Primary decomposition

$\mathbb{Z} \Rightarrow \quad 18 = 2 \cdot 9 = 2 \cdot \underline{\underline{3^2}}$ 3, 2 = prime numbers $\rightsquigarrow$ prime ideals $P \subseteq R$

$3^2 \rightsquigarrow$ ~~could generalize to P^2...~~

$3^2 \hat{=}$ „primary ideal"

$R = \boxed{\text{noetherian ring.}}$

Def: $Q \subseteq R$ ideal. Q is called „**primary**" $:\Longleftrightarrow$ in R/Q every zero-divisor is nilpotent.

Corollary: prime $\Longrightarrow$ primary

Example: Q = primary $\Longleftrightarrow$ $\forall a, b \in R$: If $ab \in Q$, and $a \notin Q$, then $\exists k: b^k \in Q$

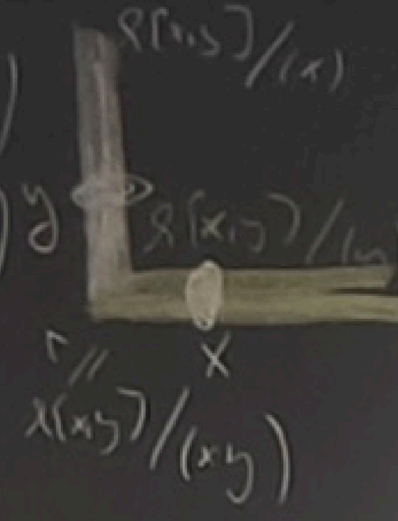
$R = k[x,y]$, $Q = (x^2, y) \rightsquigarrow k[x,y]/(x^2,y) = k[x]/(x^2)$

$Q' = (x^2, y^2)$ is also primary!

Remark: Q = primary $\Longrightarrow \sqrt{Q}$ = prime ideal!

Proof: Let $a, b \in R$, $ab \in \sqrt{Q} \Rightarrow \exists k: a^k b^k \in Q$ **Goal**: $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$.

If $a^k \notin Q \Rightarrow \exists l: (b^k)^l \in Q \Rightarrow b \in \sqrt{Q}$. $\square$

$\rightsquigarrow$ If Q = primary, $P = \sqrt{Q} \rightsquigarrow Q$ = „**P-primary**" 

Lemma: $Q, Q' = P$-primary $\Longrightarrow Q \cap Q' = P$-primary. („$Q, Q'$ = primary $\not\Longrightarrow Q \cap Q'$ = primary")

Example: $R = k[x,y]$ $Q = (x)$, $Q' := (y) \rightsquigarrow Q \cap Q' = (x) \cap (y) = (xy) = xy \cdot k[x,y]$

$xy \in Q \cap Q'$; $x^k, y^k \notin Q \cap Q'$

Proof (Lemma). $Q = P$-primary $\Longleftrightarrow \forall a, b \in R$:

$[ab \in Q,\ a \notin Q \Rightarrow b \in P]$, i.e. $[ab \in Q \Rightarrow a \in Q$ or $b \in P]$ $\square$

Let $ab \in Q \cap Q'$ if $b \in P \rightsquigarrow$ ok

$\Downarrow$ if $b \notin P \rightarrow a \in Q,\ a \in Q' \Rightarrow a \in Q \cap Q'$.

$\in Q, \in Q'$

Remark: $Q = P$-primary $\Longrightarrow \exists k.\ P^k \subseteq Q \subseteq P$. $\quad \sqrt{Q} = P$

$\begin{pmatrix} P = (p_1, \dots, p_l) \Rightarrow \forall i: \boxed{\exists k_i: p_i^{k_i} \in Q} \\ \| \qquad\qquad\qquad (k = \max k_i) \\ \sqrt{Q} \qquad \Longrightarrow P^{l(k-1)+1} \subseteq Q: \end{pmatrix}$

$\boxed{p_1^{a_1} \cdots p_l^{a_l} \quad \sum a_i \geq l(k-1)+1 \rightarrow \exists i: a_i \geq k}$

Conj: P = prime ideal $\Longrightarrow \forall k: Q := P^k$ is P-primary

WRONG: **Example**: $R = k[x,y,z]/(xy - z^2)$, $P := (x, z)$ prime ideal

$Q := P^2 = (x^2, xz, z^2) \rightsquigarrow R/Q = k[x,y,z]/(xy - z^2, x^2, xz, z^2) = k[x,y,z]/(xy, x^2, xz, z^2)$: $\bar{y}$ = zero divisor, $\neq$ nilpotent!

