$A \xrightarrow{f} B$ ring hom, i.e. $B = A$-algebra

s.th. $B =$ f.g. A-algebra

Extremal cases: ① $A \overset{f}{\twoheadrightarrow} B$ surjective

$\Downarrow \implies f$ is automatically finite

$I := \ker f$, $A/I \xrightarrow{\sim} B$

② $A \overset{f}{\hookrightarrow} B$ injective, and still finite

Def $f =$ finite $\iff B =$ integral A-algebra, i.e. $\forall b \in B$ integral over $f(A)$.

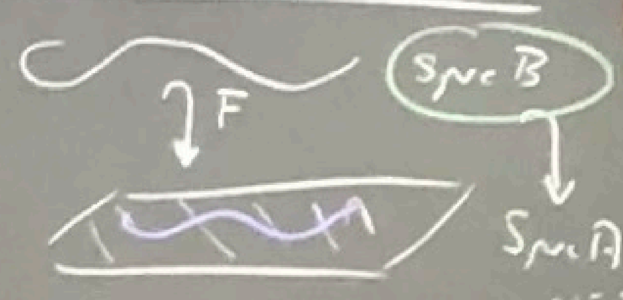
$\Big\} \rightsquigarrow \underset{F}{(\operatorname{Spec} f)} : \operatorname{Spec} B \to \operatorname{Spec} A$, $\operatorname{Spec} B = \operatorname{Spec} A/I = V(I)$

$\rightsquigarrow \bullet \overset{F}{(\operatorname{Spec} f)} : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective

$\bullet$ fibers: $P \in \operatorname{Spec} A \rightsquigarrow F^{-1}(P) \neq \emptyset$

$= \boxed{\text{finite sets.}}$

$$F^{-1}(P) = \{ Q \in \operatorname{Spec} B \mid \boxed{F(Q) = f^{-1}(Q)} = P \}$$

① 

$\operatorname{Spec} A = \mathbb{A}^n_k$

Exple. $B =$ f.g. k-algebra

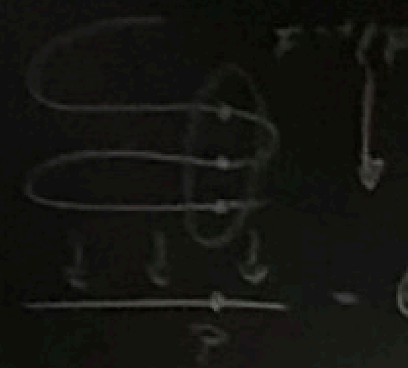
$\Updownarrow$

$B = k[x_1, \dots, x_n]/I$ i.e. $A = k[x_1, \dots, x_n]$

$F : \operatorname{Spec} B \to \operatorname{Spec} A$

$Q \longmapsto f^{-1}(Q)$

② $\operatorname{Spec} B \xrightarrow{F} \operatorname{Spec} A$ finite fibers



Def $F : X \to Y$ (...) is called quasi-finite $\iff$ all fibers $F^{-1}(y)$ $(y \in Y)$ are finite

Prop: $f : A \to B$ finite $\implies (\operatorname{Spec} f) : \operatorname{Spec} B \to \operatorname{Spec} A$ is quasi-finite

Proof. $\nearrow$ Exercise.

Theorem $A =$ domain, $\operatorname{Quot} A \hookrightarrow L$ - finite field extension. (e.g. $A = \mathbb{Z}$, $L = \mathbb{Q}(\sqrt{2})$)

$B := \overline{A}^{(L)} := \{ \ell \in L \mid \ell \text{ is integral over } A \}$ (special case: $L = \operatorname{Quot} A \rightsquigarrow \overline{A}^{(\operatorname{Quot} A)} = \overline{A}$ „Normalization of A")

$\Downarrow$ ($\rightsquigarrow$ $A \xhookrightarrow{\text{integral}} B$, $\operatorname{Quot} A \hookrightarrow L$)

$B =$ f.g. A-algebra, and, thus: $B =$ f.g. A-module, i.e. $A \to B$ is finite.

