R-modules

$M =$ · abelian grp

· $R \curvearrowright M$, i.e.: $\exists\ R \times M \longrightarrow M$ st.:

· reminder: $M \hookrightarrow R$ submodule $\Longleftrightarrow$ ideal

· 2nd -"- : $M, N = R$-modules $\rightsquigarrow M \oplus N := \{(m,n) \mid m \in M,\ n \in N\}$ — "direct sum"

· 3rd -"- : $M+N$ does only make sense if $M, N \subseteq L$ submodules ($L = R$-mod)

$M, N \subseteq M+N := \{m+n \mid m \in M,\ n \in N\} \subseteq L$

$\rightsquigarrow \exists$ can. map
$$M \oplus N \xrightarrow{\pi} M+N \quad \text{"sum"}, \qquad (m,n) \longmapsto m+n$$
there might be $\operatorname{Ker}\pi \neq 0$.

Def $M, N \subseteq L \rightsquigarrow$ their sum is direct $:\Longleftrightarrow \pi$ is an isomorphism

Remark: $M, N \subseteq L$, if $M+N$ is a direct sum $\Longrightarrow M \cap N = (0)$

Ex: $R = \mathbb{Z}$, $M = 2\mathbb{Z} \subseteq \mathbb{Z}$, $N = 3\mathbb{Z} \subseteq \mathbb{Z}$ $= L$

$2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z} \neq 0$ — $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$ not direct

no ISO — $2\mathbb{Z} \oplus 3\mathbb{Z}$ still exists!!

· $M, N = R$-modules $\rightsquigarrow \operatorname{Hom}_R(M,N) := \{\varphi : M \longrightarrow N \mid R\text{-linear}\}$

· $\varphi, \psi \in \operatorname{Hom} \rightsquigarrow (\varphi + \psi)(m \in M) := \varphi(m) + \psi(m)$

· $r \in R$, $\varphi \in \operatorname{Hom} \rightsquigarrow (r\varphi)(m) := r \cdot \varphi(m)$ $\Longrightarrow$ is an R-module.

Ex $R = K =$ field $\rightsquigarrow V, W = K$-vector spaces $\rightsquigarrow \operatorname{Hom}(V, W) = K$-vs.

ex: $W := K$ $\rightsquigarrow \operatorname{Hom}_K(V, K) =: V^*$ "dual v.s.".

Ex. $R = \mathbb{Z} \rightsquigarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \{0\} = 0$

$\varphi : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}$
$$\left.\begin{array}{l} 1 \longmapsto ?\ g \\ 2 \longmapsto 2g \\ \ \| \qquad\quad \|\ !! \\ 0 \longmapsto 0 \end{array}\right\} \Longrightarrow g = 0$$

Properties.
$$\operatorname{Hom}(M \oplus N, P) \xrightarrow{\sim} \operatorname{Hom}(M,P) \oplus \operatorname{Hom}(N,P)$$
M, N

$\operatorname{Hom}(M, P \oplus Q) = \operatorname{Hom}(M,P) \oplus \operatorname{Hom}(M,Q)$

$M, N, L = R$-modules $\rightsquigarrow$ **Def** $f : M \times N \longrightarrow L$ is called bilinear

$\Longleftrightarrow$
$$\left.\begin{array}{l} \forall m \in M : f(m, \cdot) : N \longrightarrow L \\ \forall n \in N : f(\cdot, n) : M \longrightarrow L \end{array}\right\} \text{both are } R\text{-linear.}$$

$f(m+m', n+n') = f(m+m', n) + f(m+m', n') = f(m,n) + f(m',n) + f(m,n') + f(m',n')$

e.g. $f(m, n+n') = f(m,n) + f(m,n')$

$f(m+m', n) = f(m,n) + f(m',n)$

e.g: $R \times R \xrightarrow{\cdot} R$, $a, b \longmapsto ab$ is bilinear!

Careful: $M \oplus N \xrightarrow{g} L$ is R-linear

$\rightsquigarrow g(m+m', n+n') = g((m,n) + (m',n')) = g(m,n) + g(m,n')$

Goal If $L = R$-module, if $M \otimes_R N \xrightarrow{f} L$ R-linear
$$\begin{tikzcd} M \otimes_R N \arrow[r, "f"] & L \\ M \times N \arrow[u, "\Phi\ (\text{bilinear})"] \arrow[ur, dashed, "f \circ \Phi = g"'] & \end{tikzcd}$$
$f \circ \Phi =$ bilinear

i.e. $\operatorname{Hom}_R(M \otimes_R N, L) \xrightarrow{\psi} \operatorname{Hom}_R(M, N; L) = \{\text{bilinear maps } M \times N \to L\}$

$f \longmapsto f \circ \Phi$

Wish: ψ should become an isomorphism!

