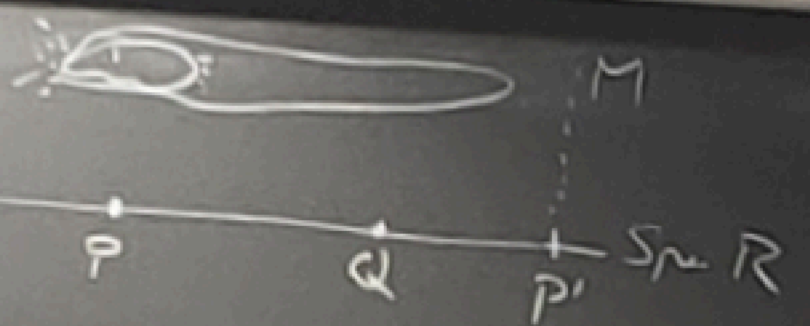
R = ring, M = finitely gen. R-module.



Def $\operatorname{Supp} M = \{P \in \operatorname{Spec} R \mid M_P \neq 0\}$

Remark: ① $0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow \operatorname{Supp} M = (\operatorname{Supp} M') \cup (\operatorname{Supp} M'')$

(proof by tensoring with $\otimes_R R_P$)

② $\underset{\substack{\| \\ \text{ideal in } S^{-1}R}}{(S^{-1}N : S^{-1}M)} = S^{-1}(N : \underline{\underline{M}})$ ← has to be $\underline{\underline{\text{f.g.}}}$!!

$\underset{\substack{\| \\ \text{ideal}}}{(N:M)} := \{a \in R \mid a \cdot M \subseteq N\}$

Proof. $\frac{a}{s} \in S^{-1}(N:M) \iff \exists\, t \in S : (t \cdot a) \cdot M \subseteq N$

$\frac{a}{s} \in (S^{-1}N : S^{-1}M) \iff \forall m \in M\ \exists\, \underset{\substack{\| \\ t(m)}}{t} \in S : t \cdot a \cdot m \in N$

Special case. $N = 0$: $(0:M) = \operatorname{Ann}(M) = \{a \in R \mid a \cdot M = 0\} \Longrightarrow \operatorname{Ann}_{S^{-1}R}(S^{-1}M) = S^{-1}\operatorname{Ann}_R(M)$

Corollary. $\operatorname{Supp}_{(M)} = V(\operatorname{Ann}(M)) = \{P \in \operatorname{Spec} R \mid P \supseteq \operatorname{Ann}(M)\}$.

Proof. **Lemma.** $I \subseteq R$ ideal, $S \subseteq R$ mult. cl. $\rightsquigarrow [S^{-1}I = (1) \iff I \cap S \neq \emptyset]$

$(\Leftarrow)$ $s \in I \cap S \Rightarrow s \in S^{-1}I$, $s \in S^{-1}R$ is a unit

$(\Rightarrow)$ $S^{-1}I = (1) \Rightarrow \exists\, \frac{a}{s}, a \in I : \frac{a}{s} = 1 \Rightarrow t(a-s) = 0$ $(\exists\, t \in S)$

(not true: $\frac{b}{s} \in S^{-1}I \Rightarrow b \in I$)

$t(a-s) = \underset{\substack{\cap \\ I}}{ta} - \underset{\substack{\cap \\ S}}{ts} \Rightarrow I \cap S \neq 0$ $(\ni ta = ts)$,

$P \in \operatorname{Supp} M \iff M_P \neq 0 \iff \operatorname{Ann}(M_P) \neq (1)$

$\operatorname{Ann}(M_P) = (\operatorname{Ann}(M))_P \neq (1) \overset{\text{Lemma}}{\iff} \underset{\substack{\| \\ \operatorname{Ann}(M)}}{I} \cap \underset{\substack{\| \\ (R \setminus P)}}{S} = \emptyset \iff \operatorname{Ann}(M) \subseteq P$ □

Let $R \to S$ be an algebra, M = R-module, N =

$\rightsquigarrow \operatorname{Hom}_R(M,N) \otimes_R S \xrightarrow{\Phi} \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S)$ (Φ is an S-linear map)

$(\varphi, t) \longmapsto \left[\begin{array}{l} M \otimes_R S \longrightarrow N \otimes_R S \\ (m, s) \longmapsto \varphi(m) \otimes (s t) \end{array}\right]$

Prop: Φ is an isomorphism, $\underline{\underline{\text{if}}}$ M is $\boxed{\text{finitely presented}}$ and $\boxed{S = \text{flat}}$

M = f.g.

