- R = noetherian ring, M = f.g. module $\Longrightarrow \exists$ "nice" filtration:
  $M = M_0 \supset M_1 \supset \dots \supset M_k = 0$
  $M_i / M_{i+1} \cong R / P_i \quad (P_i \in \operatorname{Spec} R)$
- $\operatorname{Ass}(M) \subseteq \operatorname{Spec} R$
  $\ni P$ s.t. $\exists\, R/P \hookrightarrow M$, i.e: $\exists\, m \in M : \operatorname{Ann}(m) = P$
- $\operatorname{Min}(P) = \{ \text{PI in } R \text{ sitting minimally over } \operatorname{Ann}(M) \}$

Zero divisors: M = f.g. R-module $\rightsquigarrow$ $r \in R$ is called 0-divisor for M
$:\Longleftrightarrow \exists\, m \in M \setminus \{0\} : r \cdot m = 0 \quad (r \neq 0)$

Prop: $\{\text{zero-divisors for } M\} \cup \{0\} = \bigcup \operatorname{Ass}(M) = \bigcup_{P \in \operatorname{Ass}(M)} P$

special case $M = R$: $\{\text{zero-div in } R\} \cup \{0\} = \bigcup \operatorname{Ass}(R)$

(**Reminder**: $\sqrt{(0)} = \{\text{nilpotent elements}\} = \bigcap \operatorname{Spec} R$
$= \bigcap \operatorname{Min}(R) = \bigcap \operatorname{Ass}(R)$)

Proof: "$\supseteq$" Let $P \in \operatorname{Ass}(M)$, i.e: $\exists\, m \in M : \operatorname{Ann}(m) = P \quad (\Rightarrow m \neq 0)$
$\Rightarrow \forall x \in P : x \cdot m = 0 \Rightarrow P \subseteq \{\text{zero-divisors of } M\}$

"$\subseteq$": Let $x \in R$, $m \in M$ $(x \neq 0, m \neq 0)$ with $x \cdot m = 0 \rightsquigarrow \operatorname{Ann}(m) \neq (1)$, $x \in \operatorname{Ann}(m)$
by definition: $\operatorname{Ann}(m) \in \operatorname{Ass}(M)$ — IF $\operatorname{Ann}(m)$ is prime!!

Assume: $\operatorname{Ann}(m) \neq$ prime $\Rightarrow \exists\, a, b \in R$: $ab \in \operatorname{Ann}(m)$, $a, b \notin \operatorname{Ann}(m)$
$\Updownarrow$
$abm = 0 \qquad am, bm \neq 0$

$m' := am \in M \Rightarrow \operatorname{Ann}(m) \subsetneq \operatorname{Ann}(m')$
$x \in \operatorname{Ann}(m)$, $b \notin \operatorname{Ann}(m)$, $b \in \operatorname{Ann}(m')$

R = noetherian $\rightsquigarrow$ this ascending chain of annihilators terminates! $\square$

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4 Modules of finite length; Artin rings

(do not assume in general "R = noetherian")

Def: M = f.g. R-module; $M = M_0 \supset M_1 \supset \dots \supset M_k = 0$ filtration. This f. is called
a composition series $:\Longleftrightarrow$ it is not refinable.

$\left(\dots M_{i-1} \supset M_i \supset \dots \right.$, $M_{i-1} \supset N \supset M_i$; factors: $M_{i-1}/M_i \rightsquigarrow M_{i-1}/N,\ N/M_i \left. \right)$

Remark: refining at the i-th position $\Longrightarrow 0 \to N/M_i \to M_{i-1}/M_i \to M_{i-1}/N \to 0$

M_{i-1}/M_i = old factor; N/M_i, M_{i-1}/N: new factors

Remark: Filtration is a comp. series $\Longleftrightarrow$ all factors are
"simple" R-modules, i.e: they lack non-trivial submodules.

