Prop: $I_\nu \le R = \text{ring}$ $(\nu = 1, \dots, n)$ s.t.: $I_\nu + I_\mu = (1)$ $(\forall \nu \neq \mu)$

$$\Longrightarrow \bigcap_\nu I_\nu \overset{\text{ideal}}{=} \prod_\nu I_\nu \;;\quad R/\prod_\nu I_\nu \xrightarrow{\sim} \prod_{\nu=1}^{n} R/I_\nu .$$

Proof: • $n = 2$ ✓

• Assume ok for $I_1, \dots, I_{n-1}$; $\underline{\underline{\text{new } I_n}}$

$$\Longrightarrow \forall \nu = 1, \dots, n-1: \quad \underset{\substack{\cup\!\!\!\!\!\shortmid \\ x_\nu}}{I_\nu} + \underset{\substack{\cup\!\!\!\!\!\shortmid \\ x_n^{(\nu)}}}{I_n} = (1)$$

$$\Longrightarrow \exists \quad x_\nu + x_n^{(\nu)} = 1$$

$$\left.\right\} \quad y := \prod_{\nu=1}^{n-1} x_\nu \in \prod_{\nu=1}^{n-1} I_\nu$$

$\underline{\text{and:}}$ $y = \prod_{\nu=1}^{n-1} \underset{\substack{\cap \\ 1 + I_n}}{(1 - x_n^{(\nu)})} = 1 + \underset{\substack{\in \\ I_n}}{z}$

$$\Longrightarrow y \in \left(\prod_{\nu=1}^{n-1} I_\nu\right) \cap (1 + I_n)$$

$\underline{\text{i.e.}}$ $\underset{\substack{\| \\ J}}{\left(\prod_{\nu=1}^{n-1} I_\nu\right)} + I_n = (1) \xRightarrow[\text{ind. hyp.}]{(n=2)\text{-case}} \prod_{\nu=1}^{n} I_\nu = J \cdot I_n = J \cap I_n = \left(\bigcap_{\nu=1}^{n-1} I_\nu\right) \cap I_n = \bigcap_{\nu=1}^{n} I_\nu$

$$R/J \cap I_n \xrightarrow{\sim} R/J \times R/I_n \longrightarrow \left(R/I_1 \times \dots \times R/I_{n-1}\right) \times \left(R/I_n\right) \quad //$$

---

geometry $\xrightarrow{I(-)}$ algebra $\quad (?) = \Phi$

$\xleftarrow{V(-)}$

$$\boxed{Z} \subseteq \mathbb{A}^n_k \longmapsto I(Z) \le k[x_1, \dots, x_n] \text{ ideal} \longmapsto \boxed{A(Z)} = k[x_1, \dots, x_n]/I(Z) = \text{f.g. } k\text{-algebra}$$

(even better: $A(Z) = \underline{\text{reduced}}$, i.e. $\sqrt{0} = (0)$ because: $I(Z) = \sqrt{I(Z)}$)

<u>Recipe for Φ.</u> Start with $A =$ f.g. k-algebra (reduced, $k = \mathbb{C}$)

$\Downarrow$ $\underline{\text{Goal:}}$ which $Z \subseteq \mathbb{A}^n_k$ is associated to A? (i.e. $\Phi(A) = ?$)

• find n, J: $k[x_1, \dots, x_n]/J \xrightarrow{\sim} A \rightsquigarrow \underline{\sqrt{J} = J}$ $\quad \left(\underset{\substack{\| \\ 0}}{\sqrt{0}} = \sqrt{J/J} = \sqrt{J}/J \quad (\dots\dots) \ \sqrt{J} = J\right)$

• $Z = V(J) \subseteq \mathbb{A}^n_k = k^n$

$A =$ ring (comm, with 1) $\rightsquigarrow$ $\operatorname{Spec} A := \{P \le A \mid P = \text{prime ideal}\}$

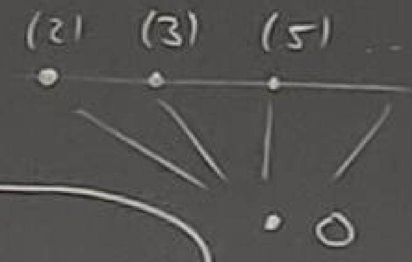
$\qquad \cup\!\!\shortmid$

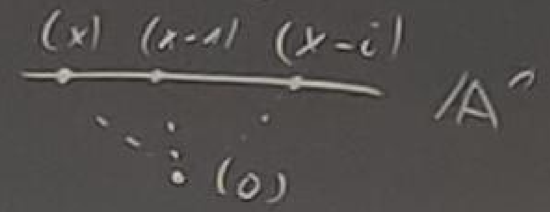
$\operatorname{Max}\operatorname{Spec} A := \{m \le A \mid m = \text{max. ideal}\}$

