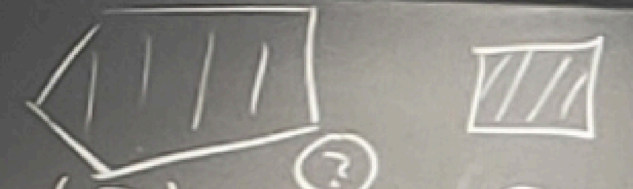
Task: Polygons in $\mathbb{R}^2$



A, B are (...), $\text{area}(A) = \text{area}(B) \xrightarrow{?} A \sim B$ ("scissor equiv.")

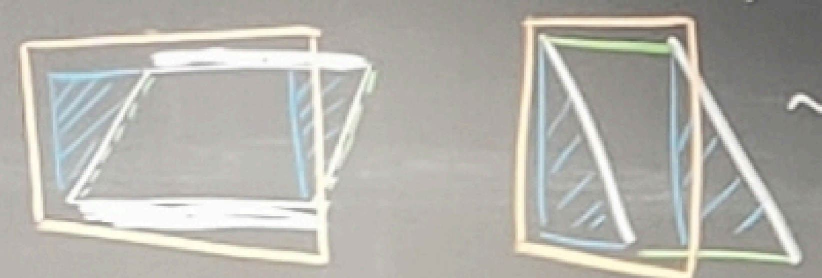
YES: (≙ Game Tangram) Proof: Cut A into: ① triangles

$A \rightsquigarrow$ triangles $T_1, \dots, T_k$

② reach rectangular triangles:

③ $\rightsquigarrow$ rectangles $R_1, \dots, R_\ell$

④



⑤ $\rightsquigarrow R_1', \dots, R_\ell'$: height $= 1$

$\rightsquigarrow R'$ has a pre chosen height (e.g. "1")

$R \rightsquigarrow R'$

$\rightsquigarrow R^* =$ one rectangle of height 1:

($\rightsquigarrow$ length (other edge) $= \text{area}(A) = \text{area}(B)$) □

Q: What about dim $= 3$? NO!! Need a new invariant (besides vol): "Dehn-Invariant"
(Hilbert 1900)

Def. S = solid (3-dim polytope), e.g.

$e \in$ edge $\rightsquigarrow \ell(e) =$ "length of e" $\in \mathbb{R}_{\geq 0} \subseteq \boxed{\mathbb{R} = \mathbb{Q}\text{-vs}}$

$a(e) =$ "angle of e": angle between the 2 faces containing $e \in \mathbb{R}/2\pi\mathbb{Z}$ (abelian group)

$D := \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/2\pi\cdot\mathbb{Q} = \mathbb{Q}\text{-vs.}$

$\sum_e \ell(e) \otimes a(e) =: d(S)$

Claim: $d(S)$ is invariant under scissor equiv., i.e.: $A \sim B \Rightarrow d(A) = d(B)$.

$\mathbb{Q}\text{-vs.} = \boxed{\mathbb{R}/2\pi\cdot\mathbb{Q}}$

Proof: What happens with 1 cut? Considered for one fixed edge e:

e cut $\Rightarrow e \rightsquigarrow e_1, e_2$
$\cdot\ \ell(e) = \ell(e_1) + \ell(e_2)$
$\ \ a(e) = a(e_1) = a(e_2)$

$\boxed{\ell(e) \otimes a(e)} = (\ell_1(e) + \ell_2(e)) \otimes a(e) = \boxed{\ell(e_1) \otimes a(e_1)} + \boxed{\ell(e_2) \otimes a(e_2)}$

2nd case: cut

$e \rightsquigarrow [e \text{ in solid } S_1] = e_1 \quad \cdot\ \ell(e) = \ell(e_1) = \ell(e_2)$

$[e \text{ in solid } S_2] = e_2 \quad \cdot\ a(e) = a(e_1) + a(e_2)$

$\Rightarrow \boxed{\ell(e) \otimes a(e) = \ell(e_1) \otimes a(e_1) + \ell(e_2) \otimes a(e_2)}$

adding up $\rightsquigarrow d(-)$ stays invariant. □

$F: \text{Mod}_R^{(\text{opp})} \longrightarrow \text{Mod}_S$ $(R, S$ rings$)$, $\text{Mod}_R =: \mathcal{C}$, $\text{Mod}_S =: \mathcal{D}$

Examples: $N = R$-module $\rightsquigarrow$

$\text{Mod}_R \xrightarrow{-\otimes_R N} \text{Mod}_R$, $M \longmapsto M \otimes_R N$

$\text{Mod}_R \xrightarrow{\text{Hom}(N, \cdot)} \text{Mod}_R$, $M \longmapsto \text{Hom}_R(N, M)$