Weierstraß preparation theorem: „W. Vorbereitungssatz" (trivial version)

$k[x_1, \dots, x_n] = k[\underline{x}]$; $k =$ field with $\# k = \infty$ (e.g. $k = \mathbb{C} \implies$ ok)

$f \in k[\underline{x}] \rightsquigarrow$ „degree":

$\bullet$ monomial $x_1^{r_1} \cdot \ldots \cdot x_n^{r_n} = x^r$ $(r_1, \dots, r_n \in \mathbb{N},\ r = (r_1, \dots, r_n) \in \mathbb{N}^n)$

$\rightsquigarrow \deg(x^r) := \sum_{i=1}^n r_i$ $(\deg x^3 y^2 z^5 = 10)$

$\bullet$ $f = \sum_{\substack{r \in \mathbb{N}^n \\ \text{finite}}} \lambda_r \cdot x^r$ $(\lambda_r \in k)$

Ex $\deg(x^2 + y^3 z - z^3) = 4$

$\cdot$ homogeneous, e.g.: $x^2 y^5 - 2 \cdot x y^6 + 3 z^7$

$\cdot$ general: $\deg f := \max \{ \deg x^r \mid \lambda_r \neq 0 \}$

Theorem: Let $f \in k[\underline{x}]$ with $\deg f = N$.

$\implies \exists\, a_i \in k \rightsquigarrow \psi : k[\underline{x}] \xrightarrow{\sim} k[\underline{x}]$

$x_i \longmapsto x_i + a_i x_n$ $(i = 1, \dots, n-1)$

$x_n \longmapsto x_n$

$\Big\}$ linear coordinate change

alternative writing: substitution

$z_n = x_n$

$z_i := x_i + a_i x_n$

s.t. $\psi(f) = (\boxed{c \neq 0}) \cdot x_n^N + \sum_{v=1}^N c_v(x_1, \dots, x_{n-1}) \cdot x_n^{N-v}$

$c_v \in k[x_1, \dots, x_{n-1}][x_n]$, $\deg c_v \leq v$

(ex $f = x_1^2 + x_2 x_3 + 0 \cdot x_3^2$

$a_1 = a_2 = 2$

$\psi(f) = (x_1 + 2x_3)^2 + (x_2 + 2x_3) \cdot x_3 = \dots$

$(z_1 + 2z_3)^2 + (z_2 + 2z_3) z_3 \in k[z_1, z_2, z_3]$

Proof: $\psi(x^r = x_1^{r_1} \cdot \ldots \cdot x_n^{r_n}) = (x_1 + a_1 x_n)^{r_1} \cdot \ldots \cdot (x_{n-1} + a_{n-1} x_n)^{r_{n-1}} \cdot x_n^{r_n} = (?) \cdot x_n^N +$ remaining terms

such that $\sum r_i = N$; $a_1 x_n$, $a_{n-1} x_n$, x_n, $a_1^{r_1} \cdot \ldots \cdot a_{n-1}^{r_{n-1}}$

i.e. $\psi(x^r) = a^r \cdot x_n^N +$ lower terms with $a = (a_1, a_2, \dots, a_{n-1}, \underset{a_n}{1})$; $a^r = a_1^{r_1} \cdot \ldots \cdot a_{n-1}^{r_{n-1}} \cdot a_n^{r_n}$

$\psi(f) = \psi(f_{|\deg N}) +$ lower x_n-terms

$= f_{|\deg N}(a) \cdot x_n^N +$ lower terms

$f_{|\deg N}(a) = f_{|\deg N}(a_1, a_2, \dots, a_{n-1}, 1)$

NOW: Choose a, s.t. $f_{|\deg N}(a) \neq 0$. $\square$

$\boxed{\mathbb{Z} \subset \mathbb{Q};\ L = \mathbb{Q}(\sqrt{-5}),\ \overline{\mathbb{Z}} = \mathbb{Z}[\sqrt{-5}] \supset \mathbb{Z} = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}}$

Prop. („Noether normalization") Let $A =$ f.g. k-algebra, i.e. $\exists\, k[x_1, \dots, x_n] \twoheadrightarrow A$.