but: **Lemma**: $m \in \underline{\text{Max Spec}}\, R$, if $Q \subseteq R$ ideal with $\sqrt{Q} = m \Longrightarrow Q$ is primary

Proof: $\sqrt{Q} \subseteq R/Q$ becomes $\sqrt{0} = \{\underline{\text{nilpotent elements}}\}$

$\| \quad$ · is contained in every prime ideal of R/Q

$m \subseteq R/Q \quad$ · is maximal ideal

$\{\text{zero divisors}\} \cup \{0\} = \bigcup \text{Ass}(R/Q)$ $\quad \Longrightarrow m = \sqrt{0}$ = only prime ideal in R/Q.

$= m = \sqrt{0} \rightsquigarrow$ all zero-divs are nilpotent! $\square$

(Indeed: $P = (x, z)$ from the example was not max!)

Goal: all ideals are intersections of primary ideals.

Def: I is „$\cap$"-irreducible $:\Longleftrightarrow$ If $I = I_1 \cap I_2 \Longrightarrow I = I_1$ or $I = I_2$

Prop: Every ideal I can be written as a finite intersection of $\cap$-irred. ideals

Proof: If I is not $\cap$-irred $\Longrightarrow \exists\, I = I_1 \cap I_2$, $I_1, I_2 \supsetneq I$ „noetherian" $\Downarrow$

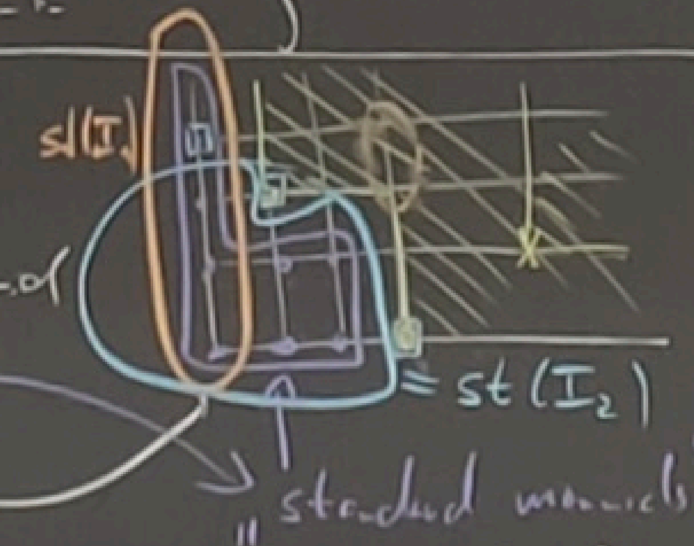
If either I_1 or I_2 are not $\cap$-irred $\Longrightarrow I_1 = I_{11} \cap I_{12}$, $I_{11}, I_{12} \supsetneq I_1$ ... terminates!

alternative / cleaner version of proof. $S := \{I \subseteq R \mid I$ is not $\underbrace{\text{repr. as a finite}}$ intersection of $\cap$-irred. ideals$\}$

Claim: $S = \emptyset$, if not $\Rightarrow \exists$ maximal element $I \in S$ $\hookrightarrow$ (-||-) $\Rightarrow I \neq (\cap$-irred$)$

$\Rightarrow I = I_1 \cap I_2 \quad (I_1, I_2 \supsetneq I)$

$\Rightarrow I_1, I_2 \notin S \Rightarrow \left.\begin{array}{l} I_1 = \text{finite int.} \\ I_2 = \text{-"-} \end{array}\right\}$ of $\cap$-irred. ideals $\Rightarrow I$ is!! ↯ $\square$

Example: monomial ideals in $k[x,y]$: 

$[I = I_1 \cap I_2$, I, I_1, I_2 = monomial

$\Downarrow$

$st(I) = st(I_1) \cup st(I_2)$

$\{\text{mon. in } k[\underline{x}]/I\}$ draw $\hat{=} \{\text{monomials in } k[\underline{x}]\} \setminus \{\text{monomials in } I\}$

