R = ring, $I \subseteq R$ ideal $\rightsquigarrow R/I$: $R \xrightarrow{\pi} R/I$

Prop: There is a 1-1-correspondence:

$$\{\text{ideals } J \subseteq R \text{ with } J \supseteq I\} \longleftrightarrow \{\text{ideals } P \subseteq R/I\}$$

$$\{I \subseteq J \subseteq R\}$$

Proof:
- $J \longmapsto J/I =: P$ (is an ideal in R/I) $\pi(J)$
- $\pi^{-1}(P) \longleftarrow P$
- both operations are mutually inverse. □

Example: • Ideals in $\mathbb{Z}$: $n \cdot \mathbb{Z}$ for $n \in \mathbb{Z}$. $n = 1, 2, 5, 10$

• Ideals in $\mathbb{Z}/10\mathbb{Z}$ $\hat{=}$ ideal $J \subseteq \mathbb{Z}$ with $(10) \subseteq J \subseteq \mathbb{Z}$, $J = n \cdot \mathbb{Z}$, $10 \cdot \mathbb{Z} \subseteq n \cdot \mathbb{Z}$ i.e. $n \mid 10$

$\mathbb{Z}/(10)$: $(1)/(10)$, $(2)/(10)$, $(5)/(10)$, $(10)/(10) = 0$

Remark: Let $I \subseteq J \subseteq R$ $\rightsquigarrow$ $J/I \subseteq R/I$ is an ideal $\rightsquigarrow$ $(R/I)/(J/I)$ = new quotient ring of R/I

Prop: $(R/I)/(J/I) \cong R/J$

Proof: $R/I \xrightarrow{p} R/J$, $(\bar{r} = r+I) \longmapsto r+J$

$$\begin{tikzcd} R \arrow[r, "\pi_I"] \arrow[dr, "\varphi"'] & R/I \arrow[d, dashed, "p"] \\ & R/J \end{tikzcd}$$

$\varphi = \pi_J$

$I \subseteq \operatorname{Ker} \varphi$ (i.e. $\varphi(I) = 0$) because $I \subseteq J = \operatorname{Ker} \varphi$

$p: R/I \to R/J$ surjective

$$\operatorname{Ker} p = \{r+I \mid p(r+I) = r+J = J\} = \{r+I \mid r \in J\} = J/I \subseteq R/I \Big\} \Rightarrow (R/I)/\operatorname{Ker} p \overset{\sim}{\to} R/J$$ □

$I \subseteq R$ ideal $\Downarrow$ R/I

Q. How can we understand R/I-modules M? $\bar{r} \cdot m = r \cdot m$, $\bar{r} = r+I$

A: $\longleftrightarrow$ R-modules M such that $I \cdot M = 0$.

Remark. $R \xrightarrow{\psi} S$ algebra, $M = S$-module $\Rightarrow$ M becomes an R-module: $r \cdot m := \psi(r) \cdot m$. //

Example. (0) k = field $\rightsquigarrow$ [k-modules] = [k-vector spaces]

(1) $R = k[x]$ $\rightsquigarrow$ [$k[x]$-modules] = [k-VS with a linear map φ]

$k \hookrightarrow$; M; x provides $(\cdot x): M \to M$ $= \varphi$

Let k = field, e.g. $k = \mathbb{R}$, or $k = \mathbb{C}$. Best: assume $k = \bar{k}$ (i.e. every $f \in k[x]$, $\deg f \geq 1$ has a zero in k)

$$\mathbb{A}^n_k = k^n = \{(c_1, \dots, c_n) \mid c_i \in k\}$$ $\boxed{\text{n-dim. affine space over } k}$

$k[x_1, \dots, x_n] \ni f \rightsquigarrow$ gives a polynomial function $f: \mathbb{A}^n \to k$. („regular functions")

$J \subseteq k[x_1, \dots, x_n] \rightsquigarrow V(J) := \{(c_1, \dots, c_n) \mid f(c) = 0 \text{ for all } f \in J\}$, $= c$

