$I \leq R \rightsquigarrow R/I,\ M/IM$

$S \subseteq R$ m.c. $\rightsquigarrow S^{-1}R,\ S^{-1}M$

$\pi: R \longrightarrow S^{-1}R$

$\overset{\cup\!\!|}{I} \rightsquigarrow S^{-1}I = \pi(I)\cdot S^{-1}R =: I\cdot S^{-1}R$

$\mathfrak{J} \leq S^{-1}R$ ideal

$\rightsquigarrow \pi^{-1}\mathfrak{J} \leq R$ (-"-)

$\quad = \mathfrak{J}\cap R$

Fact: $\cdot\ \mathfrak{J} \leq S^{-1}R \rightsquigarrow S^{-1}(\mathfrak{J}\cap R) = \mathfrak{J}$

$\cdot\ I \leq R \rightsquigarrow \underbrace{S^{-1}I \cap R}_{(*)} \supseteq I$ $\left(a\in I \Rightarrow \frac{a}{1}\in S^{-1}I,\ \pi(a) = \frac{a}{1}\in S^{-1}R \Rightarrow a \in (*)\right)$

"=" is in several wrong.

essential property:

$\boxed{I\cap S = \emptyset} \iff S^{-1}I \neq (1) = S^{-1}R.$

$I \cap S \neq \emptyset \iff S^{-1}I = (1)$

Proof: $(\Rightarrow)$ Let $s \in I \cap S$

$\Rightarrow \frac{s}{1}\in S^{-1}I$ and $\frac{s}{1}$ = unit, inverse $= \frac{1}{s}$

$\Rightarrow S^{-1}I = (1).$

$(\Leftarrow)$ $1 \in S^{-1}I$, i.e. $\exists a \in I: \frac{1}{1} = \frac{a}{1}$

$\Rightarrow \exists t \in S: t(a-1) = 0 \Rightarrow \underset{I}{\overset{\cap}{ta}} = \underset{S}{\overset{\cap}{t}}.$

$\mathbb{Z}/2\mathbb{Z} \neq \mathbb{Z}_2$

But: $I\cap S = \emptyset$ does not suffice for "=" in $(*)$

Example: $R = \mathbb{Z},\ I = (6),\ S = \{2^k \mid k \geq 0\}$ $\quad \underset{\not\,\!\!\!\!\cup}{I}$

$\mathbb{Z} \xrightarrow{\pi} S^{-1}\mathbb{Z} = \mathbb{Z}_2 = \mathbb{Z}[\tfrac{1}{2}],\ S^{-1}I \ni \frac{3}{1} \Rightarrow 3 \in S^{-1}I\cap\mathbb{Z}$

Lemma $P \leq R$ PI, assume that $P\cap S = \emptyset$. If $\frac{a}{s}\in S^{-1}P$ $(a\in R,\ s\in S) \Rightarrow a \in P.$

Proof: $\frac{a}{s}\in S^{-1}P \Rightarrow \exists p \in P, t \in S: \frac{a}{s} = \frac{p}{t}$ in $S^{-1}R \Rightarrow \exists r \in S: r(at-ps) = 0$

$r\in S \rightsquigarrow r \notin P \Rightarrow at - \underset{P}{\overset{\cap}{ps}} \in P \Rightarrow \left.\begin{array}{l} at \in P \\ t\in S \Rightarrow \notin P\end{array}\right\} \Rightarrow a \in P.$ □

Corollary $P = PI$, disj. to $S \Rightarrow \boxed{S^{-1}P \cap R = P}$ **Proof:** $a\in R;\ \frac{a}{1}\in S^{-1}P \Rightarrow a\in P.$ □

Remark: (-"-) $\Rightarrow S^{-1}P \leq S^{-1}R$ is a prime ideal.

Proof + further remark: $S^{-1}(R/I) = R/I \otimes_R S^{-1}R = S^{-1}R/I\cdot S^{-1}R = S^{-1}R/S^{-1}I$ □

$S^{-1}R/S^{-1}P = S^{-1}(R/P) =$ domain.