$\| \\ \text{„}\mathbb{N}\text{"}$

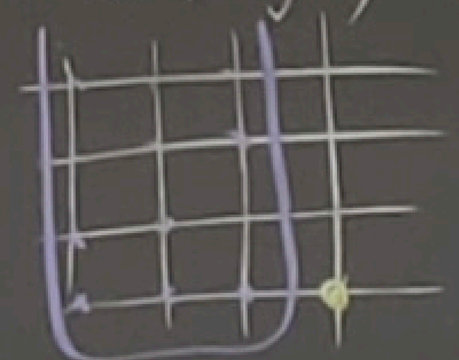
e.g. $I_1 = (x)$

$I = (x^3, xy^2)$

$I_2 = (x^3, xy^2, y^3) \neq$ irreducible

$(x^3, xy^2) = I \Longrightarrow$ · $\sqrt{I} = (x)$ = prime

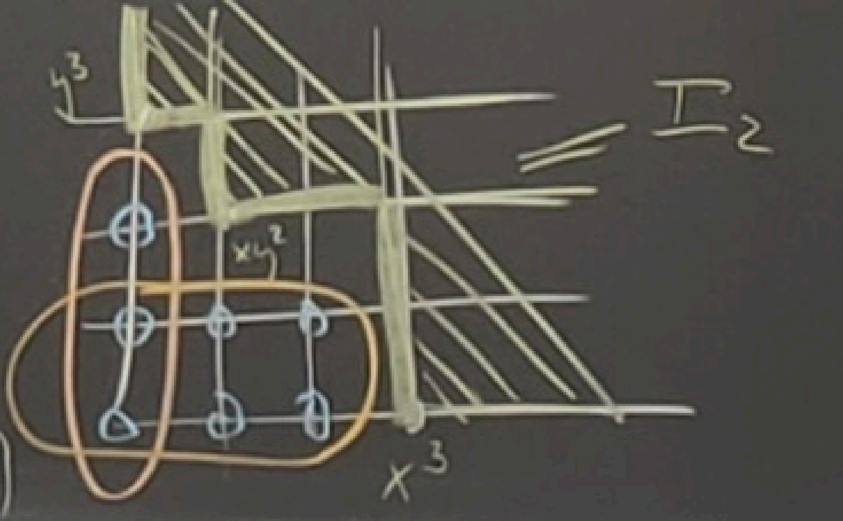
· $I \neq$ primary



$(x^3) \rightsquigarrow \sqrt{(x^3)} = (x)$

primary

$\exists$ chance: „irred" $\Longrightarrow$ „primary" $\left\{\begin{array}{l} \underline{\text{no}} \text{ — irreducible?} \\ \text{yes} \text{ — primary?} \\ \quad \text{ — } \sqrt{I_2} = (x, y) \end{array}\right.$



Lemma. Every $\cap$-irred. ideal is primary.

Proof. Let $I = \cap$-irred., goal: R/I : [zero div $\Rightarrow$ nilpotent]

$R := R/I$, i.e. $I = (0)$) THAT is: $(0) = \cap$-irreducible; show: [z.d. $\Rightarrow$ n.p. in R]

Warm up: $y \in R \rightsquigarrow \mathrm{Ann}(y^k) \subseteq R$ ideal $(\forall k \in \mathbb{N})$

$\mathrm{Ann}(y^k) \subseteq \mathrm{Ann}(y^{k+1})$ ascending $\Rightarrow \exists k : \boxed{\mathrm{Ann}(y^k) = \mathrm{Ann}(y^{k+1})}$.

this means: $\mathrm{Ann}(y) \cap (y^k) = (0)$ $\Longrightarrow$ either $y^k = 0$ or $\mathrm{Ann}(y) = 0$.

$y^k \cdot z \in (y^k)$

$$\boxed{\begin{aligned} &y^{k+1} z = y \cdot (y^k z) = 0 \\ &\hookrightarrow z \in \mathrm{Ann}(y^{k+1}) = \mathrm{Ann}(y^k) \Rightarrow y^k z = 0 \end{aligned}}$$

Let y = zero divisor in R

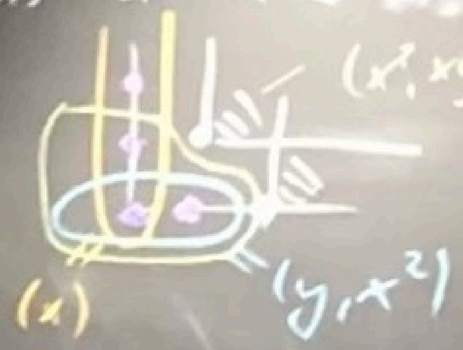
$\Rightarrow y' \neq 0 : y'y = 0 \Rightarrow \mathrm{Ann}(y) \neq 0$

$\Rightarrow y^k = 0$ for $k \gg 0$

i.e. nilpotent !! $\square$

Corollary. Each $I \subseteq R$ = noeth. r. is repr. as a finite ~~[illegible]~~ of primary ideals $I = Q_1 \cap \ldots \cap Q_m$.