<u>Example.</u> $A = \mathbb{Z}$ (other ex: $A = \mathbb{C}[x]$)

• $\operatorname{Max}\operatorname{Spec} \mathbb{Z} = \{p\mathbb{Z} \mid p = \text{prime number}\}$, $\operatorname{Spec} \mathbb{Z} = \operatorname{Max}\operatorname{Spec} \mathbb{Z} \amalg \{(0)\}$

(ideals in $\mathbb{Z}$: $n \cdot \mathbb{Z}$, $n \in \mathbb{Z}$)



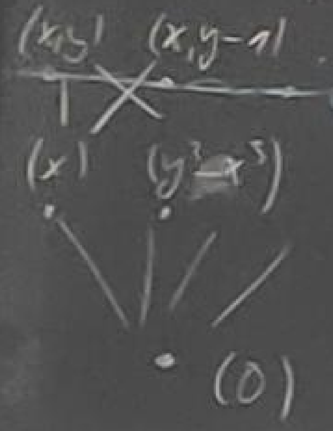


• $\operatorname{Max}\operatorname{Spec} \mathbb{C}[x] = \{(x - c) \cdot \mathbb{C}[x] \mid c \in \mathbb{C}\}$

(ideals in $\mathbb{C}[x]$: $f(x) \cdot \mathbb{C}[x]$, $f(x) \in \mathbb{C}[x]$) $\longrightarrow \operatorname{Spec} \mathbb{C}[x] = \operatorname{Max}\operatorname{Sp.} \amalg \{(0)\}$

• $\operatorname{Max}\operatorname{Spec} \mathbb{C}[x, y] = \{(x - c, y - d) \mid c, d \in \mathbb{C}\}$ ($\underline{\text{ex.}}$ $c = d = 0 \rightsquigarrow (x, y)$)

$\qquad (c, d) \in \mathbb{C}^2$



$\operatorname{Spec} \mathbb{C}[x, y] \ni (x), (y), (x^2 - y^3)$ $\qquad x \mapsto 0,\ y \mapsto 1$ $\qquad \underline{\text{ex:}}$ $\mathbb{C}[x, y]/(x) = \mathbb{C}[y]$

$\underline{\text{ex}}$. $(y^2 - x^3) \subsetneq (x, y - 1)$: $\mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y]/(x, y - 1) \xrightarrow{\sim} \mathbb{C}$ $\quad y^2 - x^3 \longmapsto 1 \neq 0$!!

$\operatorname{Spec} A$ becomes a top space (in a natural way, $\underline{\text{i.e.}}$ in the following way):

• Closed subsets (by def): $J \le A$ ideal $\rightsquigarrow$ $V(J) := \{P \in \operatorname{Spec} A \mid P \ge J\}$.

• Check the axioms: $\emptyset = V((1))$, $\operatorname{Spec} A = V((0))$ $\;\Big|\;$ $V(I) \cup V(J) = V(I \cap J) = V(IJ)$

$\left(P = \underline{\text{prime}} \text{ ideal} \Longrightarrow \left[(P \supseteq I \text{ or } P \supseteq J) \Longleftrightarrow P \supseteq I \cap J \Longleftrightarrow P \supseteq IJ\right]\right.$ $\quad \underline{\text{“Zariski top on } \operatorname{Spec} A\text{”}}$ $\Big]$

$\underline{\text{Show}}$ ideals $J_\nu \rightsquigarrow \bigcap_\nu V(J_\nu) = V\left(\sum_\nu J_\nu\right)$

$\underline{\text{Ex}}$. • $\operatorname{Spec} \mathbb{Z}$, $\underline{\text{closed subsets}}$: $V(n\mathbb{Z}) = \{p\mathbb{Z} \mid p \mid n\} = \{(\underset{\text{prime}}{\text{divisors}} \text{ of } n) \cdot \mathbb{Z}\} = \text{finite set}$.