Consequence: $S \subseteq R \rightsquigarrow \{$prime ideals in R, disj. to $S\} \longleftrightarrow \{$prime ideals in $S^{-1}R\}$

$P \longmapsto S^{-1}P$

$Q\cap R \longleftarrow Q$

$\operatorname{Spec} S^{-1}R = \{P \in \operatorname{Spec} R \mid P\cap S = \emptyset\}$

Special case: $S = f^{\mathbb{N}}$

Recall $I \leq R$ ideal

$\{PI \subset R \text{ over } I\} \longleftrightarrow \{PI \text{ in } R/I\}$

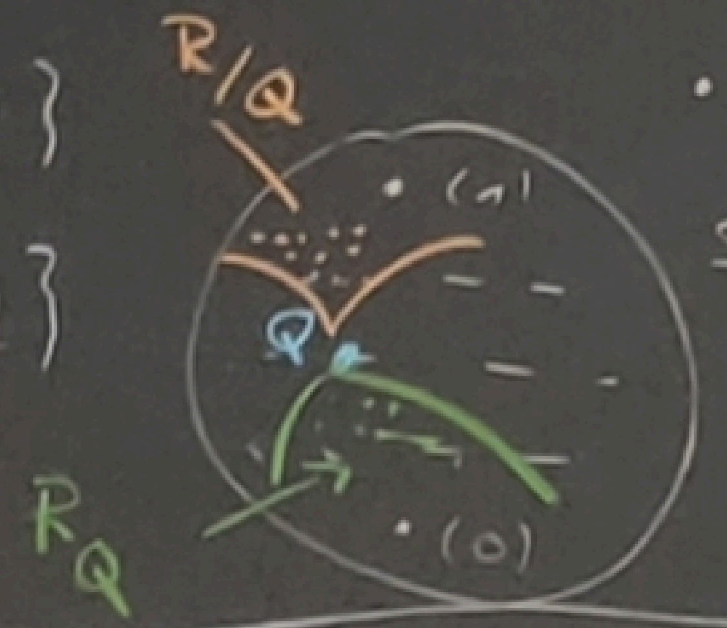
$P \longmapsto P/I$

$\operatorname{Spec} R/I = \{P\in\operatorname{Spec} R \mid P \supseteq I\} = V(I)$ $\quad \operatorname{Spec} R_f = D(f)$ $\left(= \operatorname{Spec} R \setminus V(f)\right)$

Other interpretation: $Q \subseteq R$ prime ideal $\rightsquigarrow$

$\operatorname{Spec} R/Q = \{P \in \operatorname{Spec} R \mid P \supseteq Q\}$

$\operatorname{Spec} R_Q = \{P \in \operatorname{Spec} R \mid P \subseteq Q\}$



$\cdot\ R \longrightarrow R/Q$

$\cdot\ R \longrightarrow R_Q = (\overset{S}{\overbrace{R\setminus Q}})^{-1}\cdot R$

$\operatorname{Spec} R \cup \{(1),(0)\}$

$\cdot\ R/Q \supset Q/Q = (0)$ smallest PI $\Rightarrow$ domain

local ring. $\Leftarrow R_Q \supset Q_Q = S^{-1}Q$ largest PI

$\boxed{M \mapsto S^{-1}M \text{ exact !!!}}$

If $m \neq 0 \Rightarrow \operatorname{Ann}(m) \subsetneq R \Rightarrow \exists \mathfrak{m} \in \operatorname{Max}\operatorname{Spec}: \operatorname{Ann}(m) \subseteq \mathfrak{m}$, i.e. $\operatorname{Ann}(m)\cap S = \emptyset$ $(S := R\setminus\mathfrak{m})$

$\Rightarrow \boxed{S^{-1}\operatorname{Ann}_R(m) \neq (1)}$ is $S^{-1}R = R_{\mathfrak{m}}$

$(*)\ \overset{\|}{\operatorname{Ann}_{R_{\mathfrak{m}}}\left(\frac{m}{1}\right)} \Rightarrow \frac{m}{1} \neq 0$ in $M_{\mathfrak{m}}$.