Problem: not unique !! Ex: $I = (x^2, xy)$



$(x^2, xy) = (x) \cap (y, x^2)$
$= (x) \cap (x^2, xy, y^2)$

Recall: $\sqrt{I} = \bigcap \mathrm{Min}(R/I)$

$\sqrt{0} = \bigcap \mathrm{Min}(R) = \bigcap \mathrm{Spec} R$

Def. $I = Q_1 \cap \ldots \cap Q_m$ (Q_i = primary ideals) is „reduced" / „unshortenable" $:\Longleftrightarrow$ • $\forall i : Q_1 \cap \ldots \cap \hat{Q}_i \cap \ldots \cap Q_m \supsetneq I$ (left out)

• $i \neq j \Rightarrow \sqrt{Q_i} \neq \sqrt{Q_j}$, $\sqrt{Q_i} = P_i$, $\sqrt{Q_j} = P_j$

($P_1, \ldots, P_m$ should be mutually different)

Remark $\sqrt{I} = \sqrt{Q_1} \cap \ldots \cap \sqrt{Q_m} = P_1 \cap \ldots \cap P_m \Rightarrow$ among the $\{P_i\} : \supseteq \mathrm{Min}(R/I) = \mathrm{Min}(R/\sqrt{I})$

($\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$?)

„$\subseteq$" ✓ YES

„$\supseteq$" $a \in \sqrt{I} \cap \sqrt{J} \Rightarrow \exists k : \left.\begin{aligned} a^k &\in I \\ a^k &\in J \end{aligned}\right\} \Rightarrow a^k \in I \cap J \Rightarrow a \in \sqrt{I \cap J}$)

Theorem (1^{st} uniqueness) $I = Q_1 \cap \ldots \cap Q_m$ reduced primary decomposition

$\Rightarrow \{P_1, \ldots, P_m\} = \mathrm{Ass}(R/I) \supseteq \mathrm{Min}(R/I)$ ← might be different (recall $\mathrm{Ass}(R/\sqrt{I}) = \mathrm{Min}(R/\sqrt{I})$)

Proof. Preparation. $Q = P$-primary ideal $\rightsquigarrow \sqrt{Q} = P$

$x \in R \rightsquigarrow (Q : x) := \{a \in R \mid ax \in Q\} \supseteq Q$

$\hookrightarrow$ • $\boxed{x \in Q} \Rightarrow (Q : x) = (1)$

• $\boxed{x \notin Q} \Rightarrow Q \subseteq (Q : x) \subseteq P$, $\sqrt{Q} = \sqrt{(Q : x)} = P$, P-primary

and $(Q : x)$ is P-primary.

$\left(\left.\begin{aligned} a &\Rightarrow ax \in Q \\ x &\notin Q \end{aligned}\right\} \Rightarrow a \in P \right)$

$(Q : x)$ = primary:

$ab \in (Q : x)$, $a \notin P$

$\Downarrow$

$a(bx) \in Q \Rightarrow bx \in Q \Rightarrow b \in (Q : x)$

1. step $\mathrm{Ass}(R/I) \subseteq \{P_1, \ldots, P_m\}$

First: Assume that $I = 0$.

$\hookrightarrow (0) = Q_1 \cap \ldots \cap Q_m$.

Let $x \in R \rightsquigarrow$ look at $\mathrm{Ann}(x) = (0 : x)$ $(\in \mathrm{Ass}(R)$ if $\mathrm{Ann}(x)$ = prime)

$(Q.\ (I \cap J : (x)) \overset{?}{=} (I : (x)) \cap (J : (x))$ ✓ $) \Rightarrow (0 : x) = (Q_1 : x) \cap \ldots \cap (Q_m : x)$

$\Rightarrow \mathrm{Ann}\, x = \bigcap_{Q_i \not\ni x} (Q_i : x) \Rightarrow \sqrt{\mathrm{Ann}\, x} = \bigcap_{Q_i \not\ni x} \sqrt{(Q_i : x)} = \bigcap_{Q_i \not\ni x} P_i$

If $\mathrm{Ann}\, x \in \mathrm{Ass}(R) \Rightarrow \mathrm{Ann}\, x$ = prime $= \bigcap_{Q_i \not\ni x} P_i \Rightarrow \exists i : \boxed{\mathrm{Ann}\, x = P_i}$ (otherwise: $\exists y_i \in P_i \setminus \mathrm{Ann}\, x$ $\Rightarrow \prod y_i \in \bigcap P_i \setminus \mathrm{Ann}\, x$)