Def M is finitely presented $:\iff \exists\, m, n \in \mathbb{N}$, $\exists\ R^m \longrightarrow \boxed{R^n \xrightarrow{\psi} M \longrightarrow 0}$.

$\iff$ • $\boxed{M = \text{f.g.}}$

• $\forall\, R^n \overset{\psi}{\twoheadrightarrow} M$: $\operatorname{Ker} \psi$ = f.g.

R^n

$\cup$

Proof. w.l.o.g.: $\boxed{M = R^m}$ (because of $(*)$)

$\operatorname{Hom}_R(R^m, N) = N^m$

$\varphi \longmapsto [\varphi(e_1), \ , \varphi(e_m)]$

$\Rightarrow$ l.h.s.: $N^m \otimes_R S = (N \otimes_R S)^m = \operatorname{Hom}(\underset{\substack{\| \\ R^m \otimes S}}{S^m}, N \otimes_R S)$ □

$$\begin{tikzcd}
\operatorname{Hom}_R(\overset{\overset{0}{\downarrow}}{M}, N) \otimes S \arrow[r, "\Phi_M"] \arrow[d] & \operatorname{Hom}_S(M \otimes_R \overset{\overset{0}{\downarrow}}{S}, N \otimes_R S) \arrow[d] \\
\operatorname{Hom}_R(R^n, N) \otimes_R S \arrow[r, "\Phi_{R^n}"] \arrow[d] & \operatorname{Hom}_S(R^n \otimes_R S, N \otimes_R S) \arrow[d] \\
\operatorname{Hom}(R^m, N) \otimes_R S \arrow[r, "\Phi_{R^m}"] & \operatorname{Hom}(R^m \otimes_R S, N \otimes_R S)
\end{tikzcd}$$

} Φ for free R-modules (R^m, R^n)

If Φ = isom for free modules (finite rank) $\Longrightarrow \Phi_M$ = isom (5-lemma) $(*)$

(III) Noetherian rings/modules

Example. $R = k[x, y]$ — R-module, generated by $\{1\}$

$\cup$

$m = (x, y)$ = —"— (sub mod) —"— by x, y.

$R = k[x_1, x_2, \ldots]$ still gen by 1 !!

$m = (x_1, x_2, \ldots)$ not fin gen.

$(P, \le)$ = poset. **Lemma**: Every $\neq\emptyset$ subset $Q \subseteq P$ has a maximal element ① ⟺ each ascending chain $\underbrace{p_0 \le p_1 \le p_2 \le \dots}_{(\text{in } P)}$ ② terminates. $\boxed{ACC}$ ascending chain condition

Proof (⟹) $C = \text{chain} = \{p_0, p_1, \dots\} \subseteq P$ ⟹ C contains a max element p_m ⟹ $[p_m \le p_{m+1} \Rightarrow p_m = p_{m+1}]$

(⟸) Let $Q \subseteq P$ subset ⇝ choose $q_0 \in Q$. If not max ⟹ $\exists q_1 \in Q$, $q_1 \gneq q_0$

If not max ⟹ $\exists q_2 \gneq q_1 \gneq q_0 \dots$ – this chain has to terminate ⟹ $\exists m$: q_m = maximal. □

Def: M = R-module is called Noetherian :⟺ {submodules} =: P satisfies ACC.

Prop: M is Noetherian ⟺ ③ every submodule $N \subseteq M$ is finitely generated.
(⟹ M = f.g.)