$\left(\frac{m}{s} = \frac{0}{1} \iff \exists \boxed{\underset{S}{\overset{\cap}{t}}(m} 1 - 0\cdot s) = 0\right)$

Proof of $(*)$:

$\cdot$ Let $\frac{a}{s}\in S^{-1}\operatorname{Ann}$, i.e. $a\in\operatorname{Ann}(m) \Rightarrow a\cdot m = 0$ in M

$\Rightarrow \frac{a}{s}\cdot\frac{m}{1} = \frac{am}{s} = 0$ in $M_{\mathfrak{m}}$

$\cdot$ Let $\underset{\frac{am}{s}}{\underbrace{\frac{a}{s}\cdot\frac{m}{1}}} = 0$ in $M_{\mathfrak{m}} \iff \exists t \in S: t\cdot(am) = 0$ in M

$\Rightarrow ta \in \operatorname{Ann}(m)$

$\Rightarrow \frac{a}{s} = \frac{ta}{ts} \in S^{-1}\operatorname{Ann}(m).$ □

Local tests **Lemma:** Let $f: M \longrightarrow N$ be an R-linear map.

f is 0 / injective / surjective / isomorphism $\iff$ if $f_{\mathfrak{m}}$ is so $\forall \mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R$

$0 \to M \to N$ exact $(\iff f_P$ is so $\forall P \in \operatorname{Spec} R)$

Recall $f_{\mathfrak{m}}: M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}}$ ($R_{\mathfrak{m}}$-mod)

$(s\in S = R\setminus\mathfrak{m})\ \frac{m}{s} \longmapsto \frac{f(m)}{s}$

Proof: $m\in M$. **Claim:** $m = 0 \iff \frac{m}{1} = 0$ inside $M_{\mathfrak{m}}$ $\forall \mathfrak{m}\in\operatorname{Max}\operatorname{Spec} R$

Proof of claim: $(\Leftarrow)$ $\cdot \operatorname{Ann}_R(m) := \{a\in R \mid a\cdot m = 0\} \subseteq R$ ideal

$\cdot\ m = 0 \iff 1 \in \operatorname{Ann}_R(m) \iff \operatorname{Ann}(m) = R.$

Proof of Lemma: $f: M \longrightarrow N$ $\quad 0 \to M \xrightarrow{f} N$ $\quad 0 \to M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}}$

$\cdot\ (\Rightarrow)$ is always ok: $[f_{\mathfrak{m}}: M_{\mathfrak{m}} \to N_{\mathfrak{m}}] = [f: M \to N]\otimes_R R_{\mathfrak{m}}$; $R_{\mathfrak{m}}$ = flat over R

$\cdot\ f_{\mathfrak{m}}: M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}}$ is 0 $\forall\mathfrak{m}$; **Claim:** $f: M \longrightarrow N$ is 0

$m \longmapsto f(m)$

Show: $f(m) = 0$ in N

Know: $f\left(\frac{m}{1}\right) = \frac{f(m)}{1} = 0$ in $N_{\mathfrak{m}}$ //

$\cdot\ f_{\mathfrak{m}}$ = inj $\forall\mathfrak{m}$. Let: $0 \to K \to M \xrightarrow{f} N$ (exact)

$(f = \text{inj} \iff K = 0)$

$\Downarrow$

$K_{\mathfrak{m}} = 0 \Leftarrow 0 \to K_{\mathfrak{m}} \to M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}}$ exact $\forall\mathfrak{m}$ □

$S = R \setminus \mathfrak{m}$
↑ max ideal, not the element

$[f: M \longrightarrow N] \otimes_R \cancel{R_\mathfrak{m}}$
$\otimes\, \mathrm{id}_{R_\mathfrak{m}}$

$S \subseteq R$

$S^{-1}R \otimes_R M \xrightarrow{\sim} S^{-1}M$ isom of $S^{-1}R$-mod

$\left(\frac{a}{s}, m\right) \longmapsto \frac{am}{s}$ ‖ $\frac{1}{s} \otimes m \longleftarrow \frac{m}{s}$

Corollary: (*) $L \xrightarrow{f} M \xrightarrow{g} N$ R-linear.

(*) is exact $\Longleftrightarrow$ $\forall \mathfrak{m} \in \mathrm{Max\,Spec}\, R$: (*) $\otimes\, \mathrm{id}_{R_\mathfrak{m}}$ is exact over $R_\mathfrak{m}$.

$$\| \quad \left[L_\mathfrak{m} \xrightarrow{f_\mathfrak{m}} M_\mathfrak{m} \xrightarrow{g_\mathfrak{m}} N_\mathfrak{m}\right]$$

Proof. (⟹) ≙ exactness of $\otimes_R R_\mathfrak{m}$

(⟸) Start with

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[rr, bend right, "0 ? \underline{\underline{ok}}"'] & M \arrow[r, "g"] & N \end{tikzcd}$$

