R = noetherian ring ; M = f.g. R-module.

Theorem $\exists$ „nice" filtration: $\underline{M} = M_0 \supseteq M_1 \supseteq \dots \supseteq M_k = 0$ s.t.:

i-th factor $= M_i / M_{i+1} \cong R/P_i$ for some PI $P_i \in \operatorname{Spec} R$.

Examples: (0) $R = \mathbb{Z}$, $\mathbb{Z} = M$ $\rightsquigarrow$ $\exists$ trivial filtration: $\mathbb{Z} = M = M_0 \supset M_1 = 0$

$M_0/M_1 = \boxed{\mathbb{Z}} = \mathbb{Z}/(0)$

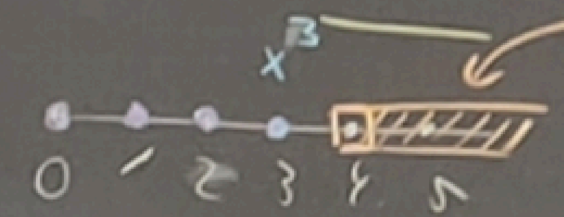
$0 \to 2\mathbb{Z}$, $\cong \downarrow 2$, $\cdot 1$

recall: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, with $0 = M_2$, $\mathbb{Z} = M_1$, $\mathbb{Z} = M_0$

$M_0 = \mathbb{Z}$, $M_1 = 2\mathbb{Z}$, $M_2 = 0$; factors: $\mathbb{Z}/2\mathbb{Z}$; $2\mathbb{Z}/0 \cong \mathbb{Z}$

$2\mathbb{Z} \xrightarrow[\cong]{\sim} \mathbb{Z}$, $2k \longmapsto k$

(1) $R = \mathbb{C}[x]$; $M = \mathbb{C}[x]/(x^4) = \mathbb{C} \oplus \mathbb{C}x \oplus \dots \oplus \mathbb{C}\cdot x^3$



$\mathbb{N} \ni k \longleftrightarrow x^k \in \mathbb{C}[x]$

$k + l \longleftrightarrow x^k \cdot x^l = x^{k+l}$

i.e.: $\mathbb{N} \longrightarrow \mathbb{C}[x]$, $k \longmapsto x^k$ hom. of semigroups (PLUS → MULT)

- filtration $\rightsquigarrow$ $R/P_{k-1} = M_{k-1}/M_k = M_{k-1} \hookrightarrow M$ $\rightsquigarrow$ embedding $\boxed{R/P_{k-1} \overset{\iota}{\hookrightarrow} M}$, $1 \longmapsto m$

$\operatorname{Ann}_R(m) = \{ r \in R \mid r \cdot m = 0 \text{ in } M \} = \{ r \in R \mid \iota(r) = 0 \}$

$= \operatorname{Ker} \iota = P_{k-1}$ if ι is understood as $R \xrightarrow{\iota} M$

$\hat{=}$ $m \in M$ with $\operatorname{Ann}(m) = P_{k-1}$.

- $\{P_0, \dots, P_{k-1}\} \subseteq \operatorname{Supp} M = \{ P \in \operatorname{Spec} R \mid M_P \neq 0 \}$

Claim: $\boxed{M = M_0 \supset \dots \supset M_{k-2} \supset M_{k-1}} \supset M_k = 0 \iff \boxed{M/M_{k-1} = M_0/M_{k-1} \supset \dots \supset M_{k-2}/M_{k-1} \supset M_{k-1}/M_{k-1} = 0}$

$\boxed{M_{k-1} \supset M_k = 0}$; $\boxed{\text{factors: } M_i/M_{i+1} \ (i \leq k-2)}$; $\boxed{M_{k-1}/M_k}$

$0 \to M_{k-1} \to M \to M/M_{k-1} \to 0$

$\operatorname{Supp} M = \operatorname{Supp}(M_{k-1}) \cup \operatorname{Supp}(M/M_{k-1})$

