)_{E'}$, i.e. for all $x \in X$ there is $m_x \in M^{**}$ such that $\overline{T_{E'}}x \longrightarrow m_x$ with respect to $\sigma(M^{**}, M^*)$. Define

$$\begin{array}{rcl} T : M^* & \longrightarrow & X^* \\ m^* & \longmapsto & \big(x \longmapsto m_x(m^*) = \lim \overline{T_{E'}}x(m^*)\big) \end{array}$$

Noting that "for all $x \in X$ there is $E \in I$ such that $x \in E$" and "for $E, F \in I$ with $E \subset F$ and $E \in I_n$, $F \in I_m$ one obtains $n \le m$" it is routine to verify that T is the required Hahn-Banach extension operator. For example, the following establishes that Tm^* is additive:

$$\begin{aligned} Tm^*(x) + Tm^*(y) &= \lim_{E'} \overline{T_{E'}}x(m^*) + \lim_{E'} \overline{T_{E'}}y(m^*) \\ &= \lim_{E'} (\overline{T_{E'}}x + \overline{T_{E'}}y)(m^*) \;=\; \lim_{E' \supset \{x,y\}} (T_{E'}x + T_{E'}y)(m^*) \\ &= \lim_{E' \supset \{x,y\}} T_{E'}(x+y)(m^*) \;=\; \lim_{E'} \overline{T_{E'}}(x+y)(m^*) \\ &= Tm^*(x+y) \end{aligned}$$

Since clearly $TM^* \supset F$, we have completed the proof of Lemma 4.3. □

We recall that the density character of a Banach space X, $\operatorname{dens} X$, is defined to be the least cardinal $\mathfrak{m}$ for which X has a dense subset of cardinality $\mathfrak{m}$. If α, β are ordinals we use $\alpha < \beta$ as $\alpha \in \beta$, hence $\beta = \{\alpha \mid 0 \le \alpha < \beta\} = [0, \beta)$, and we write $\operatorname{card} \alpha$ for the cardinal number of α. The first infinite ordinal is denoted ω. We assume some familiarity with transfinite induction arguments and ordinal and cardinal arithmetic (to the extent given e.g. in [325]).

Lemma 4.4 *Let X be a Banach space, L a subspace of X, F a subspace of X^* with* $\operatorname{dens} F \le \operatorname{dens} L$. *Then there is a subspace M of X containing L and a Hahn-Banach extension operator $T : M^* \longrightarrow X^*$ such that* $\operatorname{dens} M = \operatorname{dens} L$ *and $TM^* \supset F$.*

Proof: We use transfinite induction on $\operatorname{dens} L$. The base case $\operatorname{dens} L = \aleph_0$ was treated in the previous lemma. So assume $\mathfrak{m} \le \operatorname{dens} L$ and that the assertion holds for all cardinals $< \mathfrak{m}$. We will show that it holds for $\mathfrak{m}$, too.
Let μ be the first ordinal of cardinality $\mathfrak{m}$, and let $\{x_\alpha \mid \alpha < \mu\}$ and $\{f_\alpha \mid \alpha < \mu\}$ be dense in L and F respectively (we repeat some f_α at the end if $\operatorname{dens} F < \operatorname{dens} L$).

> Claim: *For all α with $\omega \le \alpha < \mu$ there are subspaces M_α of X and Hahn-Banach extension operators $T_\alpha : M_\alpha^* \longrightarrow X^*$ such that*
>
> - $M_\beta \subset M_\alpha$ *for* $\omega \le \beta \le \alpha$,
> - $\operatorname{dens} M_\alpha \le \operatorname{card} \alpha$,
> - $M_\alpha \supset \{x_\beta \mid \beta < \alpha\}$, $\qquad T_\alpha M_\alpha^* \supset \{f_\beta \mid \beta < \alpha\}$.

This is again proved by transfinite induction: $\alpha = \omega$ is Lemma 4.3. Assuming M_β, T_β are given as in the claim for all β with $\omega \le \beta < \alpha$ we put

$$\begin{aligned} L &:= \overline{\operatorname{lin}} \Big(\{x_\beta \mid \beta < \alpha\} \cup \bigcup_{\beta < \alpha} M_\beta \Big) \\ F &:= \overline{\operatorname{lin}} \, \{f_\beta \mid \beta < \alpha\}. \end{aligned}$$

Since $\operatorname{dens} L \le \operatorname{card}\alpha + \sum_{\beta<\alpha} \operatorname{card}\beta = \operatorname{card}\alpha < \mathfrak{m}$ and $\operatorname{dens} F \le \operatorname{dens} L$ we find M_α and T_α as in the claim by the (outer) induction hypothesis.
To finish the proof of the first induction with the help of the claim put

$$M := \overline{\bigcup_{\alpha<\mu} M_\alpha}.$$

Then $M \supset L$ and $\operatorname{dens} M \le \sum_{\alpha<\mu} \operatorname{card}\alpha = \operatorname{card}\mu = \mathfrak{m}$, thus $\operatorname{dens} M = \operatorname{dens} L$. Clearly $\|T_\alpha i_\alpha^*\| \le 1$ where $i_\alpha : M_\alpha \longrightarrow M$ stands for the inclusion map. So by compactness of $B_{L(M^*,X^*)}$ in the weak*-operator topology, there is a subnet $(T_{\alpha'} i_{\alpha'}^*)$ of $(T_\alpha i_\alpha^*)_{\alpha<\mu}$ such that

$$(T_{\alpha'} i_{\alpha'}^* m^*)(x) \longrightarrow (Tm^*)(x) \qquad \forall m^* \in M^* \quad \forall x \in X.$$

Again, it is routine to verify that T is the desired Hahn-Banach extension operator. □

After the above preparation we can finally prove the main result of this section.

Theorem 4.5 *Every M-embedded space X admits a shrinking projectional resolution of the identity. That is, there are contractive projections P_α on X ($\omega \le \alpha \le \mu$, where μ denotes the first ordinal with cardinality* $\operatorname{dens} X$*) such that*

(a) $P_\mu = Id_X$,

(b) $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ *for* $\beta \le \alpha$,

(c) $\operatorname{dens} P_\alpha X \le \operatorname{card}\alpha$.

(d) $\overline{\bigcup_{\beta<\alpha} P_\beta X} = P_\alpha X$ *for limit ordinals α,*

(e) $\overline{\bigcup_{\beta<\alpha} P_\beta^* X^*} = P_\alpha^* X^*$ *for limit ordinals α.*

Moreover the projections P_α come from subspaces M_α which admit Hahn-Banach extension operators $T_\alpha : M_\alpha^* \longrightarrow X^*$ in the sense that $P_\alpha^* = T_\alpha i_\alpha^*$, where i_α denotes the inclusion mapping from M_α to X.

Proof: We will prove by transfinite induction:

> Claim. *For all α with $\omega \le \alpha \le \mu$ there are subspaces M_α of X and Hahn-Banach extension operators $T_\alpha : M_\alpha^* \longrightarrow X^*$ such that*
>
> (a′) $M_\mu = X$,
>
> (b′) $M_\beta \subset M_\alpha$ *and* $T_\beta M_\beta^* \subset T_\alpha M_\alpha^*$ *for* $\beta \le \alpha$,
>
> (c′) $\operatorname{dens} M_\alpha \le \operatorname{card}\alpha$,
>
> (d′) $\overline{\bigcup_{\beta<\alpha} M_\beta} = M_\alpha$ *for limit ordinals α,*
>
> (e′) $\overline{\bigcup_{\beta<\alpha} T_\beta M_\beta^*} = T_\alpha M_\alpha^*$ *for limit ordinals α.*

Assuming the claim, we know from Lemma 4.1 that $T_\alpha i_\alpha^*$ is the adjoint of a contractive projection P_α on X with range M_α. Note that $\operatorname{ran} P_\beta \subset \operatorname{ran} P_\alpha$ gives $P_\alpha P_\beta = P_\beta$ and that $\operatorname{ran} P_\beta^* = T_\beta M_\beta^* \subset T_\alpha M_\alpha^* = \operatorname{ran} P_\alpha^*$ yields $P_\alpha^* P_\beta^* = P_\beta^*$, hence $P_\beta P_\alpha = P_\beta$. Trivially the other primed properties imply the corresponding unprimed ones in the statement of the theorem.