$\boxed{\text{Vanishing locus of } J}$

Example: $k = \mathbb{R}$, $J = \{f = x^2 + y^2 - 1\}$, $n = 2$, $x_1 = x$, $x_2 = y$ $\rightsquigarrow$ $\mathbb{A}^2_{\mathbb{R}}$ = real plane

$V(J) = V(f) = \{(c,d) \mid c^2 + d^2 = 1\}$

Observation: • $J \subseteq k[x_1, \dots, x_n] \rightsquigarrow \langle J \rangle \subseteq k[\underline{x}]$ ideal generated by J

$\rightsquigarrow V(J) = V(\langle J \rangle)$ $\rightsquigarrow$ $V(J)$ does only matter for J = ideal

$\supseteq$ trivial

• J = ideal $\rightsquigarrow \sqrt{J} := \{f \in k[\underline{x}] \mid \exists m: f^m \in J\}$ radical (is also an ideal)

$\rightsquigarrow V(J) = V(\sqrt{J})$ $\supseteq$ ✓ "$\subseteq$": Let $c \in V(J)$, let $f \in \sqrt{J} \Rightarrow \exists m: f^m \in J \Rightarrow (f^m)(c) = 0$, $= f(c)^m$; $f(c) = 0 \Leftarrow f(c)^m$ (k = field) □

$Z \subseteq \mathbb{A}^n_k$ subset (e.g. $Z = V(J)$ for some J)

Def. $I(Z) := \{f \in k[x_1, \dots, x_n] \mid f(z) = 0 \ \forall z \in Z\} \subseteq k[x_1, \dots, x_n]$ (is a radical ideal)

(Let $f \in \sqrt{I(Z)} \rightsquigarrow \exists m: f^m \in I(Z) \Rightarrow \forall z \in Z: f^m(z) = 0 \Rightarrow f(z) = 0 \Rightarrow f \in I(Z)$) i.e. $\sqrt{I(Z)} = I(Z)$.

$\left\{ \text{ideals in } k[\underline{x}] \right\}$ (radical) $\underset{I(\ldots)}{\overset{V(\cdot)}{\rightleftarrows}}$ $\left\{ \text{subsets of } \mathbb{A}^n_k = k^n \right\}$ closed alg. subsets

Properties: $\cdot\ I \subseteq J \Rightarrow V(I) \supseteq V(J)$

$\cdot\ Z \subseteq Z' \Rightarrow I(Z) \supseteq I(Z')$

$\cdot\ V(I+J) = V(\langle I \cup J \rangle) = V(I \cup J) = V(I) \cap V(J)$

$\cdot\ V(I) \cup V(J) = \begin{cases} V(IJ) \\ V(I \cap J) \end{cases}$ candidates (both !!)

Lemma: $V(I \cap J) = V(IJ) = V(I) \cup V(J)$

Proof: $IJ \subseteq I \cap J \subseteq I \Rightarrow V(I) \subseteq V(I \cap J) \subseteq V(IJ)$

$V(J) \subseteq$

$\Rightarrow V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$

left: show that $V(IJ) \subseteq V(I) \cup V(J)$.

Assume: $c \in V(IJ)$, $c \notin V(I)$, $c \notin V(J)$

$\Downarrow$ $\exists f \in I : f(c) \neq 0$ $\quad \exists g \in J : g(c) \neq 0$

$\underset{\in IJ}{(fg)}(c) \neq 0$ ↯ $\Leftarrow$

Lemma: Let $J_\nu \subseteq k[x]$ ideals ($\nu \in \Lambda$ = index set)

$V\left(\sum_{\nu\in\Lambda} J_\nu\right) = \bigcap_{\nu\in\Lambda} V(J_\nu)$

Proof: $V\left(\bigcup_{\nu\in\Lambda} J_\nu\right) = \bigcap_{\nu\in\Lambda} V(J_\nu)$ //

Consequence: Sets of the form $V(J)$ (J = ideal) are called „closed algebraic subsets of $\mathbb{A}^n$".