$\hookrightarrow \operatorname{Supp} M = \bigcup_{i=0}^{k-1} \operatorname{Supp}(M_i/M_{i+1}) = \bigcup_{i=0}^{k-1} \operatorname{Supp}(R/P_i) = \bigcup_{i=0}^{k-1} V(P_i) \supseteq \{P_0, \dots, P_{k-1}\}$

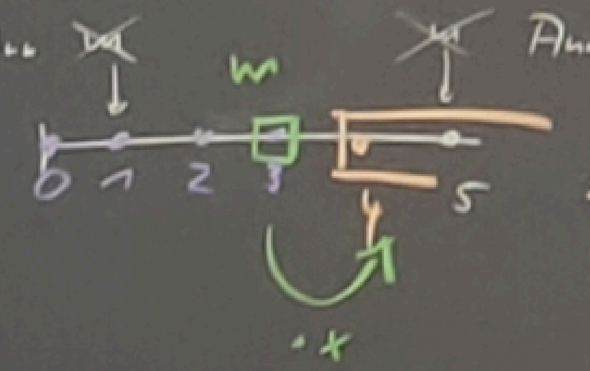
back to Ex: $R = \mathbb{C}[x]$; $M = \mathbb{C}[x]/(x^4)$

$P \in \operatorname{Supp} M \iff M_P \neq 0 \iff P \supseteq (x^4) \iff P \supseteq (x) \iff P = (x)$.

Thus: We look for embeddings $\mathbb{C}[x]/(x) \overset{\iota}{\hookrightarrow} M = \mathbb{C}[x]/(x^4)$ (inj. hom. of $\mathbb{C}[x]$-modules)

$\updownarrow$

$1 \longmapsto m \in \mathbb{C}[x]/(x)^4$: $\operatorname{Ann}(m) = (x)$



$\rightsquigarrow x^3 =: m$ fulfills: $\operatorname{Ann}_{\mathbb{C}[x]}(m \in M) = (x)$.

$\hookrightarrow \mathbb{C}[x]/(x) \overset{\iota}{\hookrightarrow} \mathbb{C}[x]/(x^4)$, $1 \longmapsto x^3$ $\rightsquigarrow$ filtration starts as $\begin{cases} M_k = M = \mathbb{C}[x]/(x^4) \\ M_{k-1} = \mathbb{C}[x]/(x) \longrightarrow M \end{cases}$

do not know yet what k is.

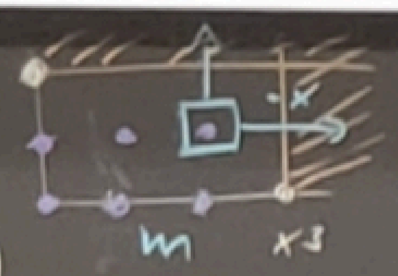
$0 \to M_{k-1} \to M \to M/M_{k-1} \to 0$

$0 \to \mathbb{C}[x]/(x) \xrightarrow{\cdot x^3} \mathbb{C}[x]/(x^4) \to \mathbb{C}[x]/(x^3) \to 0$

$\mathbb{C}[x]/(x^3) \ \cong\!\downarrow \cdot x$; $\mathbb{C}[x]/(x^2) \ \cong\!\downarrow \cdot x^2$; $\boxed{\mathbb{C}[x]/(x) \ \cong\!\downarrow \cdot x^3}$

$\rightsquigarrow$ final result: $\mathbb{C}[x]/(x^4) \supseteq x \cdot \mathbb{C}[x]/(x^4) \supseteq x^2\mathbb{C}[x]/(x^4) \supseteq x^3\mathbb{C}[x]/(x^4) \supseteq 0$

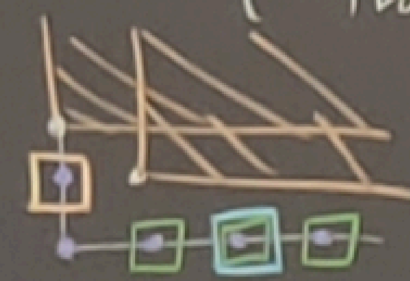
Example 2 $\mathbb{C}[x,y] = R$, $M = \mathbb{C}[x,y]/(x^3, y^2)$



