geometry $\underset{I}{\overset{V}{\rightleftarrows}}$ algebra

$k^n \supseteq V(J) = Z$ $\qquad$ $k[x_1, \dots, x_n] \supseteq J$

$\downarrow$ $\qquad$ $A(Z) = k[x]/J$

topological space: · closed alg. subsets, ZARISKI-top. $\qquad$ $I(Z)$

$\rightsquigarrow$ $f \in A(Z)$ become $f : Z \longrightarrow k$

· $\sqrt{J} = I(V(J))$ $\quad$ IF $k = \bar{k}$

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closed in $\mathbb{A}^n$: $V(J) \subseteq \mathbb{A}^n$ $\rightsquigarrow$ Q: how do the open subsets look like?

of course: $\mathbb{A}^n \setminus V(J)$

basis of the topology: We just need special open subsets $\underbrace{\{U_i\}}_{\mathcal{B}}$ of $\mathbb{A}^n$:

i.e. $\forall p \in X$ $(= \mathbb{A}^n)$, $\forall V \subseteq X$ open, $p \in V$ (i.e. $p \in V \subseteq X$)

$\exists U_i \in \mathcal{B}$: $p \in U_i \subseteq V$.

In our case: $\mathcal{B} = \{ D(f) \mid f \in k[x] \}$

$\mathbb{A}^n \setminus V(f)$

Claim: If $p \in \mathbb{A}^n$, and $V \subseteq \mathbb{A}^n$ is open $\Longrightarrow \exists f \in k[x]$: $p \in D(f) \subseteq V$.

Proof V = open $\rightsquigarrow \exists J \subseteq k[x]$: $V = \mathbb{A}^n \setminus V(J)$ $\rightsquigarrow p \in V$, i.e. $p \notin V(J)$

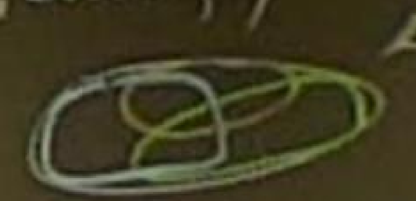
$\Longrightarrow \exists f \in J$: $f(p) \neq 0$. $\quad$ **Claim:** $p \in D(f) \subseteq \mathbb{A}^n \setminus V(J)$ $\quad$ $V(J) \subseteq V(f)$

$D(f) = \mathbb{A}^n \setminus V(f)$

$f(p) \neq 0 \rightsquigarrow p \notin V(f)$ $\qquad \square$

$$\boxed{D(f) := \{ p \in \mathbb{A}^n \mid f(p) \neq 0 \}}$$

Q: $\nu \in \Lambda$, $f_\nu \in k[x]$ $\rightsquigarrow$ when is $\bigcup_{\nu \in \Lambda} D(f_\nu) = \mathbb{A}^n$? $\quad$ ($\hat{=}$ „covering")



$\mathbb{A}^n$

Claim: covering $\Longleftrightarrow$ $J := (f_\nu \mid \nu \in \Lambda) = k[x]$ $\quad$ IF $k = \bar{k}$

Proof: $\mathbb{A}^n = \bigcup D(f_\nu) \Longleftrightarrow \mathbb{A}^n = \bigcup (\mathbb{A}^n \setminus V(f_\nu)) = \mathbb{A}^n \setminus \left(\bigcap_{\nu \in \Lambda} V(f_\nu) \right) = \mathbb{A}^n \setminus V(J)$

$\Longleftrightarrow V(J) = \emptyset \Longleftrightarrow \underbrace{I(V(J))}_{\overset{\text{HNS}}{=} \sqrt{J}} = k[x]$ $\quad$ $\sqrt{J} = (1) \Longleftrightarrow 1 \in \sqrt{J} \Longleftrightarrow \exists k: 1^k \in J$

$\Longleftrightarrow J = (1)$. $\qquad \square$

Rem $(f_\nu) = (1) \Longleftrightarrow$
$\exists$ fin. many f_ν, $\lambda_\nu \in k[x]$: $\sum \lambda_\nu f_\nu = 1$

Remark: $X := V(J) \subseteq \mathbb{A}^n$ $\rightsquigarrow$ $f \in A(Z) = k[x]/I(Z)$ gives rise to