$\cdot\ V(J) \cup V(J') = \text{cl. alg.}$

$\cdot\ \nu \in \Lambda : \bigcap_{\nu\in\Lambda} V(J_\nu) = \text{closed. alg.}$

$\Big\}$ topology on $\mathbb{A}^n$: „Zariski - topology"

Def: a subset $S \subseteq \mathbb{A}^n$ is (called) closed

$:\Leftrightarrow \exists J : S = V(J)$ ($\hat{=}$ closed alg. subsets)

Remark: $\mathbb{A}^n = V((0))$

$\emptyset = V((1))$

Example: $\mathbb{A}^1_k = k \Rightarrow$ closed subsets = finite subsets ($k = \mathbb{R}$, 0, 1) or: the whole $\mathbb{A}^1$.

closed subsets ?

$\cdot\ k = \mathbb{C}$ closed subsets $\hat{=} V(J)$, $J \subseteq \mathbb{C}[x] \rightsquigarrow \exists f(x) \in \mathbb{C}[x] : J = (f(x))$ (principal ideal domain)

$\hookrightarrow \exists \alpha_1, \ldots, \alpha_k \in \mathbb{C}$: $f(x) = (x-\alpha_1) \cdot \ldots \cdot (x-\alpha_k)$

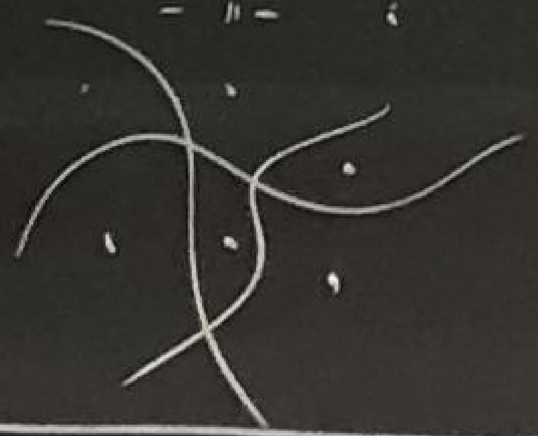
$V(f) = \bigcup_{\nu=1}^{k} V(x-\alpha_\nu) = \bigcup_{\nu=1}^{k} \{\alpha_\nu\} = \{\alpha_1, \ldots, \alpha_k\}$

Example: $\mathbb{A}^2_\mathbb{R}$: $V(x^2+y^2-1)$ = circle; $V(ax+by+c) = \underset{\text{line}}{\text{affine}}$ $(a,b,c \in k)$

$\rightsquigarrow$ eg: $\cdot$ „alg. curves" are closed subsets

$\cdot$ points - " - ; eg: $V(x,y) = \{(0,0)\}$

$V(x-3, y-5) = \{(3,5)\}$



Remark: $\mathbb{R}^2_{\text{classical}} \underset{\text{No}}{\overset{\text{id}}{\rightleftarrows}} \mathbb{R}^2_{\text{Zariski}}$ ~~cont.~~

Q: ① Z = closed alg. subset $\rightsquigarrow I(Z) \rightsquigarrow V(I(Z)) \subseteq \mathbb{A}^n$

tautological: $Z \subseteq V(I(Z))$ $\quad \| \ ?$ YES

$\exists J.\ Z = V(J)$ $\Rightarrow$ $V(J) \subseteq V(\underset{\supseteq J}{I(V(J))}) \subseteq \overset{Z}{V(J)}$

② J = ideal (radical) $\rightsquigarrow V(J) \rightsquigarrow I(V(J)) \rightsquigarrow J \subseteq I(V(J))$

$I(V(J)) \overset{?}{=} J$; $\quad \sqrt{J} \subseteq I(V(\sqrt{J}))$

It remains to check: J = radical ideal $\Rightarrow J = I(V(J))$

Counter example: $J := (x^2+1) \subseteq \mathbb{R}[x]$ $\quad V(J) \subseteq \mathbb{R}^1$, $V(J) = \{c \in \mathbb{R} \mid c^2+1=0\} = \emptyset$