To establish the claim choose a set $\{x_\beta \mid \beta < \mu\}$ which is dense in X. First of all we find a separable subspace M_ω containing $\{x_\beta \mid \beta < \omega\}$ and a Hahn-Banach extension operator T_ω by Lemma 4.3. So assume M_β and T_β are given as in the claim for all $\beta < \alpha$. If α is a *successor ordinal* let $L := \overline{\operatorname{lin}}\,(M_{\alpha-1} \cup \{x_\beta \mid \beta < \alpha\})$ and $F := T_{\alpha-1} M_{\alpha-1}^*$. By Theorem 3.1 $\operatorname{dens} M_{\alpha-1} = \operatorname{dens} M_{\alpha-1}^*$, so $\operatorname{dens} F = \operatorname{dens} M_{\alpha-1}^* = \operatorname{dens} M_{\alpha-1} \le \operatorname{dens} L \le \operatorname{card}(\alpha-1) + \operatorname{card}\alpha = \operatorname{card}\alpha$, and Lemma 4.4 provides us with M_α and T_α satisfying (b$'$) and (c$'$). Note $x_\beta \in M_\alpha$ for $\beta < \alpha$.

If α is a *limit ordinal* let $M_\alpha := \overline{\bigcup_{\beta<\alpha} M_\beta}$. For $\alpha = \mu$ we get $M_\mu = X$ since, for successor ordinals β, $M_\beta \supset \{x_\gamma \mid \gamma < \beta\}$. Denoting by $j_\beta : M_\beta \longrightarrow M_\alpha$ the inclusion maps we find a Hahn-Banach extension operator T_α as a w^*-operator topology accumulation point of $(T_\beta j_\beta^*)$ in $B_{L(M_\alpha^*, X^*)}$. As in the proof of Lemma 4.4 it is easily verified that M_α and T_α have the properties (a$'$) – (d$'$). The nontrivial part is property (e$'$).

Writing $Q_\beta = T_\beta i_\beta^* (= P_\beta^*)$, it is sufficient for the proof of (e$'$) to show that

$$Q_\beta x^* \longrightarrow Q_\alpha x^* \quad \text{for all } x^* \in X^*. \tag{$*$}$$

Actually (e$'$) and ($*$) are equivalent since property (e) is easily seen to be the same as the s_{op}-continuity of the mapping $\alpha \longmapsto P_\alpha^*$ from $[\omega, \mu]$ to $L(X^*)$, where $[\omega, \mu]$ is equipped with the order topology. A similar remark applies to (d) so that in particular $P_\beta x \overset{w}{\longrightarrow} P_\alpha x$ for all $x \in X$, i.e.

$$(Q_\beta x^*)(x) \longrightarrow (Q_\alpha x^*)(x) \quad \text{for all } x^* \in X^* \quad \text{and for all } x \in X. \tag{$**$}$$

So our task is to improve w^*-convergence to $\|\,.\,\|$-convergence. To accomplish ($*$) it is enough to prove

$$Q_\beta x^* = 0 \quad \text{for } x^* \in \ker Q_\alpha, \tag{1}$$

$$Q_\beta x^* \longrightarrow x^* \quad \text{for } x^* \in \operatorname{ran} Q_\alpha. \tag{2}$$

Part (1) follows from the inclusion $M_\beta \subset M_\alpha$, which immediately yields

$$\ker Q_\alpha = M_\alpha^\perp \subset M_\beta^\perp = \ker Q_\beta.$$

To see (2) note that $\operatorname{ran} Q_\alpha = \operatorname{ran} P_\alpha^* = (\ker P_\alpha)^\perp = (X/\ker P_\alpha)^*$ is the dual of an M-embedded space, hence by Corollary 3.2

$$B_{\operatorname{ran} Q_\alpha} = \overline{\operatorname{co}}^{\|\,\|}\, w^*\text{-sexp}\, B_{(X/\ker P_\alpha)^*}. \tag{$\dagger$}$$

Since $T_\beta M_\beta^* \subset T_\alpha M_\alpha^*$ we may regard Q_β as an operator on $(X/\ker P_\alpha)^*$, in particular $Q_\beta x^* \in B_{(X/\ker P_\alpha)^*}$ for $x^* \in B_{(X/\ker P_\alpha)^*}$. With these identifications the convergence in ($**$) means

$$Q_\beta x^* \longrightarrow x^* \quad \text{weak}^* \text{ in } (X/\ker P_\alpha)^*.$$

So
$$Q_\beta x^* \xrightarrow{\|\,\|} x^* \qquad \text{if } x^* \in w^*\text{-sexp}\, B_{(X/\ker P_\alpha)^*}.$$
Because of (†) this shows (2).
Thus the claim, hence the theorem, is completely proved. □

We remark that the existence of a (shrinking) projectional resolution of the identity is an isometric property which depends on the norm. However the important consequences we will deduce from it appeal to the isomorphic structure.
Recall that a norm $\|\,.\,\|$ on X is said to be *locally uniformly rotund* (LUR), whenever $\|x\| = \|x_n\| = 1$ and $\lim \|x + x_n\| = 2$ imply $x_n \to x$.
If $(X, \|\,.\,\|)$ is smooth, then by definition for every $x \in S_X$ there is a unique $f_x \in S_{X^*}$ such that $f_x(x) = 1$. This support mapping $x \mapsto f_x$ from S_X to S_{X^*} is then $\|\,.\,\|$-w^*-continuous. If it is even $\|\,.\,\|$-w-continuous, X is said to be *very smooth*. The norm is called *Fréchet differentiable* in case $x \mapsto f_x$ is $\|\,.\,\|$-$\|\,.\,\|$-continuous (the latter is equivalent to the more common requirement that $\lim_{t\to 0}(\|x+ty\|-\|x\|)/t$ exists uniformly in $y \in S_X$ for every $x \in S_X$). For more details on these notions we refer to [154].
Further recall that a Banach space is called *weakly compactly generated* if it is the closed linear span of some weakly compact set. A good source of information on these spaces is Chapter Five in [154].

Theorem 4.6 *Let X be an M-embedded space. Then:*

(a) *X has a shrinking Markuševič basis, i.e. there are $(x_i)_{i\in I}$ in X, $(f_i)_{i\in I}$ in X^* such that* $\operatorname{card} I = \operatorname{dens} X$ *and*
- $f_i(x_j) = \delta_{ij}$,
- $\overline{\operatorname{lin}}\,(x_i) = X$,
- $\overline{\operatorname{lin}}\,(f_i) = X^*$.