$P \supseteq (x^3, y^2) \iff P = (x,y)$

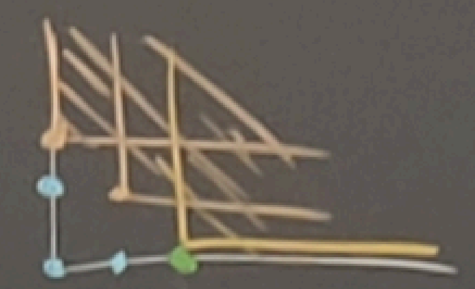
$\mathbb{C}[x,y]/(x,y) \hookrightarrow M$, $1 \longmapsto x^2 y$ ($\hat{=}$ Problem 30)

Example 3: (xy, y^2) :



$M = \mathbb{C}[x,y]/(xy, y^2) = \mathbb{C}[x] \oplus \mathbb{C} \cdot y$ (as $\mathbb{C}$-vs.)

$P \supseteq (xy, y^2) \iff P \supseteq (xy, y) = (y)$ $\rightsquigarrow$ examples: $\boxed{P = (y) ; P = (x,y)}$, $P = (x-1, y)$



$\mathbb{C}[x,y]/(y) \overset{\iota}{\hookrightarrow} \mathbb{C}[x,y]/(xy, y^2)$, $1 \longmapsto x^2$ cokernel ?

$\operatorname{Ker} \iota = \{ r(x,y) \in \mathbb{C}[x,y] \mid r(x,y) \cdot x^2 \in (xy, y^2) \} = ((xy, y^2) : (x^2))$ = monomial ideal = generated by monomials

$x^a y^b \in \operatorname{Ker} \iff x^{a+2} y^b \in (xy, y^2)$: $y^2 \mid x^{a+2} y^b \iff b \geq 2$, $xy \mid x^{a+2} y^b \iff b \geq 1$ $\rightsquigarrow$ $(b \geq 1)$ ↯

$\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ $\quad$ $\mathbb{Z}/2\mathbb{Z} \overset{\cdot 3}{\hookrightarrow} \mathbb{Z}/6\mathbb{Z}$ $\quad \hat{=} \quad R/P \hookrightarrow M$

$\{0,1\}$ $\quad$ $\{0,1,2,\dots,5\}$

$0 \mapsto 0$

$1 \mapsto 3$

R = noeth., M = f.g. module. $\overset{?}{\rightsquigarrow}$ certain prime ideals $P_1, \dots, P_\ell \in \operatorname{Spec} R$ „belonging to M"

How to do this? $\quad$ „$\operatorname{Ass}(M)$" $\quad$ „associated primes".

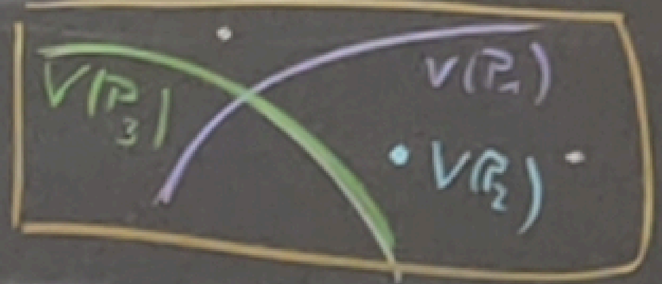
1) take a „nice" ~~filtration~~ $\rightsquigarrow$ collect the P_i occurring there [$\nexists$ canonical nice filtration]

2) $M \rightsquigarrow \operatorname{Ann}(M) \subseteq R$ ideal $\rightsquigarrow V(\operatorname{Ann} M) = \operatorname{Supp} M$ $\quad$ [Supp. is not finite]

3) take the minimal PI inside $\operatorname{Supp} M$ – $\exists$ finitely many: $V(I) \ni P_1, \dots, P_k$ minimal elements

recall: $V(P) = \overline{\{P\}}$ = irred. subset of $\operatorname{Spec} R$

4) $\operatorname{Ass}(M)$: ....