$(n \neq 0)$

• $\{(2), (3), (7)\} = V(2 \cdot 3 \cdot 7\mathbb{Z})$ $\qquad V(0\mathbb{Z}) = \operatorname{Spec} \mathbb{Z}$

• $\{p\mathbb{Z}\} = V(p\mathbb{Z})$, $\{(0)\} = \text{closed}$? If so, then: $\{(0)\} = V(J) = \{P \mid P \supseteq J\}$

$\Downarrow$

$J = (0) \Longrightarrow V((0)) = \operatorname{Spec} \mathbb{Z}$

!! $\{(0)\}$ is not closed ! $\;\Big\|\;$ $X = \text{top space}$, $S \subseteq X$ sub$\underline{\text{set}}$

$\overline{\{(0)\}} = \operatorname{Spec} \mathbb{Z}$ “(0) is dense in $\operatorname{Spec} \mathbb{Z}$” $\qquad \overline{S} := \bigcap_{\substack{F \supseteq S, \\ \text{closed}}} F$

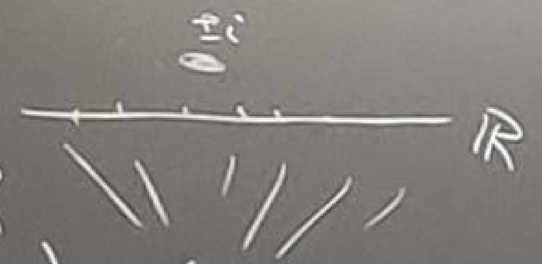
Q • $\underset{R}{\mathbb{C}[x_1, \dots, x_n]}$, $\operatorname{Max\,Spec} R = \{(x_1 - c_1, \dots, x_n - c_n) \,/\, c_i \in \mathbb{C}\}$ $\supseteq$

Exple $R = \mathbb{R}[x]$ $\rightsquigarrow$ • $(x - r) \,/\, r \in \mathbb{R}$ $\longleftrightarrow$ $r \in \mathbb{R}^1$

• (x^2+1) $\quad \mathbb{R}[x]/(x^2+1) = \mathbb{C}$

$\overset{\shortparallel}{(x+i)(x-i)}$ $\quad V((x+i)(x-i)) = \{(x-i), (x+i)\} \mathrel{\hat{=}} \{i, -i\}$ in $\mathbb{C}[x]$

$\pm i$



(0)

• Why Spec / Max Spec ?

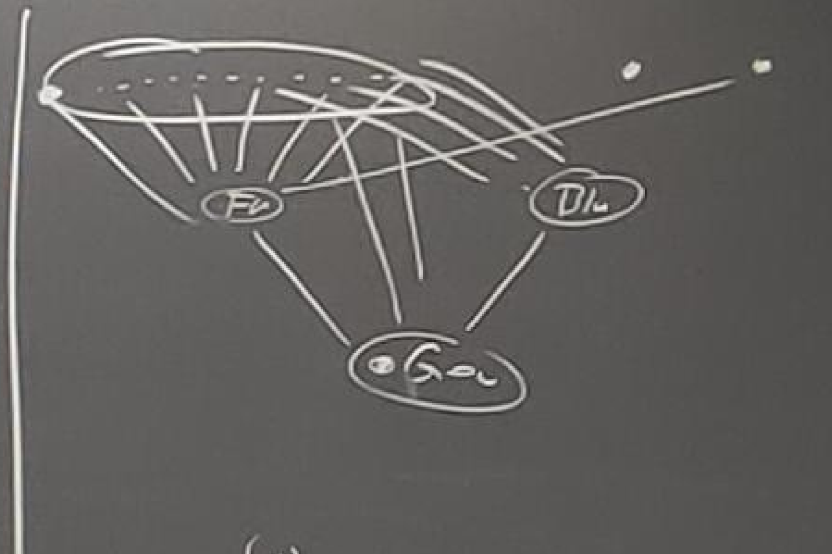
- Lemma: $P \in \operatorname{Spec} R$

$\{P\}$ is closed $\Longleftrightarrow$ $P \in \operatorname{Max\,Spec} R$

(ex: (0) in $\operatorname{Spec} \mathbb{Z}$ was not closed)

- $k = \bar{k}$ $\Longrightarrow$ $\operatorname{Max\,Spec} k[x_1, \dots, x_n] \longleftrightarrow k^n$

$(x_1 - c_1, \dots, x_n - c_n) \longleftarrow (c_1, \dots, c_n)$



$(0) = \eta$

$\mathbb{A}^n_k = k^n$ $\qquad \mathbb{A}^n_k = \operatorname{Spec} \underbrace{k[x_1, \dots, x_n]}_{R}$ $\qquad \underset{\mathbb{A}^n_k}{\operatorname{Spec} R} \;\dots\; \underset{\mathbb{A}^n_{k,\max}}{\operatorname{Max\,Spec} R} \underset{(k=\bar{k})}{\longleftrightarrow} k^n$

