⑧ Projective resolutions

R = ring, M = R-module ⇝ [M = „flat" :⟺ $\otimes_R M$ is exact]

[P = „projective" :⟺ $\mathrm{Hom}_R(P, \cdot)$ exact]

i.e. $[N \twoheadrightarrow N') \Longrightarrow [\mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N')$ stays surjective

Prop: P = R-module. Then

P = projective ⟺ each surjection $M \xrightarrow{\pi} P$ splits

(i.e. $0 \to K \to M \xrightarrow{\pi} P \to 0$ splits, i.e. $M \cong P \oplus P'$)

$$\begin{tikzcd} M \cong P \oplus P' \arrow[d, "\pi"'] \arrow[dl, phantom] \\ P \arrow[u, phantom] \end{tikzcd}$$

$\pi \downarrow \quad \swarrow pr_1$

⟺ P is a direct summand of a free R-module.

If this is the case, then P is flat. („free" ⟹ „projective" ⟹ „flat")

Proof: (proj ⟹ split) $M \xrightarrow{\pi} P$ ⇝ $\mathrm{Hom}_R(P,M) \twoheadrightarrow \mathrm{Hom}_R(P,P)$ is surj., too.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota"] & M \arrow[r, "\pi"] & P \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "\pi"] & P \\ P \arrow[u, "f"] \arrow[ur, "e.g.: id_P"'] & \end{tikzcd}$$

$\exists f \longmapsto id_P$

f: $\pi \circ f = id_P$ means splitting of the sequence!

(split ⟹ summand) $I \subseteq P$ should be a subset generating the R-module P.

$R^{\oplus I} = R^I \xrightarrow{\pi} P$ (π is surjective) ⟹ ∃ splitting: $M \cong \boxed{P \oplus P'} \xrightarrow[pr_1]{\pi} P$

$M = R^I$

(summand ⟹ projective) Let $P \oplus P' = R^I$ ⟹ $\underbrace{\mathrm{Hom}(P,N) \oplus \mathrm{Hom}(P',N)}_{\text{exact}} = \mathrm{Hom}(R^I, N)$ ⟸ exact: $N^I = N^{\oplus I}$

⟹ If $0 \to N_1 \to \boxed{N_2 \to N_3} \to 0$

$$\begin{bmatrix} \mathrm{Hom}(P,N_2) \longrightarrow \mathrm{Hom}(P,N_3) \\ \oplus \qquad\qquad \oplus \\ \mathrm{Hom}(P',N_2) \longrightarrow \mathrm{Hom}(P',N_3) \end{bmatrix}$$

⇓ together: surjective! ⟹ the single maps are surjective, too!

flatness: $P \oplus P' = R^I$ ⟹ we use that R^I = flat ⟹ $N \hookrightarrow N'$ ⟹ $N \otimes_R (P \oplus P') \hookrightarrow N' \otimes (P \oplus P')$

$N \otimes P \to N' \otimes P$ is injective! ⟸ $(N \otimes_R P) \oplus (N \otimes_R P') \hookrightarrow (N' \otimes P) \oplus (N' \otimes P')$

Remark If P = direct summand of, e.g., $R^2 = R \oplus R$

i.e. $\exists P'$ $P \oplus P' \cong R \oplus R$

$\not\Longrightarrow$ $P \cong R$ or 0 or R^2

Exmple: $I := (2, 1+\sqrt{-5}) \subseteq R = \mathbb{Z}[\sqrt{-5}]$ • $I \neq$ principal ⟹ $I \neq$ free

• but I = projective, in particular: ∃ M = R-module: $I \oplus M \cong R^2$

Prop: If M = R-module of finite presentation (e.g. f.g. + R = noetherian)

(i.e. $\exists R^k \to R^\ell \to M \to 0$)

then: M = projective ⟺ $\forall m \in \mathrm{MaxSpec}\, R$: M_m is a projective R_m-module.

$M_m = M \otimes_R R_m$

before Remark: Let $R \to S$ be an R-algebra

If P is projective over R ⟹ $P \otimes_R S$ is projective over S.