Def $P \in \operatorname{Spec} R$ is called associated to M $:\Longleftrightarrow \exists\, \iota : R/P \hookrightarrow M$ (R-linear)

$\operatorname{Ass}(M) = \{$all those PI$\}$

Remark: $\operatorname{Ass}(M) \neq \emptyset$ – just consider P_{n-1} from the „nice" filtration!

Example $R = \mathbb{Z}$, $M = \mathbb{Z}$ $\quad$ $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}$

$1 \mapsto m$: $\operatorname{Ann}(m) = (p)$ $\Big\}$ only $p = 0$ can occur, i.e. $\operatorname{Ass}(\mathbb{Z}) = \{(0)\}$

but: $\exists$ filtration $\mathbb{Z} \supset 2\mathbb{Z} \supset 0$

$\mathbb{Z}/2\mathbb{Z}$ $\quad$ $\mathbb{Z}$ $\quad \rightsquigarrow P_0, \dots, P_{n-2}$ (from the nice filtration) do not need to belong to $\operatorname{Ass}(M)$.

$p = 2$ $\quad$ $p = 0$

Prop: $0 \to M' \to M \to M'' \to 0$ exact $\rightsquigarrow \operatorname{Ass}(M') \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$

Ex. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

$(0) \quad (0) \quad (2)$

$\mathbb{Z}/2\mathbb{Z} \overset{id}{\hookrightarrow} \mathbb{Z}/2\mathbb{Z}$

$\operatorname{Ass}(\mathbb{Z}) = \{(0)\}$

$\operatorname{Ass}(\mathbb{Z}/2\mathbb{Z}) : \{(2)\}$

$\operatorname{Ass}(R)$ with R = integral domain?

$R/P \hookrightarrow R$

$1 \mapsto m \rightsquigarrow \operatorname{Ann}(m) = (0)$ (if $m \neq 0$) $\overset{!}{\Rightarrow} P = 0 \Rightarrow \operatorname{Ass}(R) = (0)$

Proof $\cdot$ $R/P \overset{\iota}{\hookrightarrow} M' \overset{j}{\hookrightarrow} M \Rightarrow j \circ \iota : R/P \hookrightarrow M \Rightarrow P \in \operatorname{Ass}(M)$.

$\Updownarrow$

$P \in \operatorname{Ass}(M')$

$\cdot$ Start with $P \in \operatorname{Ass}(M) \rightsquigarrow R/P \overset{\iota}{\hookrightarrow} M \overset{\pi}{\twoheadrightarrow} M''$, $f = \pi \circ \iota$

Case 1: f = injective $\Rightarrow P \in \operatorname{Ass}(M'')$

Case 2: $f \neq$ injective $\Rightarrow \exists\, \bar{r} \in R/P$, $\bar{r} \neq 0$ ($\hat{=}\ r \in R$, $r \notin P$): $f(r) = 0 \in M''$

$\iota(r) \in M \Rightarrow \iota(r) \in M'$!! (since $\pi(\iota(r)) = f(r) = 0$)

$\Rightarrow$ $R \xrightarrow{g} M' \hookrightarrow M$

$1 \mapsto \iota(r)$

$\hookrightarrow P \in \operatorname{Ass}(M') \Leftarrow P = \operatorname{Ann}_R(\bar{r} \in R/P)$, $\bar{r} \neq 0$

$\operatorname{Ann}_R(\iota(r) \in M') = \operatorname{Ann}_R(\iota(r) \in M)$

$R/P \overset{\iota}{\hookrightarrow} M$

$r\ (\notin P) \mapsto \iota(r)$ $\quad \square$

Let $I \subseteq R$ be an ideal $\underset{\text{Claim}}{\Longrightarrow} \exists\, P_1, \dots, P_k \in \operatorname{Spec} R$: $\sqrt{I} = P_1 \cap \dots \cap P_k$

Proof: $\cdot$ if we like: w.l.o.g. $\sqrt{I} = (0)$ $\rightsquigarrow$ Claim: R = ring without nilpotent elements ($\sqrt{0} = 0$)

$\Rightarrow \exists\, P_1, \dots, P_k$: $P_1 \cap \dots \cap P_k = (0)$.