(b) *X is weakly compactly generated.*

(c) *There are operators $T : X \longrightarrow c_0(I)$ and $S : X^* \longrightarrow c_0(I)$,* $\operatorname{card} I = \operatorname{dens} X$, *such that T^{**} is injective and S is w^*-w-continuous and injective.*

(d) *X has an equivalent LUR-norm whose dual norm is also LUR. In particular this norm is Fréchet-differentiable und LUR.*

(e) *X has an equivalent very smooth norm, whose dual norm is strictly convex and under which X is still M-embedded.*

PROOF: (a) We will establish this by transfinite induction on $\mathfrak{m} = \operatorname{dens} X$. For $\mathfrak{m} = \aleph_0$ the dual X^* is separable by Theorem 3.1; so an old result of Markuševič, see [421, Proposition 1.f.3 and the remark thereafter] settles the base case.
Assume the statement holds for all M-embedded spaces Y with $\operatorname{dens} Y < \mathfrak{m}$. Choose a shrinking projectional resolution of the identity $(P_\alpha)_{\omega\le\alpha\le\mu}$ for X as in Theorem 4.5 and put

$$\begin{aligned} Y_\omega &:= P_\omega X, \\ Y_{\alpha+1} &:= (P_{\alpha+1} - P_\alpha)X \qquad \text{for } \omega \le \alpha < \mu. \end{aligned}$$

Then $\operatorname{dens}(P_{\alpha+1}-P_\alpha)X \le \operatorname{dens} P_{\alpha+1}X \le \operatorname{card}(\alpha+1) = \operatorname{card}\alpha < \operatorname{card}\mu = \mathfrak{m}$, so Y_ω and $Y_{\alpha+1}$ are as in the induction hypothesis. Thus there exist shrinking Markuševič bases

$$(x_i^\omega,\ f_i^\omega)_{i\in I_\omega} \text{ in } Y_\omega \qquad \text{and} \qquad (x_i^{\alpha+1},\ f_i^{\alpha+1})_{i\in I_{\alpha+1}} \text{ in } Y_{\alpha+1}$$

with $\operatorname{card} I_{\alpha+1} = \operatorname{dens} Y_{\alpha+1} \le \operatorname{card}\alpha$. Defining $\overline{f_i^\omega} := f_i^\omega \circ P_\omega$ and $\overline{f_i^{\alpha+1}} := f_i^{\alpha+1}(P_{\alpha+1} - P_\alpha)$ we claim that

$$\{(x_i^\beta,\ \overline{f_i^\beta}) \mid \beta = \omega \text{ or } \beta = \alpha+1 \text{ for } \omega \le \alpha < \mu,\ \ i \in I_\beta\}$$

is the desired shrinking Markuševič basis. That $\overline{f_i^\beta}(x_j^\gamma) = \delta_{(\beta,i),(\gamma,j)}$ is clear since $Y_\beta \cap Y_\gamma = \{0\}$ if $\beta \ne \gamma$. An easy transfinite induction argument using $P_\alpha x = \lim_{\beta<\alpha} P_\beta x$ (i.e. property (d) in Theorem 4.5) shows

$$P_\alpha X = \overline{\operatorname{lin}}\left(P_\omega X \cup \bigcup_{\omega\le\beta<\alpha} (P_{\beta+1} - P_\beta)X\right) \qquad \omega \le \alpha \le \mu.$$

Applying this for $\alpha = \mu$ one obtains

$$\overline{\operatorname{lin}}\,\{x_i^\beta \mid \beta = \omega \text{ or } \beta = \alpha+1 \text{ for } \omega \le \alpha < \mu,\ \ i \in I_\beta\} = X.$$

The natural isomorphism $Y_{\alpha+1}^* = ((P_{\alpha+1}-P_\alpha)X)^* \simeq (P_{\alpha+1}^* - P_\alpha^*)X^*$ (similarly for Y_ω^*) gives $\overline{\operatorname{lin}}\,\{\overline{f_i^{\alpha+1}} \mid i \in I_{\alpha+1}\} = (P_{\alpha+1}^* - P_\alpha^*)X^*$. As above – now using property (e) in Theorem 4.5 – one establishes

$$P_\alpha^* X^* = \overline{\operatorname{lin}}\left(P_\omega^* X^* \cup \bigcup_{\omega\le\beta<\alpha} (P_{\beta+1}^* - P_\beta^*)X^*\right) \qquad \omega \le \alpha \le \mu,$$

and the rest is clear.

(b) By suitably scaling we may assume that the shrinking Markuševič basis from (a) satisfies $\|x_i\| = 1$ for all $i \in I$. We claim that

$$K := \{0\} \cup \{x_i \mid i \in I\}$$

is weakly compact; then $X = \overline{\operatorname{lin}}\,K$ is weakly compactly generated.

By the Eberlein-Smulian theorem it is sufficient to show that K is weakly sequentially compact. To prove this it is enough to establish that for all sequences (x_{i_n}) in K with each x_{i_n} appearing only finitely many times

$$x_{i_n} \xrightarrow{w} 0\,.$$

Given $f \in X^*$ and $\varepsilon > 0$, we find $g \in \operatorname{lin}(f_i)$ such that $\|f-g\| < \varepsilon$. Then $g(x_i) \ne 0$ only for finitely many $i \in I$. So

$$\Big||f(x_{i_n})| - |g(x_{i_n})|\Big| \le |f(x_{i_n}) - g(x_{i_n})| \le \|f-g\| < \varepsilon$$

shows the claim.

(c) Assuming $\|f_i\| = 1$ for the shrinking Markuševič basis from (a) we define T by

$$\begin{array}{rcl} T : X & \longrightarrow & c_0(I) \\ x & \longmapsto & \big(f_i(x)\big). \end{array}$$

As in the proof of part (b) one verifies that $\{i \in I \mid |f_i(x)| > \varepsilon\}$ is finite for all $x \in X$ and $\varepsilon > 0$, so T is well-defined. Then $T^{**} : X^{**} \longrightarrow \ell^\infty(I)$ is given by $T^{**}x^{**} = \big(x^{**}(f_i)\big)$. So it is clear that T has the required properties.
Assuming now $\|x_i\| = 1$ the desired operator S is given by

$$\begin{array}{rcl} S : X^* & \longrightarrow & c_0(I) \\ f & \longmapsto & \big(f(x_i)\big). \end{array}$$

(d) An inspection of the proof of Theorem 2.1 of [271] shows the following

> Theorem (Godefroy, Troyanski, Whitfield, Zizler): *Let Y be an LUR Banach space and X a Banach space with a locally convex topology τ such that B_X is τ-closed. If there is a bounded linear operator $L : Y \to X$ with a $\|\,.\,\|$-dense range such that $L(B_Y)$ is τ-compact, then X admits an equivalent LUR-norm which is τ-lower semicontinuous.*

First we apply this to X with $\tau = \sigma(X, X^*)$: Assuming $\|x_i\| = 1$ for the shrinking Markuševič basis $(x_i, f_i)_{i\in I}$ from part (a), we have that $L : \ell^1(I) \to X$, $(\lambda_i) \mapsto \sum \lambda_i x_i$ is continuous with dense range. The w^*-continuity of $f_i \circ L$ and $\overline{\text{lin}}\,(f_i) = X^*$ provide the w^*-w-continuity of L, hence the weak compactness of $L(B_{\ell^1(I)})$. The existence of an equivalent dual LUR-norm $|\,.\,|$ on $\ell^1(I)$ (e.g. $|(\lambda_i)| = (\|(\lambda_i)\|^2_{\ell^1(I)} + \|(\lambda_i)\|^2_{\ell^2(I)})^{1/2}$, cf. [271, p. 348]) then shows that X admits an equivalent LUR-norm.
If we apply the renorming theorem quoted above to X^* with $\tau = \sigma(X^*, X)$ and take for L the adjoint of the operator $T : X \to c_0(I)$ defined in the proof of part (c), we find that X^* admits an equivalent dual LUR-norm (cf. [271, Remark 2.4]). So the Asplund averaging technique (see [154, Chapter Four, § 3 and Chapter Five, § 9] or [213]) yields a norm with both features. For the second statement in (d) see [154, Chapter Two, §2].
(e) By part (c) there is a w^*-w-continuous and injective operator $S : X^* \to c_0(I)$. In particular S is weakly compact and w^*-w^*-continuous when considered as an operator into $\ell^\infty(I)$. Hence, S is the adjoint of a weakly compact operator from $\ell^1(I)$ into X and we are in the situation of Proposition 2.11. Using Day's strictly convex norm on $c_0(I)$ [154, p. 94ff.] we find, with the help of that proposition, an equivalent smooth norm $|\,.\,|$ on X with $|\,.\,|^*$ strictly convex and X still M-embedded. Such a norm is very smooth: every support mapping from $S_{(X,|\ |)}$ to $S_{(X^*,|\ |^*)}$ is $\|\,.\,\|$-w^*-continuous [154, p. 22], hence $\|\,.\,\|$-w-continuous by Corollary 2.15, i.e. $|\,.\,|$ is very smooth. □