Tensor product · $M \otimes_R N = R$-module.

$\boxed{M, N = \text{fixed}}$ · $M \times N \xrightarrow{\Phi} M \otimes_R N$ bilinear map

$(m, n) \longmapsto m \otimes n$.

in particular: $\Phi(m, n+n') = \Phi(m,n) + \Phi(m,n')$

i.e.: $m \otimes (n+n') = (m \otimes n) + (m \otimes n')$

Ex. $V, W = K$-vs., bases $v_1, \ldots, v_m \in V$ ($\cong K^m$), $w_1, \ldots, w_n \in W$ ($\cong K^n$)

$\rightsquigarrow V \otimes_K W :=$ vs with basis $v_i \otimes w_j$ $\rightsquigarrow \cong K^{mn}$

symbols obeying: $v \otimes (w + w') = v \otimes w + v \otimes w'$

Def's are not clear.

Comment "ψ is surjective" means. If $g : M \times N \longrightarrow L$ is bilinear
$$\Longrightarrow \exists f : M \otimes_R N \longrightarrow L : g = f \circ \Phi.$$

injective $\Longleftrightarrow f$ is uniquely determined!

M-module, N = submodule $\subseteq M$; L = module

$f : M/N \longrightarrow L$
$$\begin{tikzcd} M/N \arrow[r, "f"] & L \\ M \arrow[u, "\pi"] \arrow[ur, dashed] & \end{tikzcd}$$
for $\in \{g \in \operatorname{Hom}(M,L) \mid g(N) = 0\} \rightsquigarrow \operatorname{Hom}(M/N, L) \xrightarrow{\sim} \{g \in \operatorname{Hom}(M,L) \mid g(N) = 0\}$

M, N = R-modules. L = arbitrary R-module $\Longrightarrow \mathrm{Hom}_R(M\otimes_R N, L) \xrightarrow{\sim} \mathrm{Hom}(M,N;L)$
(is our goal) ($f \mapsto g$)

Reinterpretation:

$$\begin{tikzcd} M\times N \arrow[r, "\Phi"] \arrow[dr, "g"'] & M\otimes N \arrow[d, "f"] \\ & L \end{tikzcd}$$

$\mathcal{C}_{M,N}$-category: Objects $= \{$bilinear maps $M\times N \xrightarrow{g} L\}$ (fixed / varying)

Mor:

$$\begin{tikzcd} M\times N \arrow[r, "g_1"] \arrow[dr, "g_2"'] & L_1 \arrow[d, "\tau"] \\ & L_2 \end{tikzcd}$$

s.t. $g_2 = \tau\circ g_1$

Observ: • $\Phi \in \mathrm{Ob}\,\mathcal{C}_{M,N}$

• wish $\hat{=}$ Φ is an initial object of $\mathcal{C}_{M,N}$ "!!"

$\longrightarrow$ this ("universal") property determines $\Phi: M\times N \longrightarrow M\otimes_R N$ uniquely (up to isom)

Prop: $\otimes_R$ exists.

Ex: $\mathrm{Maps}(\{1,2\}, R) = R^2$

Proof: $R^{\oplus(M\times N)} := \{v \in \mathrm{Maps}(M\times N, R) \mid v\neq 0$ only for finitely many elements$\}$
$= \{\sum_{\text{finite}} r_i\cdot(m_i,n_i) \mid r_i\in R,\ m_i\in M,\ n_i\in N\}$

Submodule $S := \langle (m,n+n') - (m,n) - (m,n'), \ldots, (rm,n) - r(m,n), (m,rn) - r(m,n) \rangle \subset R^{\oplus(M\times N)}$ $\rightsquigarrow$ $M\otimes_R N := R^{\oplus(M\times N)}/S$

$M\times N \xrightarrow{\Phi} R^{\oplus(M\times N)}/S$ is bilinear.

$(m,n) \longmapsto 1\cdot(m,n) =: m\otimes n$

$(m,n+n') \longmapsto \ldots$
$(m,n) \longmapsto \ldots$
$(m,n') \longmapsto \ldots$
have to show: $\underbrace{\overline{(m,n+n')} - \overline{(m,n)} - \overline{(m,n')}}_{\in S} = 0$ ...