$\cdot$ Test $\sqrt{I}$ = prime ?? $\quad$ yes $\Rightarrow \sqrt{I} = P$ ✓

no: $\exists\, x, y \notin \sqrt{I}$, $xy \in \sqrt{I} \Longrightarrow \sqrt{I} \overset{(*)}{=} \sqrt{I + (x)} \cap \sqrt{I + (y)}$

$\sqrt{I + (x)} \supsetneq \sqrt{I}$, $\quad \sqrt{I + (y)} \supsetneq \sqrt{I}$

Proof of (*): „$\subseteq$" clear.

ex: $\mathbb{Z}/6\mathbb{Z}$: no nilpot. elements

„$\supseteq$": $\cdot$ w.l.o.g.: $I = (0)$, $\sqrt{0} = 0$, R is without nilpotent elements

$\cdot$ $x, y \neq 0$, $xy = 0$ $\quad$ Claim: $\sqrt{(x)} \cap \sqrt{(y)} \subseteq (0)$

$a \in \sqrt{(x)}$, $a \in \sqrt{(y)}$ $\Rightarrow \exists\, k\ a^k \in (x)$, $\exists\, \ell : a^\ell \in (y)$ $\Big\}$ $a^k \cdot a^\ell \in (xy) = (0) \Rightarrow a^k a^\ell = 0$

$a = 0 \Longleftarrow a^{k+\ell} = 0$

$R' := R/\sqrt{I} \rightsquigarrow$ goal: $(0) \overset{?}{=} P_1 \cap \dots \cap P_\ell$

$\bar{a} : \bar{a}^N = 0 \rightsquigarrow a \in R$, $a^N \in \sqrt{I} \Rightarrow \exists\, M\ (a^N)^M \in I$

$\Rightarrow a \in \sqrt{I} \Rightarrow \bar{a} = 0$

$I \subseteq R \rightsquigarrow \exists P_1, \dots, P_k \in \operatorname{Spec} R$: $\sqrt{I} = P_1 \cap \dots \cap P_k$.

Def This representation is called **non-redundant** $:\Leftrightarrow \nexists i : P_1 \cap \dots \cap \hat{P_i} \cap \dots \cap P_k = \sqrt{I}$ [NR]
(in particular $P_i \subseteq P_j \Rightarrow i = j$)

Def $\operatorname{Min}(R/I) = \{ P \in \operatorname{Spec} R \mid P \supseteq I, \nexists Q \in \operatorname{Spec} R : P \supsetneq Q \supseteq I \}$

Prop
- NR-repr $\Rightarrow \{P_1, \dots, P_k\} = \operatorname{Min}(R/I)$
- If $P \in \operatorname{Spec} R$, $P \supseteq I \Rightarrow \exists i : P_i \subseteq P$, $P_i \in \operatorname{Min}(R/I)$

Proof Recall: $A, B \subseteq R$ ideals, $P \in \operatorname{Spec} R$
$P \supseteq A \cap B \overset{(*)}{\Rightarrow} P \supseteq A$ or $P \supseteq B$

"$\supseteq$": $P \in \operatorname{Min}(R/I)$
$\Rightarrow P \supseteq I \Rightarrow P \supseteq \sqrt{I} = P_1 \cap \dots \cap P_k$
$\overset{(*)}{\Rightarrow} \exists i : P \supseteq P_i \supseteq I$
"minimal" $\Rightarrow P = P_i$ //
"$\subseteq$" Start with P_i, assume $\notin \operatorname{Min}$
$\Rightarrow \exists P_i \supsetneq Q \supseteq I$

similar to before: $\exists j : Q \supseteq P_j \supseteq I \Rightarrow P_i \supsetneq Q \supseteq P_j \supseteq I \rightsquigarrow$ contradiction.