$D(f) := X \setminus V(f)$ open in X

$f_\nu \in A(X)$

$\bigcup D(f_\nu) = X \Longleftarrow (f_\nu) = A(Z)$. $\qquad$ (i.e. $D_X(f) := D_{\mathbb{A}^n}(f) \cap X$)

Example. $\mathbb{A}^2 \leftrightsquigarrow k[x,y]$ $\qquad$ $D(x) = [x \neq 0]$ $\qquad$ $D(x) \cup D(y) = \mathbb{A}^2 \setminus \{(0,0)\}$

$D(y) = [y \neq 0]$ $\qquad$ $\Updownarrow$

$(x, y) \subsetneq k[x,y]$

need a further $f(x,y)$: $0 \in D(f)$

e.g.: $f = x + y + 1$ $\qquad$ $\Updownarrow$ $\quad f(0,0) \neq 0$

2^{nd} reason for niceness of the $D(f)$:

Slogan: $D(f)$ are algebraic sets, too! (Like the $Z = V(J)$)

Checking: $f \in k[x_1, \dots, x_n]$ given $\rightsquigarrow$ $V(f) \subseteq \mathbb{A}^n$ closed

$D(f) \subset \mathbb{A}^n$ open

$Z_f := V(f \cdot t - 1) \subseteq \mathbb{A}^{n+1}$

$\quad \cap$

$k[x_1, \dots, x_n, t]$

$$\begin{tikzcd} (x_1, \dots, x_n, \tfrac{1}{f}) & (x_1, \dots, x_n) \arrow[l, mapsto] \\ Z_f \arrow[r, "\text{bijection}"] \arrow[d, hook, "\text{closed}"'] & D(f) \arrow[d, hook, "\text{open}"] \\ \mathbb{A}^{n+1} \arrow[r, "p"] & \mathbb{A}^n \end{tikzcd}$$

$\mathbb{A}^{n+1} \xrightarrow{p} \mathbb{A}^n$ $\rightsquigarrow$

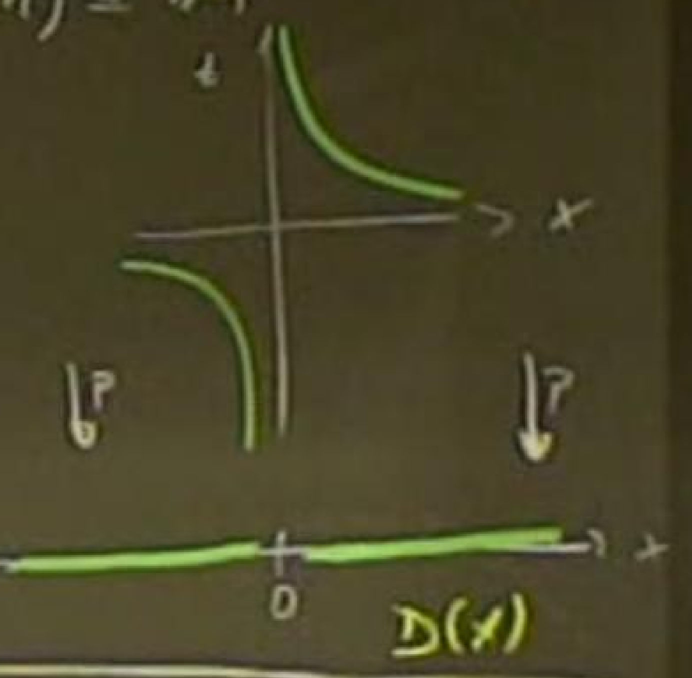
$(x_1, \dots, x_n, t) \longmapsto (x_1, \dots, x_n)$

ex: $f = x \in k[x]$, $D(x) = \mathbb{A}^1 \setminus \{0\}$

$Z_1 = V(x \cdot t - 1) \subseteq \mathbb{A}^2$

$\downarrow$?

$D(x)$



$Z \subseteq \mathbb{A}^n$ „closed alg. subsets" $\quad$ Last week: $\mathbb{A}^1 \cong V(x) < \mathbb{A}^2$

Def Z = „affine set" $:\Longleftrightarrow \exists$ $Z = V(J) \subset \mathbb{A}^n$. $\quad$ (e.g. $D(f)$ = affine sets) $\quad$ $\left(\begin{array}{l} \mathbb{A}^2 \setminus V(x,y) \\ = \mathbb{A}^2 \setminus \{(0,0)\} \end{array} \right)$