Remarks: (a) The first step of the proof of part (d), the existence of an equivalent LUR-norm on X, is of course an immediate consequence of Troyanski's theorem that weakly compactly generated spaces can be equivalently LUR-renormed (see [154, Chapter Five, § 5]). However, we gave the above argument in order to point out that the Godefroy-Troyanski-Whitfield-Zizler theorem also yields a new and simpler proof of Troyanski's renorming result. [This is indicated in the introduction of [271], but inadvertently not in the text. To obtain the renorming for weakly compactly generated spaces X recall that by the Amir-Lindenstrauss theorem (see [154, Theorem 2, p. 147] or [582]) there is a w^*-w-continuous and injective operator $T : X^* \to c_0(I)$. Then $T^* : \ell^1(I) \to X$ is w^*-w-continuous and has dense range. So an application of the Godefroy-Troyanski-Whitfield-Zizler theorem indeed yields the renorming.]

(b) Parts (a) and (b) of Theorem 4.6 give in particular a new proof that every subspace of $c_0(I)$ is weakly compactly generated and has a shrinking Markuševič basis. This is a result obtained in [345, p. 10] by different techniques.

For the large number of consequences which can be deduced for weakly compactly generated spaces we refer to [154, Chapter Five]. For convenience we state some implications which may occasionally be useful.
Recall that a Banach space is said to have the *separable complementation property* if every separable subspace is contained in a separable and complemented subspace.

Corollary 4.7 *Let X be an M-embedded space. Then:*
(a) *w^*-compact subsets of X^* are w^*-sequentially compact.*
(b) *w^*-sequentially continuous elements of X^{**} are w^*-continuous.*
(c) *X has the separable complementation property.*
(d) *If X is nonreflexive then it contains a complemented copy of c_0, in fact every copy of c_0 is complemented.*

PROOF: (a) The operator S from Theorem 4.6(c) maps these sets homeomorphically to weakly compact subsets of $c_0(I)$. So the assertion follows from the Eberlein-Smulian theorem.

(b) Corollary 4 on p. 148 in [154].

(c) Theorem 3 on p. 149 in [154]. However, this property of M-embedded spaces follows already directly from Lemmata 4.1 – 4.3; it is even enough to use Lemma 4.2 with $G = \emptyset$.

(d) Combine the separable complementation property with Corollary 3.7(a) and Sobczyk's theorem (Corollary II.2.9). □

III.5 Notes and remarks

GENERAL REMARKS. The date of birth of M-embedded spaces is hard to fix. Before the paper [292], where the first systematic study of this class was initiated, this type of space appeared in Lima's work [401] and, under the name *class* L_0, in the article [97] by Brown and Ito; there were also some unpublished results by R. Evans. In [401] Proposition 2.7 was proved, and in [97] Theorem 1.6(b), (c) and Proposition 2.10(a) were established. The most important predecessor of [292], however, is Lima's paper [403] where the connection with M-ideals of compact operators was shown and the RNP for duals of M-embedded spaces was explicitly proved; see below for more information on this result. The study of M-embedded spaces marked an important point in the development of M-ideal theory: the point of view of considering *sub*spaces of given Banach spaces, C^*-algebras, L^1-preduals and ordered spaces was abandoned in favour of investigating pure Banach space properties. In particular the fact that nontrivial isomorphic conclusions can be deduced from the geometric structure of M-embedded spaces has found some interest.

Many of the examples of M-embedded spaces in Section III.1 appeared in [623]. For C/A, however, we have followed Luecking's proof in [426]; another proof can be found in [240]. The assertion about Orlicz spaces was proved much earlier by Ando [18], establishing the L-decomposition in the dual directly. Even older is the result on $K(H)$ which was shown by Dixmier [164] in 1950. Corollary 1.5 was first proved in [38], the M-ideal argument we have presented is due to Luecking [426]. We will say more about best approximation from arbitrary Douglas algebras below and refer to [557, p. 109ff.] for the history of this problem and its relation to questions in function theory. Concerning Orlicz spaces let us mention that our proof shows that h_M is an M-ideal in ℓ_M (resp. H_M in L_M) no matter if M or M^* satisfies the Δ_2-condition; but in this generality ℓ_M need not be the bidual of h_M. There is another norm which is of interest in the theory of Orlicz spaces. The Orlicz norm is defined to be the dual norm of the Luxemburg norm with respect to the pairing $\langle f, g\rangle = \int f(t)g(t)\,dt$; i.e.,

$$\|f\|_M^0 = \sup\{|\langle f, g\rangle| \mid g \in L_{M^*},\ \|g\|_{M^*} \le 1\}$$

in the case of function spaces and similarly for sequence spaces. H. Hudzik has shown us a proof that one doesn't obtain M-ideals for this norm. In Section VI.6 we shall discuss another renorming of Orlicz sequence spaces which is of importance in the theory of M-ideals of compact operators.

The stability of M-embedded spaces (Theorem 1.6) was fully established in [292]. In the proof of this result the main difference between M- and L-embedded spaces appears clearly: in the first case the *natural* projection π_{X^*} is an L-projection, in the second case we just know that there is one. The results and techniques in [263] have inspired Proposition 1.9 which is taken from [406]. We remark that the isomorphic version of Corollary 1.10 is false, since by a result of Johnson and Lindenstrauss [355, Ex. 2] there is a $C(K)$-space which is not a subspace of a weakly compactly generated space (hence in particular not isomorphic to an M-embedded space by Theorem 4.6), yet all separable subspaces of $C(K)$ are isomorphic to subspaces of c_0.

Proposition 2.1 is implicitly contained in [97] – see also [255, Th. 1] for a more general result. Using Proposition 2.1 we obtain a more selfcontained proof of Proposition 2.2 than the one in [292]. However, to get a better understanding of the question of automatic w^*-continuity of surjective isometries in dual spaces we refer to Godefroy's work in [249], [255] and the survey chapter VII in [258]. Theorem 2.3 is from [292], the unique extension property, Lemma 2.4, and Proposition 2.5 first appeared in [269]. By Proposition 2.5 of this last paper an important class of spaces enjoying the unique extension property are those whose duals don't contain proper norming subspaces. The decomposition of M-embedded spaces in Proposition 2.6 was first proved in [291] using function module representations. The proof of Proposition 2.7 is also from this work – the result, however, first appeared in [401]. Much more than the statement in Corollary 2.8 is true: in [72] Benyamini showed that, for a first countable compact space K, $d(X, C_\sigma(K)) = 1$ implies $X \cong C_\sigma(K)$. Besides the conditions characterising M-embedded C^*-algebras which we proved in Proposition 2.9 there are several others in [76]. The construction of the predual in Proposition 2.10 comes from [256, Th. 27]. Presumably the first paper to explicitly note that the natural projection π_{X^*} from X^{***} onto X^* is not bicontractive is [96]. The example $\|Id_{(\ell^1)^{***}} - \pi_{(\ell^1)^*}\| = 2$ was extended in [103] and [263] to "If X contains a subspace isomorphic to ℓ^1 then $\|Id_{X^{***}} - \lambda\pi_{X^*}\| = 1 + |\lambda|$". In this last paper the so-called Godun set $G(X) = \{\lambda \in \mathbb{K} \mid \|Id_{X^{***}} - \lambda\pi_{X^*}\| = 1\}$ is studied. For instance it is proved that a separable space is Asplund provided that $G(X) \cap (1,2] \neq \emptyset$. The renorming of M-embedded spaces in Proposition 2.11 is from [293], the example in Proposition 2.13 essentially appeared in [337] – however, the arguments there are not at all clear; therefore, we have followed the exposition in [623].