2nd step: take $P_i \overset{?}{\rightsquigarrow} \in \mathrm{Ass}(R)$

take $x \in I_i := \bigcap_{j \neq i} Q_j = Q_1 \cap \ldots \cap \hat{Q}_i \cap \ldots \cap Q_m \supsetneq (0) \Rightarrow x \notin Q_i \Rightarrow \boxed{\sqrt{\mathrm{Ann}\, x} = P_i}$

$x \neq 0$

$\mathrm{Ann}\, x \supseteq Q_i \Rightarrow P_i = \sqrt{Q_i} \subseteq \sqrt{\mathrm{Ann}\, x}$

(m exists: $Q_i \cdot I_i = 0$, $\exists m : P_i^m \subseteq Q_i$)

improve x: $0 \neq x \in P_i^m I_i$, s.t. $P_i^{m+1} \cdot I_i = 0$

$\Rightarrow \boxed{\mathrm{Ann}\, x = P_i} \Leftarrow \begin{cases} \bullet\ P_i \cdot x = 0 \\ \bullet\ \mathrm{Ann}\, x \subseteq P_i \end{cases}$ $\square$

$y \in \mathrm{Ann} \Rightarrow yx = 0 \Rightarrow \in Q_i$

$\Rightarrow y \in P_i$

$I = Q_1 \cap \dots \cap Q_m$ reduced primary representation $\Longrightarrow$ $\{P_i = \sqrt{Q_i}\} = \mathrm{Ass}(R/I)$ $\cup$ $\mathrm{Min}(R/I)$

<u>Theorem</u>: If $P_i \in \mathrm{Min}(R/I) \Rightarrow Q_i$ is uniquely determined.

<u>Example</u>: $(xy, x^2) = I$

$(xy, x^2) = (x) \cap (y, x^2) = (x) \cap (x^2, xy, y^2)$

$\mathrm{Ass}\left({}^{k[x,y]}/_{(xy,x^2)} \right) = \{ (x), (x,y) \}$

<u>Primary decomp</u> can be done by comp. algebra: – Macaulay 2
– Singular

<u>Proof</u>. $P_i \in \mathrm{Min}(R/I)$; $I = Q_1 \cap \dots \cap Q_m \mid \otimes_R R_{(P_i)}$ $\rightsquigarrow$ only primaries below P_i survive!

• Let Q_j be such that $\boxed{P_j \not\subseteq P_i}$ (i.e. $P_j \neq P_i$) $\quad I \subseteq P_i$ sits minimally above $\subseteq P_j$

$\Rightarrow Q_j \otimes_R R_{P_i} = (R \setminus P_i)^{-1} Q_j = (1)$

<u>because</u>: $Q_j \not\subseteq P_i$ (if $Q_j \subseteq P_i \Rightarrow \sqrt{Q_j} \subseteq P_i$)

$\Rightarrow \exists\, q \in Q_j,\ q \in (R \setminus P_i) \Rightarrow q \in Q_j$ becomes a unit after localisation!

$\Longrightarrow$ $\underline{I \otimes R_{P_i}} = Q_i \otimes R_{P_i} = (Q_i)_{P_i} = (R \setminus P_i)^{-1} \cdot Q_i$ $\Longrightarrow$ not (yet) the Q_i, but $Q_i \otimes R_{P_i}$ is uniquely det.

<u>remains to show</u>: $Q, Q' = P$-primary, $\left[Q_P = Q'_P \stackrel{?}{\Longrightarrow} Q = Q' \right]$

<u>Proof</u>. Let $q \in Q \stackrel{?}{\rightsquigarrow} (q \in Q')$

$\hookrightarrow q/1 \in Q_P = Q'_P \Rightarrow \exists\, q' \in Q' : \frac{q}{1} = \frac{q'}{s}$ $\quad$ s $\notin$ P $(s \in R \setminus P) \Rightarrow \exists\, t \in R \setminus P$ $\quad$ $t \notin P$