$I(\emptyset) = \mathbb{R}[x] = (1) \Rightarrow \boxed{I(V(x^2+1)) = (1)}$

Ex: $V(x^2+y^2) \subseteq \mathbb{R}^2$

Theorem: $k = \bar{k} \Rightarrow I(V(J)) = \sqrt{J}$ „HNS" = „Hilbert - Nullstellensatz".

i.e. assume that k is alg. closed, i.e. every $f \in k[x] \setminus k$ has a zero in k.

geometry $\longleftrightarrow$ algebra $\quad k=\bar{k}$ $\quad$ 1-1 correspondence

$\mathbb{A}^n_k \qquad k[\underline{x}]$

$\cup \qquad \cup$

Z closed $\rightleftarrows$ J ideal

<u>Def</u>. X = top. space. X is called irreducible $:\Longleftrightarrow \nexists\ X = Z_1 \cup Z_2$: $\bullet\ Z_i \subseteq X$ closed $\bullet\ Z_i \neq X\ \forall i$.

<u>Exmple</u>. $\bullet$ $X = \mathbb{R}$ (classical) $\quad X = (-\infty,1] \cup [0,\infty)$ ⇝ <u>not</u>

$\bullet$ $X \subseteq \mathbb{R}$ (−"−) $\quad$ <u>if</u>, e.g.: $\underbrace{0,1 \in X}_{\text{NOT irred}}$ ⇝ $Z_1 = (-\infty,\frac{1}{2}] \cap X \ni 0, \not\ni 1$

$Z_2 = [\frac{1}{3},\infty) \cap X \ni 1, \not\ni 0$

↳ irred. subsets ≙ single points. $\quad$ NOT irred $\Longleftarrow X = Z_1 \cup Z_2$

$\bullet$ $X = \mathbb{A}^1_k$, Zariski-top $\Longrightarrow$ <u>is</u> irreducible! $(k = \text{infinite field})$

$\mathbb{A}^2 \supset V(xy-1) = \{(\alpha,\beta) \in \mathbb{R}^2 / \alpha\beta = 1\}$ <u>is</u> irreducible!

<u>Prop</u>. $Z \subset \mathbb{A}^n_k$ Zariski-closed (⇝ $Z = V(J)$).

Then Z is irreducible $\Longleftrightarrow$ $I(Z)$ is a prime ideal.

<u>Proof</u>. $(\Longrightarrow)$ $I(Z)$ NOT prime $\Longrightarrow \exists f,g \in k[\underline{x}] \setminus I(Z)$, $fg \in I(Z)$.

$Z_1 = V(I(Z) + (f)) \subseteq V(I(Z)) = Z$

$Z_1 \subsetneq Z$: f vanishes on Z_1, does <u>not</u> on Z $\quad$ ∴ <u>Same</u>: $Z_2 = V(I(Z) + (g)) \subsetneq Z$

$Z_1 \cup Z_2 = Z \Longrightarrow Z$ reducible.

$(\Longleftarrow)$ Assume $Z = Z_1 \cup Z_2$, $Z_i \subsetneq Z \Longrightarrow \exists f_i : f_i \in I(Z_i), f_i \notin I(Z) \Longrightarrow f_1 \cdot f_2 \in I(Z_1) \cap I(Z_2)$ ‖ $I(Z)$

$I(Z)$ is <u>not</u> prime $\Longleftarrow I(Z) = I(Z_1 \cup Z_2)$

<u>Ex</u>. $\bullet$ $X = \mathbb{A}^1_k$ ≙ ideal in $k[x]$: (0)

$\bullet$ $X = V(xy-1)$ ⇝ ideal $(xy-1) \subseteq k[x,y]$, (x,y)

$\boxed{k[x,y]/(xy-1)} = k[x,\frac{1}{x}] \subseteq k(x)$

$\mathbb{Z}[\sqrt{-5}] \qquad 2 \cdot 3 = 6 = (1+\sqrt{-5})(1-\sqrt{-5})$

$P_1 = (2, 1+\sqrt{-5}),\ P_3 = (3, \ldots)$

$P_2 = (2, 1-\sqrt{-5}),\ P_4 = (3, \ldots)$

$P_1P_2 = (2),\ P_3P_4 = (3)$ $\}$ $(6) = P_1P_2P_3P_4$

$P_1P_3 = (1+\sqrt{-5})$

<u>Prop</u>. $Z \subseteq \mathbb{A}^n_k \ldots$ $\quad$ Z is a single point $\Longleftrightarrow$ $I(Z)$ = maximal ideal.