, show: $\operatorname{Ker} g = \operatorname{Im} f$ (know: „⊇")

- $L \xrightarrow{f} M \longrightarrow M/\operatorname{Im} f \longrightarrow 0$, $M/\operatorname{Im} f = C$
- $0 \longrightarrow K \longrightarrow M \xrightarrow{g} N$

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, equal] & M \arrow[r] & C \arrow[d, equal] \\ \operatorname{Ker} g \arrow[rr, "\Phi"] & & M/\operatorname{Im} f \end{tikzcd}$$

(not a complex)

$(\operatorname{Ker} g = \operatorname{Im} f) \Longleftrightarrow \Phi = 0$ □

Corollary: M = R-module. Then M is R-flat $\Longleftrightarrow$ $\forall \mathfrak{m} \in \mathrm{Max\,Spec}$: $M_\mathfrak{m}$ is $R_\mathfrak{m}$-flat („flat as $R_\mathfrak{m}$-module")

Proof. M = R-flat $\Longleftrightarrow$ $\forall$ R-modules $\boxed{A \overset{\iota}{\hookrightarrow} B \text{ (injective)}}$: $A \otimes M \hookrightarrow B \otimes M$ remains injective

$$(A \otimes_R M) \otimes_R R_\mathfrak{m} = A \otimes_R (M \otimes_R R_\mathfrak{m}) = A \otimes_R M_\mathfrak{m} = A \otimes_R R_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m} = A_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m}$$

$= A_\mathfrak{m} \otimes_R M$

(⟸) $A \otimes M \hookrightarrow B \otimes M$ inj ? $\overset{\forall \mathfrak{m}}{\rightsquigarrow}$ check: $(A \otimes M) \otimes R_\mathfrak{m} \xhookrightarrow{\text{inj} ?} (B \otimes_R M) \otimes_R R_\mathfrak{m}$

$\| \quad A_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m} \longrightarrow B_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m}$ (inj ?)

(not used so far $\left[A_\mathfrak{m} \otimes_R M \longrightarrow B_\mathfrak{m} \otimes_R M\right]$)

(⟹) Let $X \hookrightarrow Y$ be $R_\mathfrak{m}$-modules ; $X \otimes_{R_\mathfrak{m}} M_\mathfrak{m} \xhookrightarrow{\text{inj} ?} Y \otimes_{R_\mathfrak{m}} M_\mathfrak{m}$.

$X \otimes_{R_\mathfrak{m}} (R_\mathfrak{m} \otimes_R M)$ $\quad R \longrightarrow R_\mathfrak{m}$

$= (X \otimes_{R_\mathfrak{m}} R_\mathfrak{m}) \otimes_R M = X \otimes_R M$

$X \otimes_{R_\mathfrak{m}} M_\mathfrak{m} = X \otimes_{R_\mathfrak{m}} (R_\mathfrak{m} \otimes_R M) = X \otimes_R M$: $X \otimes_R M \longrightarrow Y \otimes M$ inj. because M = R-flat! □

Nakayama-Lemma M = R-module ; M = finitely generated (as an R-module)
$(\Longleftrightarrow \exists k \in \mathbb{N} : R^k \twoheadrightarrow M$ (R-linear, surjective))

Cayley-Hamilton. $I \subseteq R$ ideal. Assume $\varphi : M \longrightarrow IM$ is an R-linear map. ($IM \subseteq M$)

$\Longrightarrow \exists\, p(x) \in R[x]$, namely: $p(x) = \sum_{j=0}^{n} a_j \cdot x^{n-j} = \cancel{a_0}\boxed{x^n + a_1 x^{n-1} + \dots + a_n}$ with: • $a_j \in I^j \subseteq R$ • $a_0 = 1$

such that $p(\varphi) := \sum_{j=0}^{n} a_j \cdot \varphi^{n-j}$ (with $\varphi^k := \underbrace{\varphi \circ \varphi \circ \dots \circ \varphi}_{k\text{-times}}$) satisfies $p(\varphi) = 0$ in $\mathrm{End}_R(M)$!

Proof: Reminder: R = ring ⇝ • consider $\mathrm{Mat}(a,b,R)$ = matrices $(a \times b)$ with entries in R.