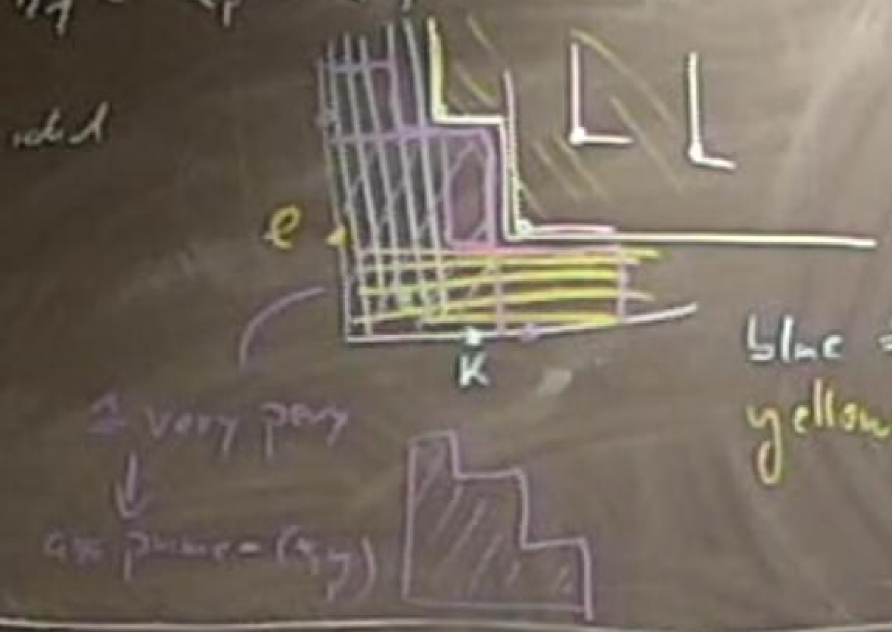
$t(qs - q') = 0 \Rightarrow \in Q'$

$\Rightarrow qs - q' \in Q'$

$\Rightarrow qs \in Q'$

$\Rightarrow q \in Q'$

<u>Example</u>: $k[x,y] \supseteq I$ = monomial ideal



blue $\hat{=}$ $(x^k) \rightsquigarrow (x)$

yellow $\hat{=}$ $(y^l) \rightsquigarrow (y)$

⑥ <u>Integral ring extensions</u>

R without zero-div. = „<u>integral domain</u>"

<u>Def</u> $A \subseteq B$ rings, $b \in B$. b is called <u>integral</u> over A $\Longleftrightarrow$
$\exists$ equation $1\, b^n + a_1 b^{n-1} + a_2 b^{n-2} + \dots + a_{n-1} b + a_n = 0 \quad \mid a_i \in A$ $(a_0 = 1)$.

<u>Example</u>: $\mathbb{Z} \subseteq \mathbb{R}$: • $\sqrt{2}$ is integral over $\mathbb{Z}$: $\sqrt{2}^2 - 2 = 0$
$a_2 = -2$ (in $\mathbb{Z}$)

• π is <u>not</u> integral because it is transcendental over $\mathbb{Q}$, i.e.:
$\nexists\ a_0 \pi^n + \dots + a_n = 0$ with $a_i \in \mathbb{Z}$

• $\frac{1}{2} \in \mathbb{R}$ is not integral: $\left(\frac{1}{2}\right)^n + a_1 \left(\frac{1}{2}\right)^{n-1} + \dots + a_n = 0 \quad (a_i \in \mathbb{Z})$

$(\cdot 2^n) \Rightarrow 1 + \underbrace{2a_1 + 4a_2 + \dots + 2^n a_n}_{\text{even}} = 0$

(but: $2 \cdot \left(\frac{1}{2}\right)^1 - 1 = 0$)

<u>Def</u>. R = domain. R = „integrally closed in $\mathrm{Quot}\, R$" = „<u>normal</u>" $\Longleftrightarrow$ If $r \in \mathrm{Quot}\, R$ is integral over R, then $r \in R$.

<u>Proposition</u>. R = factorial domain (<u>e.g.</u> R = principal ideal domain) $\Rightarrow$ R = normal.

(<u>examples</u>: • $k[x]$
• $k[x_1, \dots, x_n]$ (<u>not</u> PI-domain; still factorial: „Gauß-Lemma")
<u>counterex</u>: $\mathbb{Z}[x]/_{(x^2+5)} = \mathbb{Z}[\sqrt{-5}] \neq$ factorial)

<u>Proof</u> Let $r = \frac{a}{b} \in \mathrm{Quot}\, R$ with $\gcd(a,b) = 1$, $\quad r^n + c_1 r^{n-1} + \dots + c_n = 0 \quad (c_i \in R) \quad / \cdot b^n$

$a^n + \underbrace{c_1 a^{n-1} b + \dots + c_n b^n}_{\text{divisible by } b} = 0 \Rightarrow b \mid a^n$ ↯ $\square$

<u>Example</u>. $\mathbb{Z}[\sqrt{5}] \supset \mathbb{Z}$ not normal!!