- $P \supseteq I \overset{(\ast\ast)}{\Rightarrow} \exists i : P \supseteq P_i$ //

$\rightsquigarrow \sqrt{I} = \bigcap \operatorname{Min}(R/I)$; $\sqrt{0} = \bigcap \operatorname{Min}(R)$

Prop $S \subseteq R$ mult. closed, M = f.g. R-module ($\rightsquigarrow$ • $S^{-1}M$ = f.g. $S^{-1}R$-module • $\operatorname{Spec} S^{-1}R = \{ P \in \operatorname{Spec} R \mid P \cap S = \emptyset \}$)

$\operatorname{Ass}(S^{-1}M) = \operatorname{Ass}(M) \cap \operatorname{Spec}(S^{-1}R)$

$S^{-1}R$
$\uparrow$
R

$\operatorname{Spec} S^{-1}R \subseteq \operatorname{Spec} R$; $S^{-1}P \subseteq S^{-1}R$, P

Proof ($\supseteq$) $R/P \hookrightarrow M \Rightarrow S^{-1}(R/P) \hookrightarrow S^{-1}M$
$S^{-1}(R/P) =$ • 0 if $S \cap P \neq \emptyset$ • $S^{-1}R/S^{-1}P$ if $S \cap P = \emptyset$. $\Rightarrow S^{-1}P \in \operatorname{Ass}(S^{-1}M)$

($\subseteq$) Let $P \in \operatorname{Spec} R$: • $S \cap P = \emptyset$
• $\exists S^{-1}R/S^{-1}P \hookrightarrow S^{-1}M$, $\boxed{1 \overset{\varphi}{\longmapsto} \frac{m}{s}}$, $S^{-1}P = \operatorname{Ann}_{S^{-1}R}\left(\frac{m}{s}\right)$

If $P = (p_1, \dots, p_l) \Rightarrow \forall i : \frac{p_i m}{s} = 0$ in $S^{-1}M \Rightarrow \exists t_i \in S : t_i p_i m = 0$ in M, $t := \prod t_i \in S$
$\Rightarrow p_1, \dots, p_l$ kill $t \cdot m \in M$, i.e. $P \cdot (tm) = 0$ in M.

$\Rightarrow R/P \overset{\psi}{\longrightarrow} M$, $1 \longmapsto tm$ $\quad /S^{-1} \Downarrow \quad S^{-1}R/S^{-1}P \longrightarrow S^{-1}M$, $\boxed{1 \overset{S^{-1}\psi}{\longmapsto} tm}$ $\Rightarrow \varphi = S^{-1}\psi$ up to units!
$\Downarrow$ $S^{-1}\psi$ = injective.

$R/P \overset{\psi}{\longrightarrow} M$, $\boxed{\psi = \text{inj.}}$, $S^{-1}R/P \overset{S^{-1}\psi}{\hookrightarrow} S^{-1}M$ □

Prop $\operatorname{Min}(R/\operatorname{Ann}(M)) \subseteq \operatorname{Ass}(M)$ (special case $\operatorname{Min}(R/I) \subseteq \operatorname{Ass}(R/I)$)

Proof: Let $P \supseteq \operatorname{Ann}(M)$ be a minimal prime ideal. $\tilde{R} := R_P = (R \setminus P)^{-1} \cdot R$
(Assume that we have shown the claim for P = max ideal in a local ring) $= (*)$
$\Downarrow$ (all $Q \subseteq P$ survive $\rightsquigarrow (R_P, P)$ = local ring.)
$P \subseteq \tilde{R}$ is a minimal prime over M_P
$(\operatorname{Ann}(M_P) = (\operatorname{Ann} M)_P)$
$\overset{(*)}{\longrightarrow} P \in \operatorname{Ass}_{R_P}(M_P) \overset{\text{prop}}{\underset{\text{before}}{\Longrightarrow}} P \in \operatorname{Ass}(M)$.