Proof: (⟹) $N \subseteq M$ submodule.
Let $n_1 \in N \setminus 0$ ⇝ Q: $N_1 := \langle n_1 \rangle \subseteq N$ – is $N_1 = N$ (?) If not: choose $n_2 \in N \setminus N_1$

⇝ $N_2 = \langle n_1, n_2 \rangle \subseteq N$ ⇝ Q $N_2 = N$ (?) ... ⟹ obtain $n_1, n_2, n_3, \dots$ generating submodules $N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \dots \subseteq \underline{N} \subseteq M$

② ⟹ this terminates ⟹ $\exists k$: $\underset{\langle n_1, n_2, \dots, n_k \rangle}{\underset{\parallel}{N_k}} = N$.

(⟸) Show ②: Let $M_1 \subsetneq M_2 \subsetneq \dots M_k \subsetneq \dots \subseteq M$ ⇝ $N := \bigcup_{k=1}^{\infty} M_i \subseteq M$ ⇝ $\exists n_1, \dots, n_e \in N$ generate N

⟹ $\exists q$ $n_1, \dots, n_e \in M_q$ ⟹ $N = \langle n_1, \dots, n_e \rangle \subseteq M_q$
⟹ $N = M_q$ ⟹ $M_q = M_{q+1} = \dots$ □

Lemma. $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ exact. M = noeth ⟺ M' and M'' are noeth.
(in particular: $M = M' \oplus M''$ ⟹ (M = noeth ⟺ M', M'' are)

Proof: (⟹) M': have to show: ascending chains of submodules of M' terminate. ✓
M'': –"– –"– of M'' –"–. Take: $M_0'' \subset M_1'' \subset \dots \subseteq M''$
⟹ $\pi^{-1} M_0'' \subseteq \pi^{-1} M_1'' \subset \dots \subseteq \pi^{-1} M'' = M$ ⟹ terminates ⟹ ↳ terminates!

(⟸) $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M$ ascending chain.

- $M_i' = M_i \cap M' = \iota^*(M_i) \subseteq M'$ (–"–)
- $M_i'' := \pi(M_i) \subseteq M''$.

$\Big\}$ $\exists i$: $M_j' = M_{j+1}'$ $\forall j \ge i$
$M_j'' = M_{j+1}''$ $\forall j \ge i$

Q: $L, N \subseteq M$ ($L = M_j$, $N = M_{j+1}$), **Know:**
with: $L \subseteq N$

- $L \cap M' = N \cap M'$
- $\pi(L) = \pi(N)$
- $L \subseteq N$

$\Big\}$ $\overset{(?)}{\Longrightarrow} L = N$

$$\begin{tikzcd}
0 \arrow[r] & \boxed{L \cap M'} \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, hook, bend left] & \boxed{\left(\pi(L) = {}^{L+M'}/_{M'}\right)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \boxed{N \cap M'} \arrow[r] & N \arrow[r] & \boxed{\pi(N)} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\iota"] & M \arrow[r, "\pi"] \arrow[d, phantom, "\cup" description] & M'' \arrow[r] & 0 \\
 & & L & &
\end{tikzcd}$$

⟹ may apply the 5-Lemma! □

Def. R = „noeth. ring" :⟺ R = noeth. R-module. (⟺ R satisfies ①, ②, ③)
⟺ all ideals in R are fin. generated.

Properties of noeth rings · free modules of finite rank (i.e. $M = R^n$) are noeth ⇝ all submodules are f.g. !!
($R^n = R^{n-1} \oplus R$) ⇓

Let R = noeth ring →

- M = f.g. ⟹ all submodules $N \subseteq M$ are f.g.

(↳ $\exists$ $R^n \twoheadrightarrow M$, i.e. $\exists K$: $0 \to K \to \underset{\text{noeth}}{R^n} \to M \to 0$ exact)).

M = f.g. R-module

Example: R = p.i.d. (e.g.: $R = \mathbb{C}[x]$, $\mathbb{Z}$, ...) ⟹ R = noetherian.

$\underset{\text{field}}{\mathbb{R}} \times V \to V$ e.g. $V = \mathbb{R}^1, \mathbb{R}^2$

Lemma. R = noetherian ⟹ · ${}^{R}/_{I}$ = noeth
· $S^{-1}R$ (for $S \subseteq R$ m.c. subset) = noeth.