(⇝ implies (⟹) in Prop above)

Proof (of remark):

$$P \oplus P' = R^I \xrightarrow{\otimes_R S} (P \otimes_R S) \oplus (P' \otimes_R S) = (P \oplus P') \otimes_R S = R^I \otimes_R S = S^I$$

Proof (proposition). Recall: • $R \to S$ flat algebra } ⟹ $\boxed{\mathrm{Hom}_R(M,N) \otimes S \xrightarrow{\sim} \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)}$

• M is finitely presented

M = projective ⟸ $\forall N \twoheadrightarrow N'$: $\mathrm{Hom}_R(M,N) \twoheadrightarrow \mathrm{Hom}_R(M,N')$ surj.

$\forall m$: M_m = projective (over R_m) ⟺ $\forall \bar N \to \bar N'$: $\mathrm{Hom}_{R_m}(M_m, \bar N) \twoheadrightarrow \mathrm{Hom}_{R_m}(M_m, \bar N')$ surj

R_m-modules

e.g.: N_m N'_m

$\mathrm{Hom}_{R_m}(M_m, N_m) \twoheadrightarrow \mathrm{Hom}_{R_m}(M_m, N'_m)$

$\boxed{\mathrm{Hom}_R(M,N)}_m \longrightarrow \boxed{\mathrm{Hom}_R(M,N')}_m$ surjective ($\forall m$)

L $\qquad L'$

blue box for $S = R_m$

Recall: L, L' = R-modules ⟹ [$\varphi: L \to L'$ is surj ⟺ $\forall m$: φ_m is surjective]

back to the exmpl: $(2, 1+\sqrt{-5})$ is proj ⟺ it is locally projective ⟸ locally free: ↗ Exercise class!

⟹ it is nice/important to know how [illegible] proj. modules over local (R, m)

Prop: P = projective and of finite presentation over (R, m) = local ⟹ P = free module.

Proof: Choose $R^k \twoheadrightarrow P$ such that $(R/m)^k \overset{\sim}{\underset{\pi}{=}} P \otimes_R \boxed{R/m} =: \bar P$ ⇝ $0 \to P' \to R^k \xrightarrow{\pi} P \to 0$ splits

$R/m = k$

i.e. $R^k \cong P' \oplus P$

$(P \oplus P') \otimes_R k \xrightarrow[\pi]{\sim} P \otimes k$

⟹ $P' \otimes_R k = 0$, P' = f.g. (P = f. pres.)

⇓ NAK

$P' = 0$ ⟹ $\pi: R^k \to P$ is an isom. □

$\mathcal{A}$ = abelian category, ex: $\mathcal{A} = \{R\text{-modules}\}$ (for a fixed ring R)

Complexes: $M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1}$ $d^2=0$, i.e. $d_i \circ d_{i+1} = 0 \ \forall i$

$\Updownarrow$

i-th homology: $H_i(M_\cdot) = Z_i/B_i \Longleftarrow$ $B_i(M_\cdot) \subseteq Z_i(M_\cdot)$

$$\left.\begin{matrix} B_i(M_\cdot) & \subseteq & Z_i(M_\cdot) \\ \| & & \| \\ \mathrm{im}(d_{i+1}) & \subseteq & \mathrm{Ker}(d_i) \end{matrix}\right\} \subseteq M_i$$

H_i is functorial [complex-map Φ] $\longmapsto H_i(\Phi)$

$$\begin{tikzcd}
M_\cdot \arrow[d, "\Phi"] & \dots \arrow[r] & M_{i+1} \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d, "\Phi_i"] & M_{i-1} \arrow[r] \arrow[d, "\Phi_{i-1}"] & \dots \\
N_\cdot & \dots \arrow[r] & N_{i+1} \arrow[r] & N_i \arrow[r] & N_{i-1} \arrow[r] & \dots
\end{tikzcd}$$

Commuting squares $\Longrightarrow$

$$\begin{tikzcd}
H_i(M_\cdot) \arrow[d, "H_i(\Phi)"] \\
H_i(N_\cdot)
\end{tikzcd}$$

built from

$$\begin{tikzcd}
B_i(M) \subseteq Z_i(M) \arrow[d] \\
B_i(N_\cdot) \subseteq Z_i(N_\cdot)
\end{tikzcd}$$