Remark: M = R-module. M is simple $\Longrightarrow M$ is generated by 1 element, i.e: $R \overset{1}{\twoheadrightarrow} M$
i.e. $R/I \xrightarrow{\sim} M$

- submodules of R/I $\triangleq$ ideals J : $I \subseteq J \subseteq R$

  $\hookrightarrow R/I$ is simple $\Longleftrightarrow \nexists\, J$ (except $J = I, R$) $\Longleftrightarrow R/I$ has no non-trivial ideals
  $\Longleftrightarrow R/I$ = field, i.e $I \in \operatorname{Max} \operatorname{Spec} R$

$\to M$ = simple $\Longleftrightarrow M \cong R/\mathfrak{m}$, $\mathfrak{m}$ = max ideal in R.

Example: $R = \mathbb{C}[x]$, $M = \mathbb{C}[x] \Longrightarrow$ such a comp. series does not exist!
$\left(\mathfrak{m} < R \Longrightarrow \mathfrak{m} = \mathfrak{m}_c = (x - c),\ c \in \mathbb{C} \Longrightarrow \mathbb{C}[x]/\mathfrak{m}_c \cong \mathbb{C} \right.$ = 1-dim. $\mathbb{C}$-vs.

Example: ① $R = M = \mathbb{C}[x]/x^2 = \mathbb{C} \oplus \mathbb{C}x$, 2-dim $\mathbb{C}$-vs.
$\triangleq$ filtration: $\mathbb{C}[x]/(x^2) \supset x \cdot \mathbb{C}[x]/(x^2) \supset 0$

$0 \to x \cdot \mathbb{C}[x]/(x^2) \to \mathbb{C}[x]/x^2 \to \mathbb{C}[x]/(x) \to 0$

$x \cdot \mathbb{C}[x]/x^2 \xleftarrow{\sim} \mathbb{C}[x]/x$ isom. of $\mathbb{C}[x]/x^2$-modules
$x \cdot r \longleftarrow\!\shortmid r$

(situation in our examples: $k \hookrightarrow R$, k = field)

② $R = M = \mathbb{Z}/9\mathbb{Z} \Longrightarrow 0 \to 3 \cdot \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$
$3r \mapsfrom r$, $3 \cdot \mathbb{Z}/9\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$

③ $\mathbb{Z}/6\mathbb{Z}$: $0 \to 2 \cdot \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$
$2 \cdot \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$

Def: $\ell(M) :=$ length of the shortest composition series of M $\leq \infty$
e.g: $\ell(\mathbb{C}[x]) = \infty$ (as a $\mathbb{C}[x]$-module)

M = R-module $\rightsquigarrow$ $\ell(M) \in \mathbb{N} \cup \{\infty\}$

<u>Prop:</u> (1) ℓ is strictly monotonic increasing, <u>i.e.:</u> $N \subsetneq M \Rightarrow \ell(N) < \ell(M)$ (or $\ell(M) = \infty$)

(2) $M = M_0 \supsetneq \dots \supsetneq M_k = 0$ („filtration of length k") $\Rightarrow k \leq \ell(M)$

(3) $\ell(M) < \infty \Rightarrow$ every filtration can be refined to a composition series

((4) any comp. series has length $\ell(M)$)

(5) ℓ is additive, <u>i.e.:</u> $0 \to M_1 \to M_2 \to M_3 \to 0 \rightarrow \ell(M_2) = \ell(M_1) + \ell(M_3)$

(even more: $0 \to M_1 \to M_2 \to \dots \dots \to M_s \to 0$ exact $\Rightarrow \sum_{i=1}^{s} (-1)^i \cdot \ell(M_i) = 0$

(6) composition series are characterized by $\cdot$ length $= \ell(M)$, or

$\cdot$ length (factors) $= 1$.

<u>Proof:</u> (1) Let $N \subseteq M$. Let $M = M_0 \supset \dots\dots \supset M_\ell = 0$ be a shortest composition series of M.