Problem was: $I \subseteq R$ ⇝ $S^{-1}I \subseteq S^{-1}R$
⇝ $(S^{-1}I \cap R) \supseteq I$
$\varphi: R \to S^{-1}R$ ⟹ $S^{-1}I = \varphi(I) \cdot S^{-1}R$

Proof: · $0 \to I \to R \to {}^{R}/_{I} \to 0$ ✓
· $S^{-1}R$: $J \subseteq S^{-1}R$ ideal ⇝ $J \cap R \subseteq R$ ideal (f.g.) ⇝ $J = S^{-1}(J \cap R) = \left(\frac{a_1}{1}, \dots, \frac{a_k}{1}\right)$.
$J \cap R = (a_1, \dots, a_k)$

Hilbert's Basissatz: R = noetherian ring $\Longrightarrow$ $R[x]$ is a noeth. ring.

Consequences: $R \longrightarrow S$ algebra, s.t.: • R = noeth.
• $R \to S$ f.g. algebra (i.e. $S = R[x_1, \dots, x_n]/I$)

$\Longrightarrow$ S = noeth.

$\Longrightarrow$ f.g. $\mathbb{Z}$-algebras, $\boxed{k\text{-algebras}}$ $\Longrightarrow$ noetherian

k = field

Remark: · $R \longrightarrow R_f = R[\frac{1}{f}]$ is a f.g. algebra

· $R \longrightarrow R_p$ is not a f.g. algebra

Proof (Hilbert) Let $I \subseteq R[x]$ an ideal ⇝ look for $a_1(x), \dots, a_k(x) \in I$: $I = (a_1, \dots, a_k)$

$R \hookrightarrow R[x]$ $I_0 := \langle \{$leading coefficient $\underline{\lambda} \in R$ of elements $f(x) \in I\} \rangle$ is an ideal in R

$f(x) = \boxed{\lambda_n} x^n + \lambda_{n-1}x^{n-1} + \dots + \lambda_1 x + \lambda_0$ $(\lambda_n \neq 0)$

ex: $(\lambda x + \mu)\cdot x$ // $\lambda' x^2 + \mu' x + 0$ $\Big\} \xrightarrow{\text{add}} \underline{\lambda'}x^2 + (\lambda + \mu')x + (\mu + 0)$

$\lambda x^2 + \mu x$ $\Longrightarrow (\lambda + \lambda')x^2 + (\mu + \mu') + 0$

$\Longrightarrow I_0 \subseteq R$ is f.g.!! Choose $I_0 = (\lambda^1, \dots, \lambda^k)$ $(\lambda^i \in I_0 \subseteq R)$ $\Longrightarrow \forall i \; \exists f_i(x) \in I$ λ_i = leading coeff of f_i

$\Longrightarrow I' = (f_1(x), \dots, f_k(x)) \subseteq I$ **Goal**: Show ~~that~~ $I' = I$ wrong!

Try to show it anyway: Let $f(x) \in I$ ⇝ λ := leading coeff. of $f(x)$, i.e. $f(x) = \lambda \cdot x^n +$ lower terms

$\lambda \in I_0 \Longrightarrow \exists r_1, \dots, r_k \in R$: $\lambda = r_1\lambda^1 + \dots + r_k\lambda^k$ ‖ Choose $N := \max\{\deg f_1, \dots, \deg f_k\}$

$\boxed{\text{If } n \geq N}$ $\Longrightarrow$ compare $[f(x) = \lambda \cdot x^n + \dots]$ with $[r_1 \cdot f_1(x) \cdot x^{n - \deg f_1} + \dots + r_k \cdot f_k(x) \cdot x^{n - \deg f_k}] = \lambda x^n +$ lower terms $= g(x)$

$\underset{I}{\cap}$ $\underset{I'}{\cap}$

$\underset{I}{f(x)}$ ⇝ $\underset{I}{f(x)} - \underset{I'}{g(x)} \in I$ is better than the original f. $\deg(f - g) < n$. □

We are left with polynomials f of degree < N.