<u>Proof</u>. Z = <u>point</u> $\Longleftrightarrow$ Z is a <u>minimal</u>, <u>closed</u>, subset of $\mathbb{A}^n$. (Zariski) (<u>non-empty</u>)

$(\Longrightarrow)$ ok

$(\Longleftarrow)$ Z = minimal but not a point $\Longrightarrow$ choose $p \in Z$: $\{p\} \subsetneq Z$.

$\{p\} = V(J)$ − what is J? $\quad p = (p_1, \ldots, p_n)$

$J = (x_1 - p_1, \ldots, x_n - p_n)$

$Z \subseteq \mathbb{A}^n$ ⇝ $I(Z) \subseteq k[\underline{x}]$ ‖ $k[\underline{x}]$ = ring of regular fcts on $\mathbb{A}^n$

restrict φ_f: $\quad f \in k[\underline{x}]$ ⇝ becomes $\varphi_f : \mathbb{A}^n \longrightarrow k$, $\quad c = (c_1, \ldots, c_n) \longmapsto f(c_1, \ldots, c_n)$

$\varphi_f|_Z : Z \hookrightarrow \mathbb{A}^n \longrightarrow k$ ⇝ reg. fct on Z

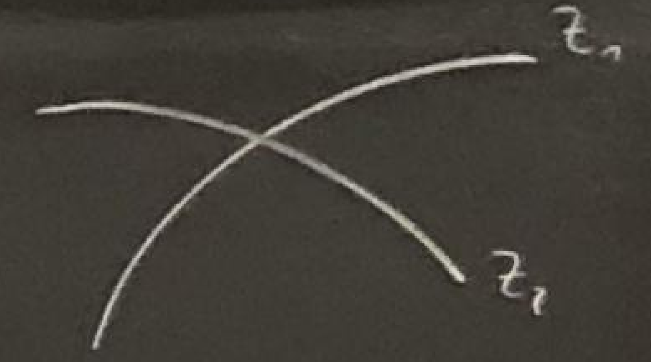
<u>ex</u>. $f \in I(Z) \Longrightarrow \varphi_f \equiv 0$ on Z !!

$\cancel{k[\underline{x}] \twoheadrightarrow I(Z)}$ $\quad k[\underline{x}]/I(Z) =: A(Z)$ = "ring of regular functions on Z".

$f + I(Z) \in k[\underline{x}]$ ⇝ $f = g$ on Z !! $\quad$ ($f \ni g$)

$\bullet$ <u>Ex</u>. $\bullet$ $X = \mathbb{A}^1$. $k[x]$, $I(X) = (0)$ ⇝ $A(\mathbb{A}^1) = k[x]/(0) = k[x]$

$\bullet$ $X = V(xy-1) \Longrightarrow A(X) = k[x,y]/(xy-1) \cong \underline{k[x,\frac{1}{x}]}$



$\bullet$ <u>Remark</u>: X = irred $\Longleftrightarrow A(X)$ = domain

X = pt $\Longleftrightarrow A(X)$ = field.

X, $A(X) = k[x]$ ⇝ $\bullet$ $k[x] = k[x]/(0)$ ⇝ $X = \mathbb{A}^1$, $\quad \{(\alpha,0) \mid \alpha \in k\} = \mathbb{A}^1 \times \{0\}$

$I(Z)$ ⇝ $A(Z) = k[\underline{x}]/I(Z)$ $\quad \bullet$ $k[x] = k[x,y]/(y)$ ⇝ $X = V(y) \subseteq \mathbb{A}^2_k$