It took a while to realize the connection between unique Hahn-Banach extension, equality of topologies on S_{X^*}, continuity of the support mapping, and the RNP for X^*. In [587] Sullivan introduced Hahn-Banach smooth spaces as those which admit unique Hahn-Banach extension from X to X^{**}, realized that no dual space is Hahn-Banach smooth and proved that the equality of norm and weak* topologies on S_{X^*} implies this property. Shortly after, Smith and Sullivan [573] showed that weakly Hahn-Banach smooth spaces X (only norm attaining functionals x^* are required to have unique Hahn-Banach extensions) have duals with the RNP. Studying the continuity of the support mapping $D : S_X \to 2^{S_{X^*}}$ Giles, Gregory, and Sims [246] proved that weak Hahn-Banach smoothness means $w^* = w$ on the set of norm attaining functionals in S_{X^*}. They also gave two arguments (one of them attributed to Phelps) that this implies the RNP for X^*. Also, their Theorem 3.3 contains that duals of M-embedded spaces have the RNP in a rather explicit manner. In [251] Godefroy called Namioka points those $x^* \in S_{X^*}$ where the w^*- and w-topologies agree on S_{X^*}, showed Lemma 2.14 and established connections to minimal norming subspaces and unique preduals – see also [645]. All this was unnoticed by those working in M-ideal theory so that the RNP for duals of M-embedded spaces was reproved twice in [403] and [252]. The argument in the text using proper norming subspaces now seems to be the shortest.

Property (V) for M-embedded spaces was first shown in [268]. The proof we have presented is a modification due to Lima of the argument in [292] which shows that M-embedded spaces contain c_0. Besides the references for properties (V) and (V^*) already cited in 3.3, we mention [116] and [117] for a study of (V) in vector valued function spaces, [83] representative of the work of Bombal and [512] for a weakening of (V) and its relation to Grothendieck spaces – see also [155]. A good summary is the survey article by E. and P. Saab [549]. The most remarkable recent result in this area is Pfitzner's theorem [493] that all C^*-algebras have property (V). Finally we mention that property (V) has occasionally been called *strict Dieudonné property*, e.g. in [451].

In [270] and [250] Godefroy and Talagrand (see also [258, p. 158ff.] and [256, p. 244ff.]) say that a Banach space X has *property* (X) if any $x^{**} \in X^{**}$ such that $x^{**}(\sum^* x_k^*) = \sum x^{**}(x_k^*)$ for every wuC-series $\sum x_k^*$ in X^*, must be in X. They show that every Banach space with property (X) is strongly unique predual of its dual; since property (X) is an isomorphic invariant this remains true for every equivalent norm on the space. In [183] Edgar introduced an ordering of Banach spaces by

$$X \prec Y \quad \text{iff} \quad X = \bigcap_{T \in L(X,Y)} (T^{**})^{-1}(Y)$$

and showed that X has property (X) iff $X \prec \ell^1$ [183, Prop. 10]. He also proved that property (X) implies property (V^*) [183, Th. 13]. Since it was shown in [250], [264] and

[258] that X^* has property (X) whenever X is a separable space with property (u) not containing ℓ^1, we get by Theorem III.3.8 that duals of separable M-embedded spaces X have property (X). By [258, Th. VII.8] we obtain that in this situation an operator $T : X^{**} \to Y^*$ is w^*-continuous if T is w^*-w^*-Borel measurable. We finish this digression by remarking that property (X) is strictly stronger than property (V^*) as shown in [594]. Accepting the existence of special cardinal numbers one also obtains an example of this from [183, Prop. 12]: $\ell^1(\Gamma) \prec \ell^1$ iff $\operatorname{card}\Gamma$ is not a real-valued measurable cardinal. (This also shows that some of the examples of spaces with property (X) in [270] and [250] have to be read with additional set-theoretic assumptions.)
Example 3.5 was first noted in [267] and Proposition 3.6 is an unpublished result of Pfitzner. Godefroy and Li proved property (u) for M-embedded spaces in [264], but only with the work in [396] the general background for this, presented in Theorem I.2.10, became apparent to us. Meanwhile the ultimate clarification was provided by [263] introducing the concept of u-ideals – see below. We mention that by the work of Knaust and Odell in [380] the hereditary Dunford-Pettis property (and equivalently property (S)) is sufficient for property (u), also the recent paper [367] studying property (u) in vector valued function spaces is of interest here. Let us remark that property (u) for M-embedded spaces can also be derived from recent deep results by Rosenthal who showed that X has (u) provided Y^* is weakly sequentially complete for all subspaces Y of X [530]. Proposition 3.9 and Corollary 3.10 are from [269]. Theorem 3.11 was obtained independently by Godefroy and Li [265] and D. Werner [620].
The main result of Section III.4, Theorem 4.5, is due to Fabian and Godefroy [212], but our proof largely follows ideas of Sims' and Yost's papers [570] and [571]. M. Fabian has shown us another proof of this result based on techniques from his article [210]. We shall have more to say on projectional resolutions of the identity later in this section.

Further examples of M-embedded spaces. We first consider the Schreier space S. This is the completion of the space of all sequences which are eventually 0 under the norm

$$\|(x_n)\|_S = \sup_E \sum_{n\in E} |x_n| \ .$$

Here the supremum has to be taken over all "admissible" finite sets E, meaning all sets $E = \{n_1, \dots, n_k\}$ where $k \le n_1 < \dots < n_k$. This Banach space has its origin in Schreier's paper [561], and some information on S can be found in [45] and [115]. Schreier used (a variant of) the space S to show that $C[0,1]$ fails the weak Banach-Saks property. (Recall that a Banach space X is said to have the *weak Banach-Saks property* if every weakly convergent sequence in X admits a subsequence with norm convergent arithmetic means.) We now present a result from [623].

- *The Schreier space S is an M-ideal in its bidual.*

To show this note first that the natural basis (e_n) is 1-unconditional and shrinking, hence $S^{**} \cong \{(x_n) \mid \|(x_n)\| := \sup_N \|\sum_{n=1}^N x_n e_n\| < \infty\}$ (cf. [421, Prop. 1.b.2]). Also $x_n \to 0$ for $(x_n) \in S^{**}$. For $x \in B_{S^{**}}$, $j_i \in B_S$ (without restriction with finite support) and $\varepsilon > 0$ choose first n_0 such that $j_i(n) = 0$ for $n > n_0$, then $m_0 \ge n_0$ such that $|x(m)| < \varepsilon/(n_0-1)$

for $m \geq m_0$. Put $j(n) = x(n)$ for $n < m_0$ and $j(n) = 0$ otherwise. Then

$$\sum_E |x(n) + j_i(n) - j(n)|$$

can be estimated by $\sum_E |x(n)| \leq \|x\| \leq 1$ if $\min E > n_0$, and by $\sum_{E\cap\{1,\dots,n_0\}} |j_i(n)| + \sum_{E\cap\{n_0+1,\dots\}} |x(n)| \leq \|j_i\| + (n_0-1)\frac{\varepsilon}{n_0-1} \leq 1+\varepsilon$ if $\min E \leq n_0$. Hence $\|x+j_i-j\| \leq 1+\varepsilon$. Actually, the space originally considered by Schreier in [561] can be represented as a sequence space by means of the norm $\|(x_n)\| = \sup_E \left|\sum_{n\in E} x_n\right|$. The above proof can clearly be modified so as to show that this space is also M-embedded.
By Corollary 3.7 every nonreflexive subspace of a quotient of S contains a copy of c_0. In the recent paper [456] E. Odell proves the stronger result that *every* infinite dimensional subspace of a quotient of S contains a copy of c_0. Thus in order to obtain the full strength of Odell's theorem one "only" has to show that no infinite dimensional reflexive specimen exists among the subspaces of quotients of S. Odell also defines a hierarchy of Schreier spaces $S_1, S_2, \dots$ as follows. Let $(S_1, \|\,.\,\|_1) = (S, \|\,.\,\|_S)$ and suppose that $(S_m, \|\,.\,\|_m)$ is already constructed. Then S_{m+1} is the completion of the finitely supported sequences with respect to the norm

