A proof
of the Markov-Kakutani fixed point theorem
via the Hahn-Banach theorem

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S. Kakutani, in [2] and [3], provides a proof of the Hahn-Banach theorem via the Markov-Kakutani fixed point theorem, which reads as follows.

Theorem *Let K be a compact convex set in a locally convex Hausdorff space E. Then every commuting family $(T_i)_{i \in I}$ of continuous affine endomorphisms on K has a common fixed point.*

In this note I wish to point out how to obtain, conversely, this theorem from the Hahn-Banach theorem. The use of the Hahn-Banach theorem necessitates formulating the Markov-Kakutani theorem in the setting of locally convex spaces. Actually, it holds in a general Hausdorff topological vector space as well and has a well-known and simple proof (see e.g. [1, p. 456]); but it is applied for the most part in locally convex spaces, for instance in order to show the existence of Haar measure on a compact abelian group. So the point of this note is rather to illustrate the power of the Hahn-Banach theorem than to simplify the proof of the Markov-Kakutani theorem.

The key to the proof of the theorem lies in the following lemma, which of course is a special case of the Schauder-Tychonov fixed point theorem. However, its assumptions are strong enough to allow a completely elementary treatment. It is here that the Hahn-Banach theorem, in the form of the separation theorem, enters.

Lemma *Let K be a compact convex set in a locally convex Hausdorff space E, and let $T : K \to K$ be a continuous affine transformation. Then T has a fixed point.*

Proof If the lemma were false, the intersection of the diagonal $\Delta := \{(x, x)\colon x \in K\}$ of $K \times K$ with the graph of T, viz. $\Gamma := \{(x, Tx)\colon x \in K\}$, would be empty. Since Δ and Γ are compact convex subsets of $E \times E$, the Hahn-Banach theorem applies to produce continuous linear functionals l_1 and l_2 on E and numbers $\alpha < \beta$ such that

$$l_1(x) + l_2(x) \leq \alpha < \beta \leq l_1(y) + l_2(Ty)$$

for all $x, y \in K$. Consequently,

$$l_2(Tx) - l_2(x) \geq \beta - \alpha$$

for all $x \in K$. Iterating this inequality yields

$$l_2(T^n x) - l_2(x) \geq n(\beta - \alpha) \to \infty$$

for arbitrary $x \in K$ so that the sequence $(l_2(T^n x))_{n \in \mathbf{N}}$ is unbounded, which contradicts the compactness of $l_2(K)$.

The Markov-Kakutani theorem is now readily established by means of a simple compactness argument: Let K_i denote the set of all fixed points of T_i. We have $K_i \neq \emptyset$ by the lemma, and K_i is compact and convex. To show $\bigcap_{i \in I} K_i \neq \emptyset$, which is our aim, it is enough to do so for finite intersections. Since T_i and T_j commute, we conclude $T_i(K_j) \subset K_j$. Hence, $T_i|_{K_j}$ has a fixed point by the lemma so that $K_i \cap K_j \neq \emptyset$. An obvious induction argument now shows $\bigcap_{i \in F} K_i \neq \emptyset$ for all finite $F \subset I$.

References

[1] N. Dunford and J. T. Schwartz. *Linear Operators. Part 1: General Theory.* Interscience Publishers, New York, 1958.

[2] S. Kakutani. *Two fixed-point theorems concerning bicompact convex sets.* In: *Selected Papers, Vol. I*, pages 144–147. Birkhäuser, 1986.

[3] S. Kakutani. *A proof of the Hahn-Banach theorem via a fixed point theorem.* In: *Selected Papers, Vol. I*, pages 154–158. Birkhäuser, 1986.

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