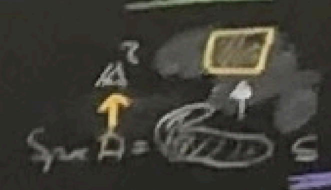
$\implies \exists$ linear combinations, i.e. $y_i \in \operatorname{Span}_k(x_1, \dots, x_n)$ $(i = 1, \dots, d \leq n)$ such that: ① $y_1, \dots, y_d$ = lin. independent

$\exists\, d$

② $k[y_1, \dots, y_d] \subseteq k[x_1, \dots, x_n] \twoheadrightarrow A$ $\varphi =$ injective and finite

$\|$ polynomial ring

φ



$\operatorname{Spec} A \subseteq \mathbb{A}^3$

(i.e. we reach the situation of ex ② of the upper black board,

Exple. $A = k[x, y, z]/xy - z^2 \rightsquigarrow k[x, y] \hookrightarrow (k[x, y, z] \twoheadrightarrow A)$

φ still inj., integral: $z^2 - xy = 0$ in A

Proof $k[x_1,\dots,x_n] \xrightarrow{\varphi} A$ ⇝ φ is finite.

1 Check: φ = injective (?) → YES ⇒ Done
→ NO ⇒ $\exists f \in \operatorname{Ker}\varphi \subseteq k[x_1,\dots,x_n]$

Weierstraß Satz ⇝ w.l.o.g.: $f(x_1,\dots,x_n) = x_n^N + \underbrace{\sum c_i(x_1,\dots,x_{n-1}) \cdot x_n^i}_{\text{lower degrees in } x_n}$

⇒ x_n = integral over $k[x_1,\dots,x_{n-1}]$ **not** in $k[x_1,\dots,x_n]$, **but** in $k[x_1,\dots,x_n]/(f)$

$$\Rightarrow k[x_1,\dots,x_{n-1}] \hookrightarrow k[x_1,\dots,x_n] \twoheadrightarrow k[x_1,\dots,x_n]/(f) \xrightarrow{\varphi} A \Rightarrow k[x_1,\dots,x_{n-1}] \xrightarrow{\varphi_1} A$$

becomes finite.

finite, finite, finite

HNS (0) Cool version
- k = field, $k \hookrightarrow A$ is a f.g. k-algebra
- A = field ⇒ $k \hookrightarrow A$ is finite, i.e. A = finite field extension of k: $[A:k] < \infty$

Proof: $A = k[x_1,\dots,x_n]/I$, $a_i := \bar{x}_i$, i.e. $a_1,\dots,a_n \in A$ generate A as a k-algebra. (k, A = fields)

Induction by n: $n=0$ ⇝ $A = k$

$(n-1) \rightsquigarrow n$: $k \hookrightarrow k[a_1] \hookrightarrow k(a_1) \hookrightarrow A$ $\xRightarrow{\text{Ind. hyp.}}$ A = finite (integral) over $k(a_1)$

$k[a_1]_f$, $k(a_1) = \operatorname{Quot} k[a_1]$, $A = k(a_1)[a_2,\dots,a_n]$ (because A = field)

⇑ $a_2,\dots,a_n \in A$ are integral over $k(a_1)$

$\Rightarrow \boxed{a_2}^K + \lambda_1 a_2^{K-1} + \dots + \lambda_K a_2^0 = 0$ ($\lambda_i \in k(a_1) = \operatorname{Quot} k[a_1]$)

⋮

$\boxed{a_n}^\ell + \dots = 0$

$k[a_1] \ni f :=$ common denominator of all $\lambda_i \in \operatorname{Quot} k[a_1]$

⇒ all $\lambda_i \in k[a_1]_f \subset \operatorname{Quot} k[a_1]$

⇒ $a_2,\dots,a_n \in A$ are finite over $k[a_1]_f$ ⇒ $k[a_1]_f \hookrightarrow A$ is finite | integral

A = field ⇒ $k[a_1]_f$ = field

1. Case: a_1 = algebraic over k ($\Leftrightarrow k[a_1] = k(a_1)$, $[k(a_1):k] < \infty$)

$k \underset{\text{finite}}{\hookrightarrow} k(a_1) \underset{\text{finite}}{\hookrightarrow} A \Rightarrow k \hookrightarrow A$ is finite.