$N_i := N \cap M_i$ $\qquad N = N_0 \supseteq \dots \supseteq N_i \supseteq \dots \supseteq N_\ell = 0$

$N_i / N_{i+1} = (N \cap M_i) / \underline{N \cap M_{i+1}} \xhookrightarrow{\varphi} M_i / M_{i+1}$ $\qquad$ <u>Ker</u> $\varphi = \{ a \in N \cap M_i \mid a \in M_{i+1} \} = N \cap M_i \cap M_{i+1} = \underline{N \cap M_{i+1}}$

$M_i / M_{i+1} \;\|\;$ simple R-module

$\Rightarrow N_i / N_{i+1} = \begin{cases} 0 & \text{or} \\ M_i / M_{i+1} \end{cases} \Longrightarrow N_\bullet$ becomes a composition series of N (after killing redundancies) $\Rightarrow \ell(N) \leq \ell(M)$

<u>Assume:</u> $\ell(N) = \ell(M) \Rightarrow \forall i: N_i / N_{i+1} \xrightarrow{\sim} M_i / M_{i+1}. \Longrightarrow \forall i: N_i = M_i$ (induction, decreasing!)

assume: $N_{i+1} = M_{i+1}$

$$\begin{tikzcd} 0 \arrow[r] & N_{i+1} \arrow[r] \arrow[d, "\wr"] & N_i \arrow[r] \arrow[d, "\wr"] & N_i / N_{i+1} \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & M_{i+1} \arrow[r] & M_i \arrow[r] & M_i / M_{i+1} \arrow[r] & 0 \end{tikzcd}$$

ind hypothesis $\qquad$ claim

(2) $M = M_0 \supsetneq M_1 \supsetneq \dots\dots \supsetneq M_k = 0 \overset{?}{\Longrightarrow} k \leq \ell(M)$

$\ell(M_{i-1}) > \ell(M_i) \Rightarrow \ell(M_0) \geq k$.

(5) $0 \to M_1 \to M_2 \to M_3 \to 0$

$\boxed{\text{take comp. series of length } \ell(M_1) \quad * \quad \text{take a comp. series of length } \ell(M_3)} \Rightarrow$ $\cdot$ filtration of M_2 of length $\ell(M_1) + \ell(M_3)$

$\cdot$ factors stay simple $\Rightarrow$ becomes a compos. series, too.

$\ell(M_2) \leq \ell(M_1) + \ell(M_3) \underset{(2)}{\leq} \ell(M_2)$ //

(3) $M = M_0 \supset \dots\dots \supset M_k = 0$ filtration, <u>not</u> composite $\Rightarrow \exists i: M_i / M_{i+1} \neq$ simple

$\Rightarrow \exists\ 0 \neq N \subsetneq M_i / M_{i+1} \rightarrow N = L / M_{i+1}, \quad M_{i+1} \subsetneq L \subsetneq M_i$

$\Rightarrow$ longer filtration, $\qquad$ (2) $\Rightarrow$ this process terminates!

(4) $M_\bullet$ = comp. series $\Rightarrow k \geq \ell(M)$

$M_0 \supset \dots \supset M_k$ $\qquad$ (2) $\Rightarrow k \leq \ell(M)$ //

If $M = M_0 \supset \dots \supset M_k = 0$, $k = \ell(M)$ is a filtration $\Rightarrow$ composition series. //

$\ell(M) = 1 \Longleftrightarrow M$ = simple.

$\square$

<u>Def:</u> M = R-module is called Artinian $:\Longleftrightarrow M$ satisfies the (DCC) for the set of its submodules.

(<u>Reminder:</u> ACC $\triangleq$ „noetherian")

<u>Examples:</u> ① $\mathbb{C}[\varepsilon]/_{\varepsilon^2} = \mathbb{C}[x]/_{x^2}$ ($\dim_{\mathbb{C}} = 2$) $\Rightarrow$ ACC <u>and</u> DCC.

② $R = \mathbb{Z}$; $M = \mathbb{Z}$. noetherian $\rightsquigarrow$ ACC ✓

DCC: $\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset \dots \supset 2^k \cdot \mathbb{Z} \supset$ ~~DCC~~

$\boxed{\begin{array}{l} \underline{\text{Lemma}}: 0 \to M' \to M \to M'' \to 0 \\ M \text{ has DCC} \Longleftrightarrow M' \text{ and } M'' \text{ have DCC.} \\ \underline{\text{Proof}}: \text{ as before.} \end{array}}$

③ $\mathbb{Z}_p = \mathbb{Z}[\frac{1}{p}]$ (p = prime number); $R = \mathbb{Z} \Rightarrow \mathbb{Z}_p$ has submodules $p^k \cdot \mathbb{Z}$ ($k \in \mathbb{Z}$)

$\| \; M$ $\qquad \dots \subseteq p^{k+1} \cdot \mathbb{Z} \subseteq p^k \mathbb{Z} \subseteq \dots$ ~~DCC~~ ~~ACC~~

④ $R = \mathbb{Z}$, $M = \mathbb{Z}_p / \mathbb{Z} \Rightarrow p\mathbb{Z} \supset p^2\mathbb{Z} \supset \dots$ do not survive!