Q. What are „the" maps between those sets/objects?

· $f : \mathbb{A}^m \longrightarrow \mathbb{A}^n$ is called „regular" $:\Longleftrightarrow$ $f = (f_1, \dots, f_n)$; $f_i : \mathbb{A}^m \longrightarrow \mathbb{A}^1$ is a polynomial, i.e. $f_i \in k[x_1, \dots, x_m]$

Def $f: \mathbb{A}^m \longrightarrow \mathbb{A}^n$ regular $\iff f = (f_1, \dots, f_n)$, $f_i \in k[x_1, \dots, x_m]$

$\mathbb{A}^n \supseteq V(J)$: $J \subseteq k[y_1, \dots, y_n]$ ideal

$f^{-1}(V(J)) = \{ p \in \mathbb{A}^m \mid f(p) \in V(J) \} = \{ p \in \mathbb{A}^m \mid g(f(p)) = 0 \ \ \forall g \in J \subseteq k[y] \}$

$g(f(p)) = (g \circ f)(p) =: (f^*(g))(p)$

Understand $g \circ f$: $\boxed{k[y_1, \dots, y_n] \xrightarrow{f^*} k[x_1, \dots, x_m]}$

$g \longmapsto (g \circ f)(x_1, \dots, x_m) = g(f_1(\underline{x}), \dots, f_n(\underline{x}))$

i.e. $y_i \longmapsto f_i(\underline{x})$

$\rightsquigarrow$ f^* = ring homomorphism !!

i.e. $J' := f^*(J) \cdot k[\underline{x}]$ „image ideal of J"

$\Rightarrow f^{-1}(V(J)) = V(J')$

$\{ p \in \mathbb{A}^m \mid h(p) = 0 \ \ \forall h \in f^*(J) \}$

$= V(\langle f^*(J) \rangle) = V(J')$

2nd step: $\begin{array}{ccc} Z & \xrightarrow{f} & Z' \\ \cap & & \cap \\ \mathbb{A}^m & \overset{F}{\dashrightarrow} & \mathbb{A}^n \end{array}$ f is called regular $\iff$ $\exists$ regular F, such that the diagram commutes.

Claim: $f: Z \to Z'$ is regular $\iff$ $f^*: A(Z') \longrightarrow A(Z)$ is a ring hom.

$\begin{array}{ccc} A(Z') & \longrightarrow & A(Z) \\ \uparrow & & \uparrow \\ k[\underline{y}] & \xrightarrow{F^*} & k[\underline{x}] \end{array}$

$g \longmapsto g \circ f$

Remark: $D(f) \subseteq \mathbb{A}^n$, regular fcts. on $D(f)$?

$[f \neq 0]$ $\quad \cup\!\shortmid \quad V(J) = Z \rightsquigarrow A(Z) = k[\underline{x}]/J$

$\left\{ \begin{array}{l} \bullet\ g \in k[x_1, \dots, x_n] \sim g: D(f) \subseteq \mathbb{A}^n \xrightarrow{g} k \\ \bullet\ \frac{1}{f} \in \mathrm{Quot}(k[x_1, \dots, x_n]) = k(x_1, \dots, x_n) \end{array} \right.$

$\frac{1}{f}: D(f) \longrightarrow k$

Proposal: $k[\underline{x}, \frac{1}{f}] \supseteq k[\underline{x}]$

(ex: $\mathbb{Z}[\sqrt{-5}] = \{ a + b\sqrt{-5} \mid a, b \in \mathbb{Z} \}$

$\mathbb{Z}[x] = \{ a_0 + a_1 x + \dots \}$

$\mathbb{Z}[x]/(x^2 + 5)$)

Recall: $D(f) \hookrightarrow \mathbb{A}^{n+1}$

$p \longmapsto (p, \frac{1}{f(p)})$

$Z_f = V(f t - 1)$

$A(D(f)) := A(Z_f) = k[\underline{x}, t]/(f(\underline{x}) \cdot t - 1)$

$\frac{1}{f(\underline{x})} \leftarrow\!\shortmid t$

R = commutative ring.

$R \setminus R^*$ is an ideal.

Def: R $(R, \mathfrak{m})$ is a local ring $\iff$ R has exactly one maximal ideal, $\mathfrak{m}$

Ex: • $k[x]$ is **NOT** a local ring: $\forall c \in k$: $\mathfrak{m}_c := (x - c)$ is maximal.

• trivial example: $R = K$ = field: Only $\underline{(0)}$ and (1) are ideals.