Prop: $(A, \operatorname{Spec} A)$

$J \subseteq A$ ideal $\rightsquigarrow$ $\left[V(J) = \emptyset \Longleftrightarrow J = (1) \right]$

Proof: Have to show: $(\Longrightarrow)$

assume: $J \neq (1) \Longrightarrow \exists$ max ideal $m \supseteq J$,

i.e. $m \in V(J)$. $\square$

• Remark: classical setup $k = \bar{k}$

$V(J) = \emptyset \Longrightarrow I(V(J)) = I(\emptyset) = (1)$

$\| \text{HNS}$

$\sqrt{J}$ $\Longrightarrow J = (1)$

• Ex: $J = (x^2+1) \subseteq \mathbb{R}[x]$ goes wrong!

$V(J) \subseteq \mathbb{R}^1 \Longrightarrow \emptyset$

$= \{ r_1 \in \mathbb{R}^1 \,/\, f(r_1) = 0 \; \forall f \in J \}$

$\Updownarrow$

$r_1^2 + 1 = 0$

$A \rightsquigarrow$ for $f \in A$ $\rightsquigarrow$ $D(f) := \operatorname{Spec} A \setminus V(f) = \{P \in \operatorname{Spec} A \,/\, f \notin P\}$

open subset $\qquad V((f))$

Claim: It is a basis of the top.:

i.e.: If $P \in U \underset{\text{open}}{\subseteq} \operatorname{Spec} A \Longrightarrow \exists f: P \in D(f) \subseteq U$

Proof: $U = \operatorname{Spec} A \setminus V(J) \rightsquigarrow P \notin V(J)$, i.e. $J \not\subseteq P$

i.e. $P \in D(f)$, $f \in J \rightsquigarrow V(J) \subseteq V(f)$ $\Longrightarrow \exists f \in J : f \notin P$

$\underset{U}{\operatorname{Spec} A \setminus V(J)} \supseteq \underset{D(f)}{\operatorname{Spec} A \setminus V(f)}$ $\square$

$f_\nu \in A \rightsquigarrow \bigcup_\nu D(f_\nu) = \operatorname{Spec} A$

$\Updownarrow$

$(f_\nu) = (1)$

$k^n \supseteq V(J) = X$ $\;\|\;$ $\operatorname{Spec} A$ $\quad$ eg: $A = k[x_1, \dots, x_n] \Longrightarrow \operatorname{Spec} \supseteq k^n$

$J \subseteq A \rightsquigarrow V(J) \subseteq \operatorname{Spec} A$ closed

$P \in V(J)$ $\mathrel{\hat{=}}$ $P \subseteq A$ PI s.t. $P \supseteq J$, i.e. $J \subseteq P \subseteq A$

$\mathrel{\hat{=}}$ Prime ideals in A/J

(we already know: $\{J \subseteq I \subseteq A\} \overset{\Phi}{\longleftrightarrow} \{\bar{I} \subseteq A/J\}$ $\qquad \pi: A \longrightarrow A/J$

$I \longmapsto I/J$

$\pi^{-1}(\bar{I}) \longleftarrow \bar{I}$

$A/I \qquad (A/J)/(I/J) = A/I$

Claim Φ respects prime ideals!

$\operatorname{Spec} A \supseteq V(J) = \operatorname{Spec} A/J$

$\{J \subseteq P \subseteq A\}$

$k^n \supseteq Z = V(J) \rightsquigarrow A(Z) = k[x_1,\dots,x_n]/J$

rings of "regular" fcts on X.

replace by $A(Z) = k[x_1,\dots,x_n]/I(X)$

• $\operatorname{Spec} A \supseteq Z$ subset $\rightsquigarrow I(Z) \subseteq A$?

$\quad = $ fcts vanishing on Z.

1^{st} quest: how do elements $a \in A$ become functions on $\operatorname{Spec} A$?

$\operatorname{Spec} A \to k$??

Start: • $a \in A$ • $P \in \operatorname{Spec} A$ $\Big\}$ $a(P) = \bar a \in K(P)$.