<u>but:</u> $\left(\underset{\underset{0}{\|}}{\mathbb{Z}} \subset \frac{1}{p}\mathbb{Z} \subset \frac{1}{p^2}\mathbb{Z} \subset \dots\dots \subset \mathbb{Z}[\frac{1}{p}] \right) / \mathbb{Z} \Rightarrow$ ~~ACC~~

<u>Claim:</u> DCC is ok

<u>Proof:</u> <u>Claim</u>. $\{ \frac{1}{p^k} \cdot \mathbb{Z} / \mathbb{Z} \}$ is all submodules of $\mathbb{Z}_p / \mathbb{Z}$ ②

<u>Proof:</u> $\frac{a}{p^k} \in \mathbb{Z}_p$, $\gcd(a, p^k) = 1 \Rightarrow \exists b, c \in \mathbb{Z}: ab + cp^k = 1 \mid \cdot \frac{1}{p^k}$

$\Rightarrow \frac{a}{p^k} \cdot b + \cancel{c} = \frac{1}{p^k}$

$\Rightarrow \frac{a}{p^k} \sim \frac{1}{p^k}$

Def R = „artinian ring" $\iff$ R = artinian R-module $\iff$ DCC for ideals.

Prop R = artinian $\iff$ $\ell_R(R) < \infty$ $\iff$ R = noetherian and $\operatorname{Max Spec} R = \operatorname{Spec} R$.

And, if this is the case, then $\# \operatorname{Spec} R < \infty$.

Ex • $R = \mathbb{C}[x]$ $\hat{=}$ $\operatorname{Spec} \mathbb{C}[x] - \mathbb{A}^1_{\mathbb{C}}$: $I_k := (x^k)$ is decreasing.

$\operatorname{Spec} R \ni (0)$, is not maximal!

• $R = \mathbb{C}[\varepsilon]/_{\varepsilon^2}$: $\operatorname{Spec} \mathbb{C}[\varepsilon]/_{\varepsilon^2} = \{(\varepsilon)\}$

$P \subseteq R$ prime ideal

$\operatorname{ht}(P) := \{\text{longest } k \mid \exists\ \underbrace{P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_k = P}_{\text{prime ideals}}\}$

$\dim R := \sup\{\operatorname{ht}(P) \mid P \in \operatorname{Spec} R\}$

$\shortparallel$ $\dim \operatorname{Spec} R$

$\rightsquigarrow$ „Max Spec = Spec" $\iff$ $\dim R = 0$

$\dim \mathbb{C}[x] = \dim \mathbb{A}^1 = 1$

$\dim \mathbb{C}[\varepsilon]/_{\varepsilon^2} = 0$

Proof. • $\ell_R(R) < \infty \implies$ DCC, ACC //

• Assume R = noetherian, but $\ell_R(R) = \infty$ $\xrightarrow{(!)}$ $\operatorname{Spec} R \supsetneq \operatorname{Max Spec} R$.

$\{I \subseteq R \text{ ideal} \mid R/_I \text{ has infinite length}\} \ni (0)$; noetherian $\rightsquigarrow$ let $I \subsetneq R$ be maximal with $\ell(R/_I) = \infty$.

$\boxed{\text{Claim: } I = \text{prime ideal.}}$ If not $\rightsquigarrow$ $\exists\, x, y \in R$: $xy \in I$, $x, y \notin I$ $\rightsquigarrow$ $I + (x), I + (y) \supsetneq I$.

$0 \to I+(x)/_I \to R/_I \to R/_{I+(x)} \to 0$ $\implies$ $0 \to R/_{\boxed{(I:x)}} \to R/_I \to R/_{\boxed{I+(x)}} \to 0$

R, 1, x, $\operatorname{ker} = (I : x)$

$(I : x) \supseteq I + (y) \supsetneq I$ both $\supsetneq I$.