$\text{Mod}_R^{\text{opp}} \xrightarrow{\text{Hom}(\cdot, N)} \text{Mod}_R$, $M \longmapsto \text{Hom}_R(M, N)$

$\forall F$: $A, B \in \text{Mod}_R \Rightarrow \text{Mor}(A,B) \xrightarrow{\Phi_{AB}} \text{Mor}(FA, FB)$, $\varphi \longmapsto F\varphi$

In our cases:
- are abelian groups
- Φ_{AB} = group homomorphism $\rightsquigarrow \Phi_{AB}: 0 \longmapsto 0$

Ex. $M \xrightarrow{0} M' \Rightarrow M \otimes N \xrightarrow{0} M' \otimes N$

$F: \mathcal{C} \longrightarrow \mathcal{D}$ additive functor

$M' \xrightarrow{d'} M \xrightarrow{d''} M''$ complex in $\mathcal{C}$ (i.e.: $d' \circ d'' = 0$)

$\Downarrow$

$FM' \xrightarrow{Fd'} FM \xrightarrow{Fd''} FM''$ complex in $\mathcal{D}$

0, too!

Recall: $H := \text{Ker}(d'')/\text{Im}\, d' = 0 \Longleftrightarrow$ complex is exact

F sends all exact sequences to exact sequences.

Def: F is exact $:\Longleftrightarrow [\forall\ M' \to M \to M''$ exact $\Rightarrow FM' \to FM \to FM''$ exact$]$

Counterex: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, $F := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \cdot)$

$F(-): 0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$ is not exact!

ok

homework

Remark: F is exact $\Longleftrightarrow \forall$ "s.e.s" $\boxed{0} \to M' \to M \to M'' \to \boxed{0}$: $0 \to FM' \to FM \to FM'' \to 0$ exact.

Ex. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \cdot)$ is not exact

Def. · $F: \mathcal{C} \to \mathcal{D}$ is left exact $\iff \forall\ 0 \to M' \to M \to M'' \cancel{\to 0}$ exact

the image $0 \to FM' \to FM \to FM''$ is exact

· F right exact $\iff$ $M' \to M \to M'' \to 0$ (exact) gives $FM' \to FM \to FM'' \to 0$ exact.

(left + right exact $\iff$ exact)

Lemma: $\boxed{\mathrm{Hom}_R(K, \cdot)}$ is left exact.

$f \circ \tilde{\varphi} = \varphi$

Proof: $0 \to M' \xrightarrow{f} M \to M'' \Rightarrow 0 \to \mathrm{Hom}(K,M') \to \mathrm{Hom}(K,M) \to \mathrm{Hom}(K,M'')$

$\tilde{\varphi}$? $\Longleftarrow$ $\varphi \longmapsto 0$

$\varphi \longmapsto 0$

$\forall x \in K: f(\varphi(x)) = 0 \Rightarrow \varphi(x) = 0 \Rightarrow \varphi = 0$ [f = inj]

$0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ $\Leftrightarrow$ $f(M') = \mathrm{Ker}\, g$ $\qquad \exists y \in M'$: $\varphi(x) = y$.

$\forall x \in K: g(\varphi(x)) = 0 \Rightarrow \varphi(x) \in \mathrm{Ker}\, g$, i.e.: $\varphi(x) \in f(M')$ □

Lemma': $0 \to M' \to M \to M''$ complex, if $\forall K$: $0 \to \mathrm{Hom}(K,M') \to \mathrm{Hom}(K,M) \to \mathrm{Hom}(K,M'')$ is exact

$\Rightarrow$ the original seq. was exact!

Proof: E.g.: Take $K = R$. Then $\mathrm{Hom}_R(R, M) = M$, $1 \mapsto ?$ □

Functor $\boxed{\mathrm{Hom}_R(\cdot, N)}$ Lemma: If $M' \to M \to M'' \to 0$ is exact in Mod_R, then, $\forall N$:

$M' \leftarrow M \leftarrow M'' \leftarrow 0$ in Mod^{opp}

$0 \longleftarrow \varphi$

$\mathrm{Hom}(M',N) \leftarrow \mathrm{Hom}(M,N) \leftarrow \mathrm{Hom}(M'',N) \leftarrow 0$ is exact.

Remark: $\boxed{\mathrm{Hom}_R(\cdot, N): \mathrm{Mod}_R^{opp} \to \mathrm{Mod}_R \quad \text{left exact}}$

—"— $: \mathrm{Mod}_R \to \mathrm{Mod}_R^{opp}$ right exact

Proof e.g.: exactness at $\mathrm{Hom}(M,N)$:

$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$, $M/f(M')$, Homomorph. satz.

Reverse:

Lemma': $M' \to M \to M'' \to 0$ complex, if $\forall N$: $0 \to \mathrm{Hom}(M'',N) \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N)$ is exact

$\Rightarrow$ the original is exact.