- $\mathrm{Mat}(a,a,R) \ni A \rightsquigarrow \det A \in R$
- $A \rightsquigarrow \operatorname{adj}(A) := \left[(i,j)\text{-th entry} = (-1)^{i+j} \cdot \det A_{ij}\right]^T$, A_{ij} = A after erasing i-th row j-th col.

$A \cdot \operatorname{adj}(A) = \boxed{\operatorname{adj}(A) \cdot A = \det A \cdot I_a}$ $\quad (\Longrightarrow [A \text{ invertible} \Longleftrightarrow \det A \in R^*])$

Proof. • R = domain ⇝ $R \hookrightarrow K$
- $T = \mathbb{Z}[x_{ij} \mid 1 \le i,j \le a]$ domain • $X := (x_{ij}) \in \mathrm{Mat}(a,a,T)$ $\xRightarrow{\text{ok for } X \text{ (instead } A)}$
- $T \longrightarrow R$
  $x_{ij} \longmapsto a_{ij}$, i.e. $X \longmapsto A$ //

Let $m_1, \dots, m_k \in M$ be generators

$\Longrightarrow \varphi(m_i) \in IM \Longrightarrow \exists\, a_{ij} \in \cancel{R}\, I$: $\boxed{\varphi(m_i) = \sum_{j=1}^{n} a_{ij}\, m_j}$ (*)

Introduce an $R[x]$-module structure on M — by explaining what $(x\cdot)$ does ! $\left(\begin{array}{l}(x\cdot) \in \mathrm{End}\, M \\ (x\cdot) : M \longrightarrow M\end{array}\right)$

(⇝ reformulation of the claim: • $p(x) \in R[x]$ • $M \xrightarrow{\cdot p(x)} M$ is zero) namely: $\boxed{x \cdot m := \varphi(m)}$

(*) $x \cdot m_i = \sum_{j=1}^{k} a_{ij} \cdot m_j \rightsquigarrow \sum_{j=1}^{k} a_{ij} m_j - x m_i = 0$, i.e. $\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \boxed{a_{i1}} & & \boxed{a_{ik}} \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} - \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} = 0$

$B := \begin{pmatrix} a_{11} - x & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} - x & \cdots & a_{2k} \\ \vdots & & & \\ a_{k1} & \cdots & & a_{kk} - x \end{pmatrix}$ square matrix over $R[x]$

$\rightsquigarrow B \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} = 0 \longrightarrow \underbrace{(\operatorname{adj} B) \cdot B}_{} \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} = 0$

$\det B \cdot I_k \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} = 0$ $\Longrightarrow \forall i : \boxed{\det B \cdot m_i} = 0$, $\det B = p(x) \in R[x]$

$\Downarrow$ $p(x) \cdot M = 0$

□

C-H: $\varphi: M \longrightarrow IM$, $\boxed{M = f.g.}$ $\Rightarrow \exists p(x)$: highest coeff $= 1$, $a_i \in I^i$ $\Big\}$: $p(\varphi) = 0$ in $End(M)$
i.e. $p(\varphi)(M) = 0$

3 Consequences: ① $I \subseteq R$ ideal, $IM = M \Rightarrow \exists \boxed{p \in 1 + I} \subseteq R : p \cdot M = 0$

Proof: $\varphi: M \longrightarrow \underset{M}{\overset{\|}{IM}}$, $\varphi(m) := m$; i.e. $\varphi = id_M$

C-H $\Rightarrow \exists p(x)$: $p(id)(M) = 0$

$$\underset{a_0 = 1}{(x^k + a_1 x^{k-1} + \dots + a_k)}(id) = \sum_{i=0}^{k} a_i (id)^{k-i} = \sum a_i \cdot 1^{k-i} = \sum_{i=0}^{k} a_i = 1 + \overbrace{\sum_{i=1}^{k} a_i}^{\in I} \;\; \| \; p$$

①' Let $I \subseteq Jac(R) = \bigcap_{\mathfrak{m} \in MaxSpec R} \mathfrak{m} = \{a \in R \mid 1 + aR \subseteq R^*\} \Rightarrow 1 + Jac(R) \subseteq R^*$

$1 + I \subseteq R^*$

$\Rightarrow p \in R^* \Rightarrow M = 0!$

That is: $I \subseteq Jac R$, $M = IM \Rightarrow M = 0$

①'' $(R, \mathfrak{m})$ = local ring $\Rightarrow Jac(R) = \mathfrak{m} \Rightarrow [M = \mathfrak{m} \cdot M$ implies $M = 0]$
IF M was f.g. !!