"initial"-property:

$$\begin{tikzcd} M\times N \arrow[r, "\Phi"] \arrow[dr, "\text{bilinear} = g"'] & R^{\oplus(M\times N)}/S \ni \sum r_i(m_i,n_i) \arrow[d] \\ & L \quad \sum r_i\, g(m_i,n_i) \end{tikzcd}$$

$\square$

Examples: ① $M\otimes_R R = M$

Proof: $M\otimes_R R \xrightarrow{?} M$
$\Updownarrow$
bilinear $M\times R \longrightarrow M$, $m, r \longmapsto r\cdot m$

$M \longrightarrow M\otimes_R R$, $m \longmapsto m\otimes 1$

$m\otimes r \longmapsto rm \longmapsto (rm)\otimes 1$, $m\otimes r = r\cdot(m\otimes 1)$, id?

⓪ $m\otimes 0 \in M\otimes N$, $m\otimes 0 = 0$
because: $m\otimes 0 = m\otimes(0+0) = m\otimes 0 + m\otimes 0$

② $(M\oplus M')\otimes_R N = (M\otimes_R N)\oplus(M'\otimes_R N)$
$(m,m')\otimes n \longleftrightarrow [(m\otimes n), (m'\otimes n)]$

$\mathrm{Hom}(A\oplus B, C) = \mathrm{Hom}(A,C)\oplus \mathrm{Hom}(B,C)$

expanded version: $(M\oplus M')\otimes_R N \longrightarrow (M\otimes_R N)\oplus(M'\otimes_R N)$
$(m,m'), n \longmapsto ((m\otimes n), (m'\otimes n))$

$(M\otimes_R N) \longrightarrow (M\oplus M')\otimes N$
$m\otimes n \longmapsto (m,0)\otimes n$ ...

3) $(\otimes_R N)$: $M \longmapsto M\otimes_R N$ is a functor

$$\begin{tikzcd} M \arrow[r, mapsto] \arrow[d, "f"] & M\otimes_R N \arrow[d, "\tilde f"] \\ M' \arrow[r, mapsto] & M'\otimes_R N \end{tikzcd}$$

$\tilde f: M\times N \longrightarrow M'\otimes N$, $(m,n) \longmapsto f(m)\otimes n$.

Corollary: $R^5\otimes_R R^3 = R^{15}$ | $R^a\otimes R^b = R^{ab}$.
$R^2\otimes_R R^2 = (R\oplus R)\otimes_R(R\oplus R) = (R\otimes_R(R\oplus R))\oplus(R\otimes_R(R\oplus R))$

4) $\mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z} \ni 1\otimes 2 = 2\cdot(1\otimes 1)$ $\quad$ $1\otimes 1$ = most interesting element!
$0,1 \quad 0,1,2$

$\boxed{\begin{matrix} 2\cdot(1\otimes 1) \\ 3\cdot(1\otimes 1)\end{matrix}}$ $= (2\cdot 1\otimes 1) = 0\otimes 1 = 0$,
$= (1\otimes 3\cdot 1) = 1\otimes 0 = 0$

$1\otimes 1 = 0 \Longrightarrow \mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z} = 0$.

5) $M\otimes_R R/I = M/I\cdot M$ ($I\subseteq R$ ideal)

Proof: $M\times R/I \longrightarrow M/IM$, $m, \bar r \longmapsto \overline{rm}$ | $M/IM \longrightarrow M\otimes R/I$, $\bar m \longmapsto m\otimes 1$

show: $m - m' \in IM$, $= \sum r_i m_i$ ($r_i\in I$)

$r_i m_i \longmapsto (r_i m_i)\otimes 1$
$\Uparrow$
$m_i\otimes r_i$
$\|$
0

$M \otimes_R N \cong N \otimes_R M$

$m \otimes n \longmapsto n \otimes m$

BUT: $M \otimes N \ni m \otimes n$

$n \otimes m$ does in general not make sense

- If $M = N$, i.e. $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \ni \sqrt{2} \otimes \pi \neq \pi \otimes \sqrt{2}$
- $\mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y] \ni x \otimes y \,,\, \not\ni y \otimes x$

$R^a \otimes R^b = R^{ab}$, $M \otimes R/I = M/IM$;

$\mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x]$

$\mathbb{R}[x] \times \mathbb{C} \longrightarrow \mathbb{C}[x]$ $\quad$ $\underline{\mathbb{R}\text{-bilinear}}$

$f(x) \otimes c \,\hat{=}\, (f(x), c) \longmapsto c \cdot f(x)$

$\sum_\nu x^\nu \otimes \lambda_\nu \longleftarrow \sum_\nu \lambda_\nu \cdot x^\nu \quad (\lambda_\nu \in \mathbb{C})$