$\pi \longmapsto 0 \Rightarrow \pi = 0$

Proof · $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ $\qquad$ 1st claim: g = surjective.

$M''/\mathrm{Im}\, g =: N$

$\pi = 0 \Rightarrow M''/\mathrm{Im}\, g = 0 \Rightarrow g$ = surjective.

· $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ $\qquad$ 2nd claim: $\mathrm{Im}\, f \overset{\supseteq ?}{=} \mathrm{Ker}\, g$

$M/\mathrm{Im}\, f =: N$

$0 \to \mathrm{Hom}(M'',N) \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N)$

$\exists \varphi \longmapsto \pi \longmapsto 0$

$\Updownarrow$

$\pi = \varphi \circ g$ $\qquad$ $\mathrm{Ker}\, g \subseteq \mathrm{Ker}\, \pi = \mathrm{Im}\, f$. □

Prop: $\otimes_R N$ is right exact.

Example: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ $\quad | \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ $\qquad \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z}) / \underbrace{2 \cdot (\mathbb{Z}/2\mathbb{Z})}_{= 0}$

$x \mapsto x$

$\Downarrow$

$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow[0]{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{id} \mathbb{Z}/2\mathbb{Z} \to 0$ $\qquad M \otimes R/I = M/IM$

$\uparrow$ not exact (not inj!)

Proof: 1st try: $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ exact $\overset{?}{\Longrightarrow}$ $M' \otimes N \xrightarrow{F} M \otimes N \xrightarrow{G} M'' \otimes N \to 0$ exact

$m \otimes n \longmapsto m'' \otimes n \Rightarrow$ ok

$\Downarrow g$

$\exists\, m \mapsto m''$

2nd try: assumption

$\mathrm{Hom}(M' \otimes N, P) \leftarrow \mathrm{Hom}(M \otimes N, P) \leftarrow \mathrm{Hom}(M'' \otimes N, P) \leftarrow 0$

Claim (~~$\forall N, \forall P$~~) $\forall W$ $\qquad \mathrm{Hom}_{\mathbb{Z}}(M, N; P)$ $\qquad W$

$\mathrm{Hom}(M', W) \leftarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \leftarrow \mathrm{Hom}(M'', W) \leftarrow 0$

$\sum_i m_i \otimes n_i \longmapsto 0$

$\sum_i g(m_i) \otimes n_i = 0$

R = ring, $S \subseteq R$ subset (of later denominators), supposed to be multiplicatively closed, i.e. $S \cdot S \subseteq S$. Conv: $1 \in S$.

$\rightsquigarrow$ define new ring $S^{-1}R$

· also: $S^{-1}M$ (for all R-modules M) $\rightsquigarrow$ becomes a $S^{-1}R$-module.

$S^{-1}R := \{(a,b) \mid a \in R,\ b \in S\}/\sim$ equ. relation: $(a,b) \sim (x,y) :\Longleftrightarrow \exists s \in S: s\cdot(ay - bx) = 0$ in R

not transitive!

$\overline{(a,b)} =: \frac{a}{b}$

$\rightsquigarrow$ · $\exists$ canonical map: $\pi: R \longrightarrow S^{-1}R$ ring hom. with $r \longmapsto \frac{r}{1}$

· operations: $\frac{a}{b} \cdot \frac{x}{y} := \frac{ax}{by}$; $\frac{a}{b} + \frac{x}{y} = \frac{ay}{by} + \frac{bx}{by} = \frac{ay+bx}{by}$.

BUT: $\pi: R \longrightarrow S^{-1}R$ does not need to be injective!

$\operatorname{Ker} \pi = \{r \in R \mid \frac{r}{1} = 0 = \frac{0}{1}$, i.e.: $\exists s \in S: s \cdot r = 0\}$ extreme: $S := \{0,1\} \rightarrow S^{-1}R = 0$

„localization“

Examples: ① $S = \{$non-zero-divisors in $R\}$ $\Longrightarrow$ $S^{-1}R$ = „total quotient ring“, $R \overset{\iota}{\hookrightarrow} S^{-1}R$

①' R = domain $\rightsquigarrow S = R \setminus \{0\} \Longrightarrow$ $(-''-) =: \operatorname{Quot} R$

② $f \in R \rightsquigarrow S = \{f^k \mid k \in \mathbb{N}\} =: f^{\mathbb{N}}$ (e.g. $\mathbb{C}[x, \frac{1}{x}] = S^{-1}\mathbb{C}[x]$ with $S = \{x^n \mid n \in \mathbb{N}\}$)