② Let $f: M \longrightarrow M$ be a surjective, R-linear map $\Rightarrow f =$ isomorphism!

Proof: $M = R[x]$-module via f, i.e. $x \cdot m := f(m)$.

$f =$ surjective $\Rightarrow M = \underset{R[x]}{\overset{\cap}{(x)}} \cdot M \xRightarrow{①} \exists p \in 1 + (x) \subseteq R[x]$: $p \cdot M = 0$ ($x g(x) = g(x) x$ in $R[x]$)

$p(x) = 1 + x \cdot g(x) \Rightarrow \forall m \in M$: $(1 + x\, g(x))(m) = 0$

$\Downarrow$

$m + f(g(f)(m)) = 0$
$m + g(f)(f(m)) = 0$

$\Rightarrow \forall m \in M$: $m = f \circ (-g(f))(m)$
$m = (-g(f)) \circ f\,(m)$

i.e. $f \circ (-g(f)) = id$, $(-g(f)) \circ f = id$ $\Big\}$ $-g(f) = f^{-1}$!!

□

③ Lemma (of Nakayama): • Let $(R, \mathfrak{m})$ = local ring.
• Let M = f.g. R-module.

If $m_1, \dots, m_k \in M$ such that $\bar{m}_1, \dots, \bar{m}_k \in M/\mathfrak{m} \cdot M$ generate $\boxed{M/\mathfrak{m}M}$ as an $\boxed{R/\mathfrak{m}}$-module (v.s.) then $m_1, \dots, m_k$ generate M as an R-module.

field $= k$ $\|$ $R/\mathfrak{m}$; $M/\mathfrak{m}M$ $\|$ fin. dim k-vs !!

(Remark: If $m_1, \dots, m_n$ generate M, i.e. $R^k \xrightarrow{m} M$ $\Big/ \otimes_R R/\mathfrak{m}$

$(R/\mathfrak{m})^k \xrightarrow{\bar{m}} M/\mathfrak{m}M$)

Example: • $\{2, 3\}$ generate $\mathbb{Z} = M$ as $\mathbb{Z}$-module.
w: $\cancel{2}$, $\cancel{3}$

•• $\exists$ generators $m_1, \dots m_k$ of $M = (R, \mathfrak{m})$-module; $\bar{m}_1, \dots, \bar{m}_k$ = gens of $\Rightarrow \dim_k M/\mathfrak{m}M \le k$ $\Downarrow$ $\exists i$: $\{\bar{m}_0', \dots, \widehat{\bar{m}_i'}, \dots, \bar{m}_k'\}$

• $\exists$ other generators $m_0', m_1', \dots, m_k'$ of $M \Rightarrow M \longrightarrow M/\mathfrak{m}M$, $m_i' \longmapsto \bar{m}_i'$ = generators of

$\{\bar{m}_0', \dots, \bar{m}_k'\} \setminus \bar{m}_i'$ = still generating $M/\mathfrak{m}M$

$\Downarrow$

$\{m_0', \dots, m_k'\} \setminus \{m_i'\}$ = gen. syst. of M.

Ex: $R := \mathbb{Z}_{(5)} = \{\frac{a}{s} \mid a, s \in \mathbb{Z}, 5 \nmid s\}$ = local ring, max ideal is (5).
$\langle 2, 3 \rangle = \mathbb{Z}_{(5)}$ as a module over $\mathbb{Z}_{(5)}$, — how: $2, 3 \in \mathbb{Z}_{(5)}^* \Rightarrow \langle 2, 3 \rangle = \langle 2 \rangle = \langle 3 \rangle$

Proof: $N := \langle m_1, \dots, m_k \rangle \subseteq M$; goal: $N = M$. Know: $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$ (surj) $\Leftrightarrow N + \mathfrak{m}M = M$

$\Updownarrow$

$\boxed{\mathfrak{m} \cdot M/N = M/N} \Longleftrightarrow \mathfrak{m}M \hookrightarrow M \twoheadrightarrow \boxed{M/N}$ (surjective)

$Q = M/N$: $\mathfrak{m}Q = Q$
$\Downarrow$
$Q = 0$

i.e. $M/N = 0$, i.e. $M = N$.