$M, N \in L$ submodules

$(M : N) := \{ a \in R \mid a . N \subseteq M \} \subseteq R$ ideal

$\|(1) \iff N \subseteq M$

Special case: $(0 : N) = \{ a \in R \mid aN = 0 \} = \mathrm{Ann}_R(N)$

Exact sequences:

M_i = modules $\rightsquigarrow \quad \dots \to M_{i+1} \xrightarrow{d_i} M_i \xrightarrow{d_{i-1}} M_{i-1} \to \dots$ $\qquad d_i : M_{i+1} \to M_i$ (R-linear)

Def: $(M_\cdot, d_\cdot)$ is called a complex $:\iff \forall i: d_{i-1} \circ d_i = 0$ $\quad$ („$d^2 = 0$")

Ex: $(*)$ $\boxed{0 \to 4\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0}$ $\quad i = 2, 1, 0$

$4 \longmapsto 4 \longmapsto \bar{4} = 0$

Complex $\iff \forall i: \mathrm{Ker}(M_i \xrightarrow{d_{i-1}} M_{i-1}) =: Z_i(M_\cdot)$ „cycles"

$M_i \supseteq Z_i(M_\cdot) \supseteq B_i(M_\cdot)$

$\mathrm{Im}(M_{i+1} \xrightarrow{d_i} M_i) =: B_i(M_\cdot)$ „boundaries"

$Z_i(M_\cdot) / B_i(M_\cdot) =: H_i(M_\cdot) =$ „homology of $M_\cdot$ at place i"

$H_2(*) = 0$, $H_0(*) = \mathbb{Z}/2\mathbb{Z} \,/\, \mathbb{Z}/2\mathbb{Z} = 0$

$H_1(*) = 2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

Def: a complex is called exact $:\iff \forall i : H_i = 0$.

Remark $0 \to M \xrightarrow{f} N$
- complex $\iff$ always
- exact $\iff \mathrm{Ker} f = 0 \iff f$ = injective.

$M \xrightarrow{g} N \to 0 \implies$ homology at $N = N/\mathrm{Im}\, g =: \mathrm{coker}\, g$

exact $\iff g$ is surjective.

$0 \to M \xrightarrow{h} N \to 0$ exact $\iff h$ = iso.

$B/\mathrm{Ker}\, g \xrightarrow{\sim} C$

Def: „short exact sequence": $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

$\iff$
- exactness at $A \,\hat{=}\, f$ = injective
- exactness at $C \,\hat{=}\, g$ = surjective

„complex" $\hat{=} f(A) \subseteq \mathrm{Ker}\, g$

exactness at $B \,\hat{=}\, f(A) = \mathrm{Ker}\, g \,\hat{=}\, B/A \xrightarrow{\sim} C$

(strictly speaking: $B/f(A) \xrightarrow[g]{\sim} C$)

Remark: „split ex. sequence"
- $\boxed{0 \to A \to A \oplus C \to C \to 0}$ is exact
- not all exact sequ. $\boxed{0 \to A \to B \to C \to 0}$ are of this form, i.e. $\not\Rightarrow B \cong A \oplus C$

ex: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ni (0,1) \neq 0$, $2 \cdot (0,1) = 0$

$\overset{?}{\cong}$

$\mathbb{Z} \ni ??$

5-Lemma:

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & M_4 \arrow[r] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5
\end{tikzcd}$$

$\exists u \mapsto z \mapsto x \mapsto 0$; $\exists v \mapsto w \mapsto 0 \mapsto 0$; $f_4 : y \mapsto y = 0$

$\implies x = 0$

- If $M_\cdot, N_\cdot$ are exact,
- and f_1, f_2, f_4, f_5 = isom

$\implies f_3$ = isom !

Proof: diagram chasing:

$0 \to M^1 \xrightarrow{f} M^2 \xrightarrow{g} M^3$ ~~$\to 0$~~ complex $\rightsquigarrow$ Q: Is it exact?

φ, K, $f \circ \varphi$

Need criteria for this!

Lemma: $M^\cdot$ is exact $\iff \forall$ R-modules K: $\mathrm{Hom}(K, \cdot)$ is exact.

$0 \to \mathrm{Hom}(K, M_1) \to \mathrm{Hom}(K, M_2) \to \mathrm{Hom}(K, M_3)$ ~~$\to 0$~~

$\varphi \longmapsto f \circ \varphi$

Ex: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

$K := \mathbb{Z}/2\mathbb{Z}$ $\quad 0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0$

$\qquad\qquad = 0 \qquad\qquad = 0 \qquad\qquad \ni \mathrm{id}$