$$\|(x_n)\|_{m+1} = \sup \sum_{j=1}^{k} \|P_{E_j}(x)\|_m$$

where the E_j are finite subsets of $\mathbb{N}$ satisfying

$$k \leq \min E_1 \leq \max E_1 < \min E_2 \leq \max E_2 < \ldots \leq \max E_{k-1} < \min E_k,$$

where $(P_E x)_n = x_n$ if $n \in E$ and $(P_E x)_n = 0$ otherwise, and the supremum is taken over all k and all admissible strings $E_1, \dots, E_k$ as above. It follows in much the same way as above that each S_m is M-embedded and hence each infinite dimensional nonreflexive subspace of a quotient of S_m contains a copy of c_0. This gives a partial answer to Problem 6 in [456].
In [567] the extreme point structure of the unit ball of S is investigated. The knowledge that S is M-embedded allows us to add the information that one cannot expect, by Proposition II.4.2 and Theorem II.4.4, B_S to possess strongly extreme points, let alone strongly exposed points.
Let us now consider the function space $L^\infty + L^p$, $1 < p < \infty$, consisting of all measurable functions on $\mathbb{R}$ which have a representation $f = f_1 + f_2$ with $f_1 \in L^\infty(\mathbb{R})$, $f_2 \in L^p(\mathbb{R})$. We equip this space with the norm

$$\|f\| = \inf\{\|f_1\|_{L^\infty} \vee \|f_2\|_{L^p} \mid f = f_1 + f_2,\ f_1 \in L^\infty,\ f_2 \in L^p\},$$

which is equivalent to the usual one [422, p. 119]. Let $(L^\infty + L^p)_f$ denote the closed subspace generated by the characteristic functions χ_E where $\lambda(E) < \infty$.

- $(L^\infty + L^p)_f$ *is an* M*-ideal in its bidual.*

This time we will directly verify the norm condition on the projection. It is well known that the dual of $(L^\infty + L^p)_f$ is isometrically isomorphic to the space $L^1 \cap L^q$, normed by

$$\|g\| = \|g\|_{L^1} + \|g\|_{L^q}$$

(of course $1/p + 1/q = 1$), and the bidual can be identified with $L^\infty + L^p$. Let us consider $X = L^1 \cap L^q$ as the diagonal in $L^1 \oplus_1 L^q$. Then the annihilator of X in $(L^1 \oplus_1 L^q)^* = L^\infty \oplus_\infty L^p$ is

$$X^\perp = \{(f, -f) \mid f \in L^\infty \cap L^p\},$$

and one obtains

$$\begin{aligned} X^{\perp\perp} &= \\ &= \left\{ (\ell, g_1, g_2) \in L^1_s \oplus_1 L^1 \oplus_1 L^q \;\middle|\; \langle \ell, f\rangle + \int (g_1 - g_2) f \, d\lambda = 0 \quad \forall f \in L^\infty \cap L^p \right\} \\ &= \left\{ (\ell, g_1, g_2) \in L^1_s \oplus_1 L^1 \oplus_1 L^q \;\middle|\; \langle \ell, \chi_E\rangle + \int_E (g_1 - g_2) \, d\lambda = 0 \quad \text{if} \quad \lambda(E) < \infty \right\}. \end{aligned}$$

Here we write $(L^1)^{**} = L^1_s \oplus_1 L^1$ where L^1_s is the space of "singular" functionals, cf. Example IV.1.1(a). Hence one deduces

$$X^{\perp\perp} = X_s \oplus_1 X$$

with

$$X_s \;\cong\; \{\ell \in (L^1)^{**} \mid \langle \ell, 1_E\rangle = 0 \quad \text{if} \quad \lambda(E) < \infty\} \;\cong\; {(L^\infty + L^p)_f}^\perp$$

which shows our claim. Clearly, L^p can be replaced by a reflexive Köthe function space in the above proposition. This examples comes from [620].
In our next example we deal with the Bergman space

$$L^1_a = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic } \;\middle|\; \int_{\mathbb{D}} |f(x + iy)| \, dxdy < \infty \right\}.$$

By definition, this is a (closed) subspace of $L^1(\mathbb{D})$, and we are going to show:

- *The Bergman space L^1_a is L-embedded, in fact, it has a predual which is M-embedded.*

To show this we use a result proved in the next chapter, viz. Theorem IV.3.10. Let us first observe that L^1_a is a dual space. We consider the topology τ of uniform convergence on compact subsets of $\mathbb{D}$. Now, the unit ball of L^1_a is a normal family. Indeed, the mean value property of analytic functions implies

$$|f(z)| \le \frac{1}{\pi r^2} \int_{B_r(z)} |f(x + iy)| \, dxdy \le \frac{1}{\pi r^2} \|f\|$$

for $|z| \le R < 1$ and $r < 1 - R$. Moreover, an appeal to Fatou's lemma reveals that $B_{L^1_a}$ is τ-closed in the space of all analytic functions on $\mathbb{D}$; for if $f_n \xrightarrow{\tau} f$, then

$$\int |f_n(\varrho e^{i\varphi})| \, d\varphi \to \int |f(\varrho e^{i\varphi})| \, d\varphi \qquad \forall \varrho < 1,$$

hence

$$\|f\| = \int \varrho\, d\varrho \int |f(\varrho e^{i\varphi})|\, d\varphi \le \liminf \int \varrho\, d\varrho \int |f_n(\varrho e^{i\varphi})|\, d\varphi = \liminf \|f_n\| \le 1.$$

Consequently, $B_{L_a^1}$ is τ-compact, and by a classical result [317, p. 211] those $\ell \in (L_a^1)^*$ whose restriction to $B_{L_a^1}$ is τ-continuous forms a predual of L_a^1. Observe next that the embedding of L_a^1 into $L^0(\mathbb{D})$, the space of all measurable functions on $\mathbb{D}$ equipped with the topology of convergence in measure, is continuous for the τ- and L^0-topologies so that the two topologies coincide on the unit ball. Hence the above predual coincides with the space $(L_a^1)^\sharp$ from Theorem IV.3.10, $(L_a^1)^\sharp$ separates L_a^1 (since $f \mapsto f(z)$ is in $(L_a^1)^\sharp$), and the unit ball of L_a^1 is L^0-closed. Thus Theorem IV.3.10 proves our claim. – This example is due to E. Werner.

The dual space of L_a^1 can be identified with the Bloch space B (see e.g. [37]); however there is only an isomorphism and not an isometry between $(L_a^1)^*$ and B. One can check that the above predual is a renorming of the little Bloch space B_0. (In Example 1.4(j) we proved that B_0 is M-embedded in its most natural norm, which is different.) The papers [36] and [37] contain more information on renormings of B_0 and B, their relationship to operators on Hilbert space and the Bergman space. As a matter of fact, B_0 is known to be isomorphic to c_0 [566]. Let us take the chance to point out how to obtain this result from the material of this chapter: Since the Bergman projection maps $C_0(\mathbb{D})$ onto B_0 [565], we see that B_0 is a separable M-embedded $\mathcal{L}^\infty$-space; now apply Theorem 3.11. In this connection Lusky's recent paper [430] is also relevant. Similar arguments work for the case of weighted spaces of analytic functions on certain subsets of $\mathbb{C}^n$ [622].