$\alpha = \frac{\sqrt{5}+1}{2} \Rightarrow 2\alpha = \sqrt{5} + 1$
$\Rightarrow (2\alpha - 1) = \sqrt{5} \Rightarrow 4\alpha^2 - 4\alpha + 1 = 5 \Rightarrow 4\alpha^2 - 4\alpha - 4 = 0$
$\Rightarrow \boxed{\alpha^2 - \alpha - 1 = 0}$ α is integral over $\mathbb{Z}$!

$A \subseteq B \rightsquigarrow \overline{A}^{(B)} := \{ b \in B \mid b \text{ integral over } A \} \Rightarrow A \subseteq \overline{A}^{(B)} \subset B$

<u>Q</u>. Is this a subring? (YES)

<u>have to show</u>: $\exists\ b^n + a_1 b^{n-1} + \dots + a_n = 0$, $c^m + a'_1 c^{m-1} + \dots + a'_m = 0$ $\}$ $a_i, a'_i \in A$, $b, c \in B$ $\stackrel{?}{\Longrightarrow}$ $(b+c)^N + \dots = 0$ (?)

Proposition. Let $A \subseteq B$, $b \in B$. Then t.f.a.e.:

1) b is integral over A

2) $A \subseteq A[b] \subseteq B$ $\rightsquigarrow$ $\boxed{A[b] \text{ is „finite } A\text{-algebra"}}$, i.e. is an A-algebra (restricting $A[b]$ into an A-module) which is f.g. as an $\underline{A\text{-module}}$!

$A[x]$ $\{f(b) \mid f \in A[x]\}$ = smallest subring of B containing A and b

3) $\exists$ finite A-algebra C: $A \subseteq A[b] \subseteq C \subseteq B$.

4) $\exists$ (f.g.) $A[b]$-module M such that: • f.g. as an A-module • $\mathrm{Ann}_{A[b]} M = 0$

(eg.: • $\mathbb{Z}[x]$ = f.g. $\mathbb{Z}$-algebra / $\neq$ finite $\mathbb{Z}$-algebra; gen. set as a module $\{x^i \mid i \in \mathbb{N}\}$

• $\mathbb{Z}[\sqrt{5}]$ is a finite $\mathbb{Z}$-algebra: // generators: $\{1, \sqrt{5}\}$ $\{a + b \cdot \sqrt{5}\}$)

Proof. $(1) \rightarrow (2)$ $b^n + \boxed{a_1 b^{n-1} + \dots + a_{n-1} b + a_n} = 0$

$= -b^n$

Claim. $A[b]$ is gen. by $1, b, \dots, b^{n-1}$ as an A-module

(Know: $A[b]$ is gen by b as A-algebra $\rightsquigarrow$ is gen as an A-module by $\{1, b, b^2, b^3, \dots\}$)

$\Uparrow$ — because: $b^n = -\boxed{\text{yellow Sum}}$

$b^{n+1} = -b \cdot \text{yellow Sum}$

$(4) \Rightarrow (1)$ Cayley-Hamilton (I-(n)): M = f.g. R-module, $\varphi: M \longrightarrow M \Rightarrow \exists f(x) \in R[x]$, highest coeff $= 1$, $f(\varphi) = 0$ inside $\mathrm{End}(M)$ ($f(\varphi)(M) = 0$)

$M = M$, $R = A$, $\varphi = (\cdot b) \Rightarrow M \xrightarrow{\cdot b} M \Rightarrow$ • $f(x) \in A[x]$, highest coef $= 1$

• $f(b) = 0$ as a hom. $M \to M$

i.e. $\forall m \in M$: $f(b) \cdot m = 0$ in M, i.e. $f(b) \in \mathrm{Ann}_{A[b]} M$

$f(b) = 0$ in $A[b] \subseteq B$ $\Longleftarrow$

Corollary: $b, b' \in B \supseteq A$

If b, b' = integral over A $\Rightarrow$ $A[b]$ = f.g. $\underline{A}$-module, $A[b']$ = (-"-) A-mod $\Rightarrow$ $A[b, b']$ = f.g. $\underline{A[b]}$-module $\}$ $A[b, b']$ = f.g. A-module $\| \; C \ni b, b' \ni b + b', b \cdot b'$

Proof. $A[b] = Ax_1 + \dots + Ax_k$, $A[b, b'] = A[b] \cdot y_1 + \dots + A[b] y_\ell$ $\}$ $A[b, b'] = \sum_{\substack{i = 1, \dots, k \\ j = 1, \dots, \ell}} x_i y_j \cdot A$

Corollary. $A \subseteq B$: b is integral over A, c is integral over $A[b]$ $\}$ c is integr. over A.