$M := \{f \in I \mid \deg f < N\} \subseteq I \subseteq R[x]$ (R-module)

$\cap$

$\langle 1, x, x^2, \dots, x^{N-1} \rangle \subseteq R[x]$ (R-module) $\Longrightarrow$ is a f.g. R-module

$\boxed{R = \text{noeth}}$ M = f.g. R-module $\Big\}$ $I = I' + M$ (I' = f.g. $R[x]$-mod, M = f.g. R-mod.) □

• R = noetherian
• M = f.g. R-module.

"an important filtration"   "factors of the filtration"

Prop: ↳ $\exists$ filtration $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_m = 0$ with $M_i/M_{i+1} \cong R/P_i$ for some PI $P_i \subset R$.

Proof: $M = \langle m_1, \dots, m_n \rangle$ $(m_i \in M)$

$0 \longrightarrow R\cdot m_1 \longrightarrow M \longrightarrow M/R\cdot m_1 \longrightarrow 0$ i.e. $M = M_0 \supseteq (M_1 = R\cdot m_1) \supseteq M_2 = 0$

$M/R\cdot m_1 = \langle \cancel{\bar m_1}, \bar m_2, \dots, \bar m_n \rangle$

Factors: · $M_1/M_2 = R\cdot m_1$ — gen. by
· $M_0/M_1 = M/R\cdot m_1$ — < n elements

induction ⇝ $\exists$ filtration: factors have only 1 generator each!

N = module with 1 gen ≙ $R \twoheadrightarrow N$ ≙ $\exists I$: $\boxed{R/I \cong N}$

Summarize so far: w.l.o.g.: $M = R/I$.

Assume $I \neq$ prime $\Longrightarrow \exists x, y \in R \setminus I$, but $xy \in I$. $I + (x) \supsetneq I$

$0 \to (I+(x))/I \to R/I \to R/(I+(x)) \to 0$ exact $R \xrightarrow{\cdot x}$ $\mathrm{Ker}(R \xrightarrow{\cdot x} R/I) = \{r \in R \mid r \cdot x \in I\} = (I : (x))$

$\Longrightarrow 0 \to R/I{:}(x) \to R/I \to R/I+(x) \to 0$ (*) $I{:}(x) \supsetneq I$, $I+(x) \supsetneq I$

filtration, factors are $R/I{:}(x)$, $R/I+(x)$.

R = noeth $\Longrightarrow$ this ascending chain has to terminate.

· $I + (x) \supsetneq I$
· $I : (x) \supseteq I$
· $I : (x) \supseteq I + (y) \supsetneq I$ analog claims

Alternatively: $S := \{I$ = ideal in $R \mid R/I$ has not a nice filtration$\}$ **Goal** $S = \emptyset$

If $S \neq \emptyset \Longrightarrow$ choose maximal $I \in S \Longrightarrow I \neq PI \Longrightarrow (*) \Longrightarrow I+(x), I{:}(x) \supsetneq I$

$\Longrightarrow R/I+(x)$ and $R/I{:}(x)$ have both nice filtrations $\Longrightarrow$ (-"-), (-"-) $\notin S$

$R/I+(x) = L$, $R/I{:}(x) = N$

(*) $\boxed{0 \to N \to M \xrightarrow{\pi} L \to 0}$, R/I h. by (*)

$\boxed{N} = N_0 \supset N_1 \supset \dots \supset N_k = \boxed{0}$ $L = L_0 \supset L_1 \supset \dots \supset L_m = 0$ N_i/N_{i+1}, L_i/L_{i+1} = nice $(\cong R/p)$

$\boxed{M} = \pi^{-1}(L) = \pi^{-1}(L_0) \supset \pi^{-1}(L_1) \supset \dots \supset \pi^{-1}(L_m) = \boxed{N}$

$\pi^{-1}(L_i)/\pi^{-1}(L_{i+1}) = M_i/M_{i+1}$ $L_i = M_i/N$ $L_{i+1} = M_{i+1}/N$

$L_i/L_{i+1} = (M_i/N)/(M_{i+1}/N) = M_i/M_{i+1}$