We finally wish to connect the examples of M-ideals of Lorentz and Orlicz sequence spaces given in Example 1.4(b) and (c) to the corresponding ideals of compact operators on a separable Hilbert space H. Our setting will be as follows. Let E be a Banach space with a 1-symmetric basis. Then $\mathcal{J}_E$ denotes the set of compact operators T on H whose sequences $(s_n(T))$ of singular numbers belong to E. This is a Banach ideal of operators under the norm

$$\|T\|_E = \|(s_n(T))\|_E$$

Various properties of E are known to pass to $\mathcal{J}_E$ (see for instance [24] and its references); for the general theory of "symmetrically normed ideals" we refer to [275] and [568]. In this vein one can prove:

- *Let E have a 1-symmetric basis, and suppose $E \neq c_0$. Then E is an M-ideal in its bidual if and only if $\mathcal{J}_E$ is.*

The proof is contained in [623].

Douglas algebras. Let H^∞ denote the algebra of bounded analytic functions on the open unit disk. As usual we shall identify H^∞ via radial limits isometrically with a closed subalgebra of $L^\infty = L^\infty(\mathbb{T})$. The algebra H^∞ has the following maximal property: If B is a weak* closed subalgebra of L^∞ strictly containing H^∞, then $B = L^\infty$. (Note that H^∞ itself is weak* closed.) In 1969 R. G. Douglas, in his work on Toeplitz operators, was led to investigate certain norm closed subalgebras between H^∞ and L^∞; more precisely

he became interested in those subalgebras B with the additional property that B is the closed algebra generated by H^∞ and those $f \in B$ for which $f^{-1} \in H^\infty$; equivalently, B is generated by H^∞ and those conjugates of inner functions which are in B. He conjectured that every closed algebra between H^∞ and L^∞ necessarily has this property, which was shown to be true by Chang and Marshall in 1976. Hence a closed algebra between H^∞ and L^∞ is called a *Douglas algebra*. (For references for this fact and related results we refer to the surveys [555], [556], [557] and the monograph [243].)
The simplest example of a Douglas algebra is $H^\infty + C$, the linear span of H^∞ and the space C of continuous functions on $\mathbb{T}$. (As we have already remarked, Sarason was the first to observe that $H^\infty + C$ is closed in that he shows that the canonical mapping $f + A \mapsto f + H^\infty$ from C/A to L^∞/H^∞ is an isometry. Here A denotes the disk algebra. Another proof is due to Rudin [543].) It is also known that $H^\infty + C$ is the smallest Douglas algebra (apart from H^∞).
In 1979 Axler, Berg, Jewell and Shields [38] proved that $H^\infty + C$ is a proximinal subspace of L^∞. In fact, they showed that the quotient space $(H^\infty + C)/H^\infty$ is proximinal in L^∞/H^∞, and the above result follows easily from this and the fact that H^∞, being weak* closed, is proximinal. Then Luecking [426] obtained the same conclusion from his result that $(H^\infty + C)/H^\infty$ is an M-ideal in L^∞/H^∞; see Corollary 1.5.
Subsequently M-ideal methods have proved useful for obtaining best approximation results for Douglas algebras. Here we shall survey some of them. Let M be the maximal ideal space of L^∞ so that $L^\infty = C(\mathsf{M})$. If $B \subset L^\infty$ is a Douglas algebra and $S \subset \mathsf{M}$ we let

$$B_S = \{f \in L^\infty \mid f_{|S} \in B_{|S}\}.$$

It is known that B_S is closed if S is a p-set for B. (See Section V.4 for the definition of a p-set.) Younis [657] shows that H^∞_S/H^∞ is an M-ideal in L^∞/H^∞ and thus that H^∞_S is proximinal if S is a p-set for H^∞. In [658] and [427] Younis and Luecking consider algebras of the form $H^\infty + L^\infty_F$ where for $F \subset \mathbb{T}$

$$L^\infty_F = \{f \in L^\infty \mid f \text{ is continuous at each } t \in F\}.$$

It is known that $H^\infty + L^\infty_F$ is always a Douglas algebra, and for closed F it is shown in [658] that $(H^\infty + L^\infty_F)/H^\infty$ is an M-ideal in L^∞/H^∞. One can translate this result as follows (cf. [427]): If $S \subset \mathsf{M}$ is a p-set for $H^\infty + C$, then $(H^\infty + C)_S/H^\infty$ is an M-ideal in L^∞/H^∞; and Luecking and Younis ask if $(H^\infty + C)_S$ is the only Douglas algebra with the above property. This was answered in the negative by K. Izuchi [330]. The paper [331] contains the general result that, for a Douglas algebra B and a p-set S, B_S/B is an M-ideal in L^∞/B. All these results contribute to the longstanding open question whether every Douglas algebra is proximinal in L^∞. Let us mention that this problem was finally solved by Sundberg [588] who constructed a shrewd counterexample in 1984. Also, the paper [240] is of interest in this connection.
There is another type of problem where M-ideal methods have turned out to be of importance. Axler et al. prove in [38] that the unit ball of $L^\infty/(H^\infty + C)$ fails to possess extreme points, thus $L^\infty/(H^\infty + C)$ is not isometric to a dual Banach space, which stands in marked contrast to L^∞/H^∞ which can be identified with the dual of the Hardy space H^1_0. In [658] and [427] it is shown that for closed or open $F \subset \mathbb{T}$, the quotient space $L^\infty/(H^\infty + L^\infty_F)$ shares this property with $L^\infty/(H^\infty + C)$ which represents the special

case $F = \mathbb{T}$. The proof relies essentially on the non-uniqueness of best approximants from M-ideals. Meanwhile it has been shown by Izuchi [328] that the above statement holds for every subset F of $\mathbb{T}$. For related papers see [329], [332], [333] and [659].

Projectional resolutions of the identity. In 1966 Corson and Lindenstrauss stated the – in Namioka's words prophetic – conjecture that every weakly compact subset of a Banach space is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for a suitable set Γ. It is easily seen that in order to prove this it is sufficient to show the following result, which was established by Amir and Lindenstrauss in 1968.

> Theorem. (Amir-Lindenstrauss)
> *For every weakly compactly generated Banach space X there are a set Γ and a continuous injective operator $T : X \to c_0(\Gamma)$.*

In fact, most of the properties of weakly compactly generated spaces already follow from the existence of such an injection T, cf. [16, Section 1]. The first step towards the above theorem was performed by Lindenstrauss in [415] and [416] where he proved it for reflexive X. As a decisive tool he showed the existence of a projectional resolution of the identity in X, and in order to get this he used what was later called a Lindenstrauss compactness argument. We recall that a Banach space X is said to admit a projectional resolution of the identity (PRI) if there is a family $(P_\alpha)_{\omega\le\alpha\le\mu}$ of projections in X, where μ is the first ordinal with cardinality dens X, satisfying

(a) $\|P_\alpha\| = 1$ for $\omega \le \alpha \le \mu$,
(b) $P_\mu = Id_X$,
(c) $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ for $\beta \le \alpha$,
(d) $\operatorname{dens} P_\alpha X \le \operatorname{card} \alpha$ for $\omega \le \alpha \le \mu$,
(e) $\overline{\bigcup_{\beta<\alpha} P_\beta X} = P_\alpha X$ for limit ordinals α.

The proof of the general Amir-Lindenstrauss theorem in [16] is much harder than in the reflexive case, but again the existence of a PRI is instrumental. A survey of weakly compactly generated spaces up to 1967 is [417]. It still furnishes a good introduction to this area, though most of the problems stated there have meanwhile been solved, see [154, Chapter Five].

Vašák [612] generalized the Amir-Lindenstrauss theorem by constructing a PRI for every weakly countably determined Banach space X. (X is called weakly countably determined if there is a sequence $(A_n)_{n\in\mathbb{N}}$ of w^*-compact subsets of X^{**} such that for all $x \in X$ there is a subset $N_x \subset \mathbb{N}$ with $x \in \bigcap_{n\in N_x} A_n \subset X$.) In 1979 Gul'ko made a major innovation by exposing the purely topological core of the problem and by providing a very simple proof of the Amir-Lindenstrauss theorem using the notion of conjugate pairs. A recommendable account of Gul'ko's ideas for the weakly compactly generated case is the article of Namioka and Wheeler [448]. Also Stegall gave a short and selfcontained proof in [582].