$\rightsquigarrow S^{-1}R =: R_f$, $\pi: R \to R_f$

③ $P \in \operatorname{Spec} R$, i.e. $R \setminus P$ = mult. cl. $\rightsquigarrow S := R \setminus P \rightsquigarrow (R\setminus P)^{-1} \cdot R =: R_P$

Careful: $\mathbb{C}[x] \rightsquigarrow \mathbb{C}[x]_x$, $\mathbb{C}[x]_{(x)}$

$\mathbb{C}[x]_x = \{\frac{f(x)}{x^k} \mid k \in \mathbb{N}\}$, $\mathbb{C}[x]_{(x)} = \{\frac{f(x)}{g(x)} \mid g(0) \neq 0\}$

$S^{-1}M = \{\frac{m}{s} \mid m \in M, s \in S\}$

$S^{-1}R$-module: $\frac{r}{s} \cdot \frac{m}{s'} = \frac{(rm)}{ss'}$

Next Compare: $I \subseteq R$ ideal: R/I, M/IM | $S \subseteq R$ m.c.: $S^{-1}R$, $S^{-1}M$

① universal property

$R \longrightarrow R/I$ is a universal ring hom. for $R \xrightarrow{\varphi} T$ (ring) with $\varphi(I) = 0$.

$R \longrightarrow S^{-1}R$ $-''-$ for $R \xrightarrow{\varphi} T$ with $\varphi(S) \subseteq T^*$

($s \in R \Longrightarrow$ becomes invertible in $S^{-1}R$: $\frac{1}{s}$ = inverse!)

② modules: R/I-module $\Longleftrightarrow$ R-module M with $I \cdot M = 0$

$S^{-1}R$-module $\Longleftrightarrow$ R-module M with: $R \longrightarrow \operatorname{End}_R(M) := \operatorname{Hom}_R(M,M)$, $r \longmapsto \begin{bmatrix} m \longmapsto rm \\ M \longrightarrow M \end{bmatrix}$

$$\begin{array}{ccc} R & \longrightarrow & \operatorname{End}_R(M) \\ \cup\!| & & \cup\!| \\ S & \longrightarrow & \text{Automorphisms} \end{array}$$

$$\begin{array}{l} R \longrightarrow R/I \\ R \longrightarrow S^{-1}R \end{array}$$

③ Relation to $\otimes$:

$M/IM = \boxed{M \otimes_R R/I}$

$S^{-1}M = \boxed{M \otimes_R S^{-1}R}$ $\quad R \xrightarrow{\pi} S^{-1}R$

$S^{-1}M \longleftarrow M \times S^{-1}R$

$\frac{rm}{s} \longleftarrow\!\shortmid (m, \frac{r}{s})$

$\frac{m}{s} \longmapsto m \otimes \frac{1}{s}$ map $S^{-1}R$-modules

(general: $R \longrightarrow T$ ring hom. i.e. $T = R$-algebra. $M = R$-module $\Longrightarrow M \otimes_R T$ = T-module, $(m \otimes t) \cdot t' := m \otimes (tt')$

③' · $\otimes R/I$ is right exact, in general not left exact (e.g. $\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$)

· $M \longmapsto S^{-1}M$ (R, S = fixed), $(R\text{-mod}) \longrightarrow (S^{-1}R\text{-mod})$ is exact!

Proof: · right exactness: ✓ (is a $\otimes$)

· remains: $M \overset{\iota}{\hookrightarrow} M'$ injective $\overset{?}{\Longrightarrow}$ $S^{-1}M \xrightarrow{f} S^{-1}M'$ stays injective

$\frac{m}{s} \in S^{-1}M$, assume $\frac{0}{1} = f(\frac{m}{s}) = \frac{\iota(m)}{s} \Longleftrightarrow \exists t \in S: t \cdot \iota(m) = 0$

$t \cdot \iota(m) = \iota(tm) = 0 \Longrightarrow tm = 0$, $t \in S \Longrightarrow \frac{m}{1} = 0$ //

i.e. $\otimes_R S^{-1}R$ is exact.

Def An R-module F is called flat $:\Longleftrightarrow$ $\otimes_R F$ is exact.

Ex: · $R \longrightarrow S^{-1}R$ is flat ($R \xrightarrow{\varphi} T$ is an algebra; φ is called flat $:\Longleftrightarrow$ T as an R-module (via φ) is flat $\Longleftrightarrow$ $\otimes_R T$ is exact)

· $\mathbb{Z}/2\mathbb{Z}$ is not flat over $\mathbb{Z}$

· $I \in \mathbb{N} \Longrightarrow R^I = R^{\oplus I}$ = flat R-module: $M \otimes_R R^I = M \otimes_R (R \oplus R \oplus \ldots \oplus R) = (M \otimes_R R) \oplus \ldots \oplus (M \otimes_R R) = M^I = M \oplus \ldots \oplus M$

$M \longmapsto M^I$ exact