• less trivial example: $R := \left\{ \frac{f}{g} \mid f, g \in k[x];\ g(0) \neq 0 \right\}$

$k[x] \leftrightarrow$ fcts on $\mathbb{A}^1$

$k[x, \frac{1}{x}] \leftrightarrow$ fcts on $D(x) = \mathbb{A}^1 \setminus \{0\}$

fcts on „some" neighbourhood of $0 \in \mathbb{A}^1$

e.g. $\frac{1}{x - 5}$

$\mathfrak{m} = \{ r = \frac{f}{g} \mid f(0) = 0 \}$

$R^* = \{ \frac{f}{g} \mid g(0) \neq 0;\ f(0) \neq 0 \}$

Remark: $(R, \mathfrak{m})$ - local $\Rightarrow R \setminus \mathfrak{m} = R^*$ $\quad (\iff \mathfrak{m} = R \setminus R^*)$

($\alpha \in R \setminus \mathfrak{m} \overset{?}{\Longrightarrow} \alpha \in R^*$, but if $\alpha \notin R^* \Rightarrow \exists$ max. id. $\mathfrak{m}'$: $\alpha \in \mathfrak{m}' = \mathfrak{m} \Rightarrow \alpha \in \mathfrak{m}$

$\alpha \in R^* \Rightarrow \alpha \notin \mathfrak{m}$ - otherwise we would get $\mathfrak{m} = (1)$)

R = ring. **Def:** $\sqrt{(0)} = \{$nilpotent elements$\}$ - „nil radical"

Prop: $\sqrt{(0)} = \bigcap_{\substack{P \subseteq R \\ \text{prime ideals}}} P$

Proof: „$\subseteq$": If $f \in \sqrt{0} \rightsquigarrow f^n = 0$ (for some n)

and $P = PI \rightsquigarrow f^n = 0 \in P \overset{PI}{\rightsquigarrow} f \in P$ ✓

„$\supseteq$": Let $f \in R$, $f \neq$ nilpotent, i.e. $S := \{1, f, f^2, \dots\} = \{ f^n \mid n \in \mathbb{N} \} \not\ni 0$

$\mathcal{C} := \{ \text{Ideals } I \subseteq R \mid I \cap S = \emptyset \}$, $(0) \in \mathcal{C}$!!

chains in $\mathcal{C}$ have an upper bound $\overset{\text{ZORN}}{\rightsquigarrow}$ $\exists$ max. element $I \in \mathcal{C}$, i.e. ideal, disj. to S, $f \notin I$

show: I = prime !

$R \ni f$ non-nilpotent $\leadsto S = \{f^n \mid n \in \mathbb{N}\} \leadsto \mathcal{C} = \{$ideals disjoint to $S\} \ni \underline{I}$ maximal

[Goal: I = prime], let $a, b \in R$: $a, b \notin I$, but $ab \in I$.

$I + (a) \supsetneq I \Longrightarrow$ $\left.\begin{array}{l} I + (a) \notin \mathcal{C} \to \exists\, m: f^m \in I + (a) \\ I + (b) \notin \mathcal{C} \Rightarrow \exists\, n: f^n \in I + (b) \end{array}\right\} f^m \cdot f^n \in I + (ab) = I$ ↯

Same game: $\bigcap_{\substack{m \subset R \\ \text{maximal ideal}}} m$ = „Jacobson radical" $\supseteq \sqrt{0}$

Prop: $\bigcap m = \{a \in R \mid 1 + \underset{aR}{(a)} \subseteq R^*\}$

Proof: ($\subseteq$) Let $a \in \bigcap m$, show that $1 + (a) \subseteq R^*$.

Let $r \in R$; (show: $1 + ar \in R^*$; if not $\Rightarrow \exists\, m'$: $1 + \underline{ar} \in m'$

$\to a \in m' \Rightarrow 1 \in m'$ ↯

($\supseteq$) Let $a \in R$ s.t. $\underline{1 + (a)} \subseteq R^*$. Let m be some max ideal, show: $a \in m$.