2nd Case: a_1 = transcendental ⇒ $k[a_1] = k[x]$ = polynomial ring ⇒ $k[x]_f$ cannot be a field! ↯ □

Corollary: If additionally: $k = \bar{k}$ ⇒ A from the claim: $A = k$ //

Theorem (HNS) Let $k = \bar{k}$.

① If $m \subseteq k[x_1,\dots,x_n]$ is a max. ideal ⇒ $\exists c = (c_1,\dots,c_n) \in k^n$: $m = m_c := (x_1 - c_1, \dots, x_n - c_n)$

(classical)

② $J \subseteq k[x_1,\dots,x_n]$, $V(J) \subseteq k^n$ $V(J) = \emptyset$ ⇒ $J = (1)$

③ $J \subseteq k[x_1,\dots,x_n]$ ⇒ $I(V(J)) = \sqrt{J}$.

Proof. ① Let m = max. ideal
$$k \hookrightarrow k[x_1,\dots,x_n] \twoheadrightarrow k[x_1,\dots,x_n]/m = A \text{ (field)} \Rightarrow \Phi = \text{Iso}$$
$\Rightarrow \exists\, c_i \mapsto \bar{x}_i$, $\bar{x}_i = c_i$ ⇒ $\bar{x}_i - c_i = 0$ in A ⇒ $x_i - c_i \in m$

⇒ $m_c \subseteq m$ ⇒ $\boxed{m_c = m}$

② Let $J \neq (1)$ ⇒ $\exists m$: $J \subseteq \overset{\text{max ideal}}{m} \overset{①}{=} m_c = (x_1 - c_1, \dots, x_n - c_n)$ with $f = x_1 - c_1, \dots, f = x_n - c_n$

$V(J) = \{(b_1,\dots,b_n) = b \in k^n \mid f(b) = 0\ \forall f \in J\}$ //

③ we already know: $\sqrt{J} \subseteq I(V(J))$, have to show the opposite:

Let $f \in I(V(J)) \subseteq k[x_1,\dots,x_n]$, i.e. $f(b) = 0$ for all $b \in V(J) \subseteq k^n$.

Consider $k[x_1,\dots,x_n,t]$ ⇝ $g(x,t) := f(x) \cdot t - 1 \in k[x_1,\dots,x_n]$

$\cup$ ideal $J' := (J \cdot k[x_1,\dots,x_n,t],\ g(x,t))$ ⇝ $V(J') \subseteq k^{n+1}$

Assume $(b_1,\dots,b_n,c) \in V(J')$ ⇒ $\forall h(\underline{x}) \in J$: $h(b_1,\dots,b_n) = 0$, i.e. $b \in V(J)$

⇒ $f(b_1,\dots,b_n) = 0$

$g(\underline{b},c) = \underset{=0}{f(\underline{b})} \cdot c - 1 \neq 0$ ⇒ $(\underline{b},c) \notin V(J')$ **THUS**: $V(J') = \emptyset$

① ⇒ $J' = (1)$ in $k[\underline{x},t]$ ⇒ $1 = \sum \mu_j(\underline{x},t) \cdot h_j(\underline{x}) + \mu(\underline{x},t) \cdot (f(\underline{x}) \cdot t - 1)$, $h_j \in J$

$t := \frac{1}{f(\underline{x})}$ ⇝ into $k(x_1,\dots,x_n)$

$k[\underline{x},t] \to k(\underline{x})$, $x_i \mapsto x_i$, $t \mapsto \frac{1}{f}$

$1 = \sum \mu_j\left(\underline{x}, \frac{1}{f(\underline{x})}\right) h_j(\underline{x}) + 0 \quad \Big| \cdot f(\underline{x})^N$

$f(\underline{x})^N = \sum (\text{Polynomials in } \underline{x}) \cdot h_j(\underline{x}) \in J$ ⇒ $f \in \sqrt{J}$ □