• $a \in A \rightsquigarrow \bar a \in \underbrace{A/P}_{\text{domain } D} \hookrightarrow \operatorname{Quot}(D) := \left\{ \frac{f}{g} \,\middle|\, f,g \in D,\ g \neq 0 \right\}$ $\quad \boxed{K(P) = \text{field}}$

$\boxed{\bar a \in K(P)}$

Example: ① $\mathbb{C}^2 \cong \operatorname{Spec} \mathbb{C}[x,y]$, $a = a(x,y) \in \mathbb{C}[x,y]$

$P = (c,d) \;\hat{=}\; (x-c, y-d) \rightsquigarrow K(P) = \mathbb{C}[x,y]/(x-c,y-d) \xrightarrow{\sim} \mathbb{C} \rightsquigarrow K(P) = \mathbb{C}$

$\mathbb{C}[x,y]$: $x \longmapsto c$, $y \longmapsto d$

$a(x,y) \longmapsto \bar a \longmapsto a(c,d)$

② $\operatorname{Spec} \mathbb{R}[x] \ni (x^2+1) = P$

$x(P) = \bar x \in \mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$

$\bar x \longmapsto i$

①' $\mathbb{C}^2 \ni P = (x)$ $\quad \operatorname{Spec} \mathbb{C}[x,y] \ni P = (x)$ $\quad a(x,y) \in \mathbb{C}[x,y]$

$a(P) = ?$

$\bar a \in \mathbb{C}[x,y]/(x) = \mathbb{C}[y] \subseteq \mathbb{C}(y)$

$\mathbb{C}[x,y] \to$: $y \mapsto y$, $x \mapsto 0$

$\boxed{\bar a = a(0,y)}$

$\operatorname{Spec} A \supseteq Z \rightsquigarrow I(Z) = \{a \in A \mid a(P) = 0 \ \forall P \in Z\} = \{a \in A \mid a \in P \ \forall P \in Z\} = \bigcap_{P \in Z} P = \bigcap Z$

$\left(\bar a \in A/P \subseteq \operatorname{Quot}(A/P) = K(P);\quad \bar a = 0 \text{ in } A/P \iff a \in P \right)$

Prop:

Q. How about: $\boxed{I(V(J)) \overset{?}{=}_{\text{thus}} \sqrt{J}}$?

Proof: $a \in I(V(J)) \iff a \in P \ \forall P \in V(J)$
$\iff a \in P \ \forall P \supseteq J$, i.e. $I(V(J)) = \bigcap_{P \supseteq J} P$

($a \in A$)

Claim: $\boxed{\bigcap_{P \supseteq J} P = \sqrt{J}}$ $(*)$

Proof of $(*)$: mod out J on both sides

$\rightsquigarrow$ claim becomes: $\bigcap_{\bar P = P I \text{ in } A/J} \bar P = \sqrt{\bar 0}$ $\quad$ we had this last week

functoriality

$$\begin{array}{ccc} A & \rightsquigarrow & \operatorname{Spec} A \\ \varphi \downarrow & & \uparrow \operatorname{Spec}\varphi \\ B & \rightsquigarrow & \operatorname{Spec} B \end{array}$$

$\varphi = $ ring homomorphism $\rightsquigarrow \operatorname{Spec}\varphi$

$(\operatorname{Spec}\varphi)(\underbrace{Q \subseteq B}_{\text{prime ideal}}) = \varphi^{-1}(Q)$

Claim: $\varphi^{-1}(Q) = PI$.

$A \xrightarrow{\varphi} B \xrightarrow{\pi} B/Q$, ψ

$\operatorname{Ker} \psi = \varphi^{-1}(Q)$

$\Rightarrow A/\varphi^{-1}(Q) \hookrightarrow B/Q \Rightarrow A/\varphi^{-1}(Q) = $ domain

$\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q}$

$(0) \in \operatorname{MaxSpec} \mathbb{Q}$

$i^{-1}(0) = (0) \in \operatorname{Spec}\mathbb{Z} \setminus \operatorname{MaxSpec}\mathbb{Z}$

Ex: $\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q}$

$2\mathbb{Z} = (2) \longmapsto i(2\mathbb{Z}) \cdot \mathbb{Q} = \mathbb{Q} = (1)$

Prop: $\operatorname{Spec}\varphi = $ continuous! (Homework)

Ex: $A \supseteq J$ ideal $\to \pi: A \twoheadrightarrow A/J \Rightarrow \operatorname{Spec}(\pi): \underbrace{\operatorname{Spec}(A/J)}_{V(J)} \longrightarrow \operatorname{Spec} A$

$D(f) = \{P \in \operatorname{Spec} A \mid f \notin P\} = \{P \in \operatorname{Spec} A \mid f(P) \neq 0\}$ $\left(= Z_f = V(ft-1) \right) \rightsquigarrow D(f) = \operatorname{Spec} A[t]/(ft-1)$

$= \operatorname{Spec} A[\tfrac{1}{f}]$ ②