An easy consequence of the Mackey-Arens theorem is that if K is any weakly compact subset of a Banach space X, then the restriction map $R : X^* \to C(K)$ is w^*-w-continuous. From this, one can show that a Banach space X is weakly compactly generated iff there exists a w^*-w-continuous linear injection $T : X^* \to Y$ for some Banach space Y; and that a $C(K)$-space is weakly compactly generated iff K is an Eberlein compact space, i.e. homeomorphic to a weakly compact subset of a Banach space. Thus an essentially equivalent, yet more topological formulation of the Amir-Lindenstrauss theorem is: if K is an Eberlein compact space, then there is an injective operator $T : C(K) \to c_0(\Gamma)$ for a suitable set Γ. In this form it was extended by Argyros, Mercourakis and Negrepontis to Corson compact spaces K (a compact space K is called Corson compact if K is homeomorphic to a subset of $\Sigma(\mathbb{R}^I) := \{x \in \mathbb{R}^I \mid \mathrm{supp}(x) \text{ is countable}\}$ for some set I); see [450, Section 6]. Further extensions – which reintroduce functional analysis and PRIs – can be found in [609] and [611]. In [469] Orihuela and Valdivia define the notion of a projective generator, thus providing a unified approach not only to the above questions but also to the problem of constructing PRIs in dual spaces, which we discuss below. In fact, the arguments in [469] and [610], when specialized to weakly compactly generated Banach spaces, seem to provide the simplest known proof of the Amir-Lindenstrauss theorem.

We now wish to relate the existence of a PRI to other properties of a Banach space. In fact, the assumption of a PRI neither implies the existence of a continuous injection $T : X \to c_0(\Gamma)$ nor the separable complementation property nor an equivalent locally uniformly rotund renorming. Let us give an example. Let X be a Banach space with $\mathrm{dens}\, X = \mathfrak{m}$ admitting a PRI $(P_\alpha)_{\omega \le \alpha \le \mu}$, and suppose Z is an arbitrary Banach space with $\mathrm{dens}\, Z = \mathfrak{n} < \mathfrak{m}$. Then the operators $Q_\alpha = P_\alpha \oplus 0$ for $\alpha \le \nu$ and $Q_\alpha = P_\alpha \oplus Id_Z$ for $\nu < \alpha \le \mu$ (where ν denotes the smallest ordinal with cardinality $\mathfrak{n}$) define a PRI on $X \oplus_\infty Z$. Suitable choices of Z provide the desired examples. (This simple construction was shown to us by D. Yost.) On the other hand, in many classes of Banach spaces for which PRIs can be constructed, the proofs automatically yield those stronger conclusions. By the way, having an injection $T : X \to c_0(\Gamma)$ is not sufficient for the existence of an equivalent locally uniformly rotund norm on X, either, e.g. $X = \ell^\infty$ even fails to have an equivalent weakly locally uniformly rotund norm [154, p. 120]. However, certain other properties less restrictive than being weakly compactly generated are enough, as shown in [602], [662], and [272]. On the other hand, the existence of an injective operator $T : X \to c_0(\Gamma)$ is sufficient for an equivalent strictly convex norm on X, but not necessary as shown by Dashiell and Lindenstrauss, see [144]. Finally, although used for isomorphic and topological problems, the existence of a PRI is an isometric property: $X = C[0, \omega_1]$ has a PRI with respect to the supremum norm, but when equipped with the equivalent Fréchet differentiable norm constructed by Talagrand in [597] it does not; see [212, p. 149] for details.

In some dual Banach spaces X^* it is possible to construct a projectional resolution of the identity even if X^* is not weakly compactly generated. Tacon [590] did this under the assumption that X is very smooth, i.e. X is smooth and has a $\|\,.\,\|$-w-continuous support mapping $S_X \to S_{X^*}$, cf. p. 142. It is easy to see that $\mathrm{dens}\, Y = \mathrm{dens}\, Y^*$ for every subspace Y of a very smooth space X, hence very smooth spaces are Asplund spaces. In [570] and [571] Sims and Yost tried to extend Tacon's result to arbitrary Asplund spaces X; however the projections P_α in X^* which they constructed could not be arranged to satisfy $P_\alpha X^* = \overline{\bigcup_{\beta<\alpha} P_\beta X^*}$ for limit ordinals α. Yet their approach gives an elementary (not model theoretic) proof of the result of Heinrich and Mankiewicz that the dual of every nonseparable Banach space contains uncountably many nontrivial complemented subspaces (i.e., they are neither finite dimensional nor finite codimensional).

In this connection let us mention the very recent spectacular counterexamples of Gowers and Maurey who found (separable) Banach spaces without nontrivial complemented subspaces.
As we said in the introduction to Section III.4 the ideas of Sims and Yost do prove successful in the special situation of M-embedded spaces. Fabian extended Tacon's result in [210] and [211]; he proves the crucial step [210, Lemma 1] without compactness arguments but with sophisticated geometrical reasoning, and he singles out the following

PROPOSITION. *Let X be a Banach space with a $\|\,.\,\|$-w lower semi-continuous set-valued mapping $D : X \to 2^{X^*}$ satisfying*

(a) *$D(x)$ is countable for every $x \in X$,*
(b) *$\overline{\text{lin}}\,\{x^*|_V \mid x^* \in D(x),\, x \in V\} = V$ for every closed subspace V of X.*

Then X^* has a projectional resolution of the identity (which comes from Hahn-Banach extension operators).

Recall that a set-valued map Φ from a topological space A into the subsets 2^B of a topological space B is said to be *lower* (*upper*) *semi-continuous*, if the set $\{a \in A \mid \Phi(a) \cap M \neq \emptyset\}$ is open (closed) whenever M is an open (closed) subset of B. Using the above proposition, Fabian and Godefroy [212] proved the general result that the dual of every Asplund space admits a projectional resolution of the identity. In order to find a suitable mapping D they use the following selection theorem of Jayne and Rogers [342, Th. 8]:

THEOREM. *Let M be a metric space and X^* a dual Banach space with the Radon-Nikodým property. Let $F : M \to 2^{X^*}$ be an upper semi-continuous set-valued map (X^* with its w^*-topology) which takes only nonempty and w^*-closed values. Then, with respect to the norm topology on X^*, the set-valued map F has a Borel measurable selector f of the first Baire class.*

This can be applied to the support map $F : X \to 2^{X^*}$, $F(x) = \{x^* \in B_{X^*} \mid x^*(x) = \|x\|\}$. The $\|\,.\,\|$-$\|\,.\,\|$-continuous functions $f_n : X \to X^*$ converging pointwise to the selector f are used to build the set-valued map $D : X \to 2^{X^*}$ required in the proposition by $D(x) = \{f_n(x) \mid n \in \mathbb{N}\}$. Note that establishing (b) for this map D is far from being obvious. In closing, we remark that the projections P_α in X^* are in general not w^*-continuous (cf. [212, p. 149]) and that clearly not every dual space admits a PRI (e.g. $X^* = \ell^\infty$).
The question under which conditions the adjoints P_α^* of a PRI (P_α) in X form a PRI in X^*, i.e. (P_α) is a shrinking PRI, was considered by John and Zizler in [345, Lemma 3]. This is the case if X has a Fréchet differentiable norm. Theorem 1 of this article says that a weakly compactly generated Banach space X admits a shrinking projectional resolution of the identity iff it has an equivalent Fréchet differentiable norm. The renorming results of Talagrand in [597] show that this equivalence fails without extra assumptions on X.