$A \subseteq B \subseteq C \Rightarrow [B|A, C|B \text{ integral} \rightarrow C|A \text{ integral}]$

Def. $A \xrightarrow{f} B$ is called integral $:\Longleftrightarrow$ all elements of B are integral over $f(A)$.

Remark: $A \to B$ integral $\rightarrow$ [f.g. as algebra $\Longleftrightarrow$ „finite algebra"]

Theorem. Let $A \subseteq B$ be integral

1. $A \subseteq B$ be domains. Then A = field iff B = field.
2. $Q \subseteq B$ is maximal $\Longleftrightarrow$ $Q \cap A$ is maximal.

$A \subseteq B \rightsquigarrow \mathrm{Spec}\, B \xrightarrow{\Phi} \mathrm{Spec}\, A$, $Q \longmapsto Q \cap A$, PI in B

Proof. 1. $(\Rightarrow)$ Let $b \in B$, goal: $\frac{1}{b} \in B$

$\Downarrow$

$b^n + a_n b^{n-1} + \dots + a_0 = 0 \quad | \cdot \frac{1}{b}$ inside $\mathrm{Quot}\, B$

$\underbrace{b^{n-1} + \dots + a_1}_{\in B} + a_0/b = 0 \Rightarrow \frac{a_0}{b} \in B$ and $a_0 \neq 0$

$\Rightarrow \frac{1}{b} = (a_0/b) / a_0$

$(\Leftarrow)$ $a \in A$, B = field

$\Downarrow$

$\frac{1}{a} \in \mathrm{Quot}\, A \subseteq \mathrm{Quot}\, B = B \Rightarrow \frac{1}{a}$ = integral over $A \Rightarrow \left(\frac{1}{a}\right)^n + a_1 \left(\frac{1}{a}\right)^{n-1} + \dots + a_0 = 0$

$(\cdot a^{n-1}) \Rightarrow \frac{1}{a} + \underbrace{a_1 + a_2 a + \dots + a_0 a^{n-1}}_{\in A} = 0$ A = field!!

2. $A \to B \rightsquigarrow A \to B \twoheadrightarrow B/Q$

$\rightsquigarrow A/A \cap Q \underset{\text{integral}}{\hookrightarrow} B/Q$

Theorem (3) Φ is injective in chains, i.e. If $Q_1 \subseteq Q_2$ (in $\mathrm{Spec}\, B$), $Q_1 \cap A = Q_2 \cap A \Rightarrow Q_1 = Q_2$.

Proof $P = Q_1 \cap A = Q_2 \cap A$ $\rightsquigarrow$ localize $\underline{\text{everything}}$ by $S := (A \setminus P)$ i.e. $\otimes_A A_P$

$A \rightsquigarrow A_P$, $B \rightsquigarrow S^{-1}B$ $\Rightarrow P \subseteq A_P$ is $\underline{\text{the}}$ max ideal

$A_P \longrightarrow S^{-1}B$ still integral

$\mathrm{U}\!\mathrm{I}$ $\qquad$ $\mathrm{U}\!\mathrm{I}$

$[P = \max]$ $\quad$ $(Q_1 \subsetneq Q_2) \overset{(2)}{\Longrightarrow} Q_1, Q_2$ = both maximal. $\lightning$ /

Φ

Theorem: (4) („going up theorem") Φ is surjective: even more

$\mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$

$P_2 \supseteq P_1$, $Q_1 \longmapsto P_1$ $\}$ $\Rightarrow \exists Q_2 \supseteq Q_1 : Q_2 \longmapsto P_2$.

(5) „going down thm" A, B = domains, A = normal

$\Rightarrow$ similarly for $Q_2 \longmapsto P_2$, $\mathrm{U}\!\mathrm{I}$, $? \longmapsto P_1$

Proof: (4) 1st case: (A, m) = local ring; $P_2 = m$, $\rightsquigarrow$ choose $\underline{\text{some}}$ max. ideal $Q_2 \supseteq Q_1 \rightsquigarrow Q_2 \cap A$ = max. id. in $A \Rightarrow = m$

gen. case: $S := (A \setminus P_2) \rightsquigarrow$ localize. //