$\ell(R/_{(I:x)}), \ell(R/_{I+(x)}) < \infty$

$\Downarrow$

$\ell(R/_I) < \infty$ ↯ contradiction!

$\hookrightarrow I \in \operatorname{Spec} R$.

$[I \in \operatorname{Max Spec} R \longrightarrow R/_I$ = field = simple module $\implies \ell(R/_I) = 1]$ //

Remarks: R = artinian $\xrightarrow{(?)}$ • noetherian

• Spec = Max Spec

goal: Write $(0) = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k$ (~~intersect~~ product of finitely many max ideals)

Let J := minimal ideal such that J can be written as (– " –)

(ex: (1) = intersection of 0 max. ideals)

Claim: $J = 0$

Properties: • $\boxed{J^2 = J}$: $J = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k \implies J^2 = \mathfrak{m}_1^2 \cdot \ldots \cdot \mathfrak{m}_k^2 \subseteq J$ $\shortparallel$ minimal.

• $\mathfrak{m} \in \operatorname{Max Spec} R \implies \mathfrak{m} \cdot J = J$ (same reason)

$\implies J \subseteq J_{ac}$ – Radical!

$J_{ac} \cdot M = M \implies M = 0$

$M = J \longrightarrow$ from $J_{ac} \cdot J = J$ it follows that $J = 0$. } works only if J = f.g. !!

Work around. (assume $J \neq 0$) Let I = smallest ideal s.t. $I \cdot J \neq 0$ (e.g. $I = (1)$)

$\implies$ • $I J = I$ $\left(\begin{matrix} IJ \subseteq I \\ (IJ) \cdot J = I \cdot J^2 = I \cdot J \neq 0 \end{matrix} \right\} IJ = I \Big)$

• $IJ \neq 0 \implies \exists\, f \in I : f \cdot J \neq 0 \implies I = (f)$

$\implies$ Nakayama to I: $IJ = I \implies I \cdot J_{ac} = I \implies \underline{\underline{I = 0}}$ ↯

$\implies (0) = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k$, $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ = maximal ideals.

Build a filtration: $R \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1 \mathfrak{m}_2 \supseteq \ldots \supseteq \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k = (0)$ ($R/_I$-modules = R-modules killed by I)

i-th factor: $V_i = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_{i-1} / \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_{i-1} \cdot \mathfrak{m}_i$ = $R/_{\mathfrak{m}_i}$-module = $R/_{\mathfrak{m}_i}$ – vs.

R = artinian $\implies$ $\hookrightarrow$ is an artinian $R/_{\mathfrak{m}_i}$-module.

k = field, V = k-vs. V = artinian k-module $\iff$ $\dim_k V < \infty$

$\implies V_i$ = finite-dim. $R/_{\mathfrak{m}_i}$-vs. $\implies V_i$ has a composition series! (as a $R/_{\mathfrak{m}_i}$-module $\implies$ as an R-module)

$\hookrightarrow$ finite length $\implies \ell_R(R) < \infty$!! $\implies R$ = noetherian!

• $P \in \operatorname{Spec} R \implies P \supseteq \boxed{(0) = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k}$

$\implies \exists\, i : P \supseteq \mathfrak{m}_i \implies \underline{P = \mathfrak{m}_i} \in \operatorname{Max Spec} R$

R = noeth., M = f.g. $\rightsquigarrow$ nice filtration: $M = M_0 \supset M_1 \supset \ldots \supset M_k = 0$, $M_i/_{M_{i+1}} = R/_{P_i}$, $P_i \in V(\operatorname{Ann} M)$ ‖ $\operatorname{Min} M \subseteq \operatorname{Ass} M \subseteq \{P_1, \ldots, P_k\}$

Prop: $P \in \underline{\operatorname{Min}} M \implies \#\{\text{occurrences of } P\} = \#\{i \mid P_i = P\} = \ell_{R_P}(M_P)$ $(< \infty)$

Proof Let $R/_Q$ be a factor $\rightsquigarrow$ $R_P/_{Q_P} = \begin{cases} 0 & \text{otherwise} \\ \neq 0 & Q \subseteq P \iff Q = P \end{cases}$ $\square$

$(Q = P_i)$ $\operatorname{Quot}(R/_Q)$