(*) $(R, \mathfrak{m})$ = local ring, M = f.g. R-module, $\mathfrak{m}$ = minimal prime above $\operatorname{Ann} M$ $\Big\} \overset{?}{\Longrightarrow} \mathfrak{m} \in \operatorname{Ass}(M)$.
$\hookrightarrow$ If $Q \in \operatorname{Spec} R$ is above $\operatorname{Ann} M \Rightarrow Q \subseteq \mathfrak{m}$ (because $\mathfrak{m}$ = only max ideal)
$\operatorname{Ann} M \subseteq Q \searrow Q = \mathfrak{m}$
$\Rightarrow \operatorname{Min}(R/\operatorname{Ann} M) = \{\mathfrak{m}\}$
$\subseteq \{ Q \in \operatorname{Spec} R \mid Q \supseteq \operatorname{Ann} M \} = \{\mathfrak{m}\}$ $\Big\} \Rightarrow \operatorname{Ass}(M) = \{\mathfrak{m}\}$.
$\operatorname{Ass}(M) \neq \emptyset$

Summary $\operatorname{Min}(R/\operatorname{Ann}(M)) \subseteq \operatorname{Ass}(M) \subseteq \{P_0, \dots, P_{k-1}$ of any nice filtration$\} \subseteq \operatorname{Supp} M = V(\operatorname{Ann} M) = \overline{\operatorname{Min}(R/\operatorname{Ann} M)}$
P_{k-1}

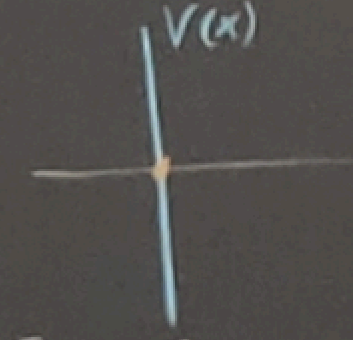
$0 \to M' \to M \to M'' \to 0 \Leftarrow M \supseteq M' \supseteq 0$ (M'', M' – factors)
$\Rightarrow \operatorname{Ass}(M) \subseteq \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$

gen. filtrations $\hat{=}$ generalization of sh. ex. seq.
$\Rightarrow \operatorname{Ass}(M) \subseteq \bigcup_i \operatorname{Ass}(M_i/M_{i+1})$
$\operatorname{Ass}(R/P_i) = \{P_i\}$

Example $R = \mathbb{C}[x,y]$, $M = \mathbb{C}[x,y]/(x^2, xy)$

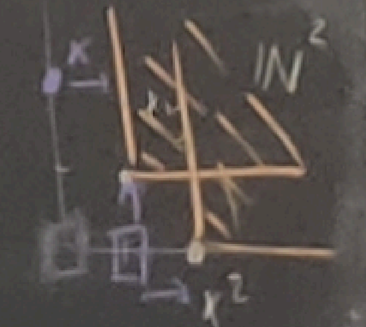
$\operatorname{Spec} \mathbb{C}[x,y]/(x^2,xy) = \{ P \in \operatorname{Spec} \mathbb{C}[x,y] \mid P \supseteq \sqrt{(x^2, xy)} \} = V((x)) = \operatorname{Spec} \mathbb{C}[x,y]/(x)$
$\sqrt{(x^2, xy)} = (x, xy) = (x)$

$\left.\begin{array}{l} x^2 = 0 \\ xy = 0 \end{array}\right\}$ if $y \neq 0 \Rightarrow x = 0$ by (1)
if $y = 0$, $x^2 = 0$ (and $x = 0$)



(ex $\operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2 \hat{=} V(\varepsilon^2) \subseteq \operatorname{Spec} \mathbb{C}[\varepsilon] = \mathbb{C}^1$
$\neq$ ($\varepsilon^2 = 0$) thick pt.
$\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon)$)

$\operatorname{Min}(\mathbb{C}[x,y]/(x^2,xy)) = \{(x)\}$
$\operatorname{Ass}(\mathbb{C}[x,y]/(x^2,xy))$



$\{(x), (x,y)\} = \operatorname{Ass}(\mathbb{C}[x,y]/(x^2,xy)) \Longleftarrow \mathbb{C}[x,y]/(x,y) \overset{\cdot x}{\hookrightarrow} \mathbb{C}[x,y]/(x^2,xy)$
$\mathfrak{m}$, $\operatorname{Ann}(\mathfrak{m}) = $ Prime : $(x), (x,y)$