If $a \notin m \Rightarrow m + (a) = (1) \Rightarrow \exists\, r \in m, s \in R$: $r + s \cdot a = 1 \Rightarrow r = \underbrace{1 - sa}_{\in R^*} \in 1 + (a)$ ↯

Lemma: 1) $P \supseteq PI$ (I, J = ideals) $\Rightarrow [P \supseteq IJ \Leftrightarrow \underline{P \supseteq I \cap J} \Leftrightarrow P \supseteq I$ or $P \supseteq J]$

2) J = ideal: $J \subseteq \bigcup_{i=1}^{k} P_i \Longrightarrow J \subseteq P_i$ for some i. IF the P_i are prime ideals with at most 2 exceptions

Proof of (2): $\underline{k = 2}$ Show: $\boxed{J \subseteq P_1 \cup P_2} \to \exists\, i: J \subseteq P_i$

Assume that this is wrong $\Longrightarrow \exists\, x_i$: $x_i \in J$, $x_i \notin P_i$ $(i = 1, 2)$

$x_1 \in P_1 \cup P_2 \to x_1 \in P_2$

$x_2 \in \ldots \to x_2 \in P_1$

$y := x_1 + x_2 \underbrace{\notin P_1, \notin P_2}$ ↯

$\hookrightarrow y \notin P_1 \cup P_2 \Rightarrow y \notin J$

$x_1 \in P_2$, $x_1 \notin P_1$; $x_2 \in P_1$, $x_2 \notin P_2$

induction: $J \nsubseteq P_i$ $(\forall i)$, show: $J \nsubseteq \bigcup P_i$, assume NOT, i.e. $J \subseteq \bigcup P_i$

we may assume: $\forall k$: $J \nsubseteq \bigcup_{i \neq k} P_i \leadsto \exists\, \underline{x_k \in J}$, $x_k \notin \bigcup_{i \neq k} P_i \Rightarrow \underline{x_k \in P_k}$

Assume P_1 = prime ideal. $\leadsto y := x_1 + (x_2 \cdots x_n)$

- $x_1, \ldots, x_n \in J \Rightarrow y \in J$.
- P_2: $x_2 \in P_2$, $x_1 \notin P_2 \Rightarrow y \notin P_2$

  $\vdots$

  $x_n \in P_n$, $x_1 \notin P_n \Rightarrow y \notin P_n$

- $\underline{P_1}$: $x_1 \in P_1$; $x_2, \ldots, x_n \notin P_1$

  PI-property $\Rightarrow x_2 \cdots x_n \notin P_1$

  $\Rightarrow y \notin P_1$

  $\Rightarrow y \notin \bigcup_i P_i$ ↯ □

Chinese Remainder theorem: classical notion: $m, n \in \mathbb{N}$, $\gcd(m, n) = 1$.

$\mathbb{Z}/mn\mathbb{Z} \xrightarrow{\Phi} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is an isomorphism.

Ex: $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ $\quad \Phi(\boxed{3}) = (1, 0)$, $\Phi(\boxed{4}) = (0, 1)$

$\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$: $1, 3 \mapsto (1, 1)$

Prop. R = ring; $I_1, \ldots, I_n$ = ideals; assume $I_i + I_j = (1)$ (if $i \neq j$)

(example: $\underbrace{m\mathbb{Z} + n\mathbb{Z}}_{(m,n)} = (\gcd(m, n) \cdot \mathbb{Z})$)

$\Rightarrow$ • $\bigcap_{i=1}^{n} I_i \overset{\supseteq}{=} \prod_{i=1}^{n} I_i$

• $R/\bigcap_{i=1}^{n} I_i \xrightarrow{\sim} \prod_{i=1}^{n} R/I_i$

Proof. $\underline{n = 2}$: • $I_1 + I_2 = (1) \leadsto \exists\, x_1 \in I_1$, $x_2 \in I_2$: $x_1 + x_2 = 1$

Let $y \in I_1 \cap I_2 \Longrightarrow y = y \cdot 1 = y \cdot (x_1 + x_2) = \underset{I_2 I_1}{y x_1} + \underset{I_1 I_2}{y x_2} \in I_1 I_2$.

• $R/I_1 \cap I_2 \xrightarrow{\Phi} R/I_1 \times R/I_2$ $\quad \ker \Phi = \{r \in R \mid r \in I_1, r \in I_2\} = I_1 \cap I_2$

$r \longmapsto (\bar{r}, \bar{r})$

Φ is surjective: $1 = x_1 + x_2$: $x_1 \overset{\Phi}{\longmapsto} (\underset{x_1}{\overset{\shortparallel}{0}}, 1 - x_2) = (0, 1)$, $x_2 \mapsto (1, 0)$ //

0 mod I_2