- new complex out of $M_\cdot$: $(M_\cdot[k])_i := M_{i-k}$, $k \in \mathbb{N}$; keep the differentials, except: $d_{M_\cdot[1]} := -d_{M_\cdot}$
- $M^i := M_{-i}$

Assume we have an exact sequence of complexes: $0 \to A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \to 0$ (means: • f, g = complex maps • $\forall i$ $0 \to A_i \to B_i \to C_i \to 0$ exact)

$$\Longrightarrow \boxed{\xrightarrow[\partial_{i+1}]{\exists} H_i(A) \xrightarrow{H_i(f)} H_i(B) \xrightarrow{H_i(g)} H_i(C) \xrightarrow[\partial_i]{\exists} H_{i-1}(A) \xrightarrow{H_{i-1}(f)} H_{i-1}(B) \to H_{i-1}(C) \xrightarrow{\exists} \dots}$$

(*) "long exact sequence of homology"

Prop: the maps ∂_i ($\forall i \in \mathbb{Z}$) constructed below have the following properties:

(i) they are functorial, i.e.

$$\begin{tikzcd}
0 \arrow[r] & A_\cdot \arrow[r] \arrow[d] & B_\cdot \arrow[r] \arrow[d] & C_\cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'_\cdot \arrow[r] & B'_\cdot \arrow[r] & C'_\cdot \arrow[r] & 0
\end{tikzcd} \rightsquigarrow \begin{tikzcd}
H_i(C) \arrow[r, "\partial_i"] \arrow[d] & H_{i-1}(A) \arrow[d] \\
H_i(C') \arrow[r, "\partial'_i"] & H_{i-1}(A')
\end{tikzcd}$$

(ii) $\forall$ $0 \to A \to B \to C \to 0 \Longrightarrow$ (*) is exact!

Proof (construction of ∂)

$$\begin{tikzcd}
0 \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & B_{i+1} \arrow[r] \arrow[d] & C_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_i \arrow[r] \arrow[d] & B_i \arrow[r] \arrow[d] & \boxed{C_i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r] & B_{i-1} \arrow[r] & C_{i-1} \arrow[r] & 0
\end{tikzcd}$$

$\exists y \longmapsto x$, $y \longmapsto z \longmapsto 0$, $z \longmapsto z$

$\partial: H_i(C_\cdot) \to H_{i-1}(A_\cdot)$

$x \in Z_i(C_\cdot)$

□

- Def: $f: M_\cdot \to N_\cdot$ is called a quasi-isomorphism ("qis") $\Longleftrightarrow \forall i: H_i(f)$ = isom.
- M = module $\rightsquigarrow$ becomes a complex: $0 \to 0 \to 0 \to M \to 0 \to \dots$ $M_\cdot$
  0-th spot. $\rightsquigarrow H_0(M_\cdot) = M$

Warning: $\exists$ functors $F: \mathrm{Mod} \to \mathrm{Mod}$ which are not exact

ex: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ exact $\quad / \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$

$\Downarrow$

$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow[0]{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{id} \mathbb{Z}/2\mathbb{Z} \to 0$

$\Uparrow$ not exact

$$\begin{tikzcd}
 & {[2]} & {[1]} & {[0]} & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 & M_\cdot \arrow[d, "f"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 & N_\cdot
\end{tikzcd}$$

$H_i(f) = 0 \ \forall i$

$\otimes \mathbb{F}_2$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_2 \arrow[r] \arrow[d, equal] & \mathbb{F}_2 \arrow[r] \arrow[d, equal] & \mathbb{F}_2 \arrow[r] \arrow[d, equal] & 0 & M_\cdot \otimes \mathbb{F}_2 = \bar{M}_\cdot \arrow[d, "f \otimes id_{\mathbb{F}_2}"] \\
0 \arrow[r] & \mathbb{F}_2 \arrow[r] & \mathbb{F}_2 \arrow[r] & \mathbb{F}_2 \arrow[r] & 0 & N_\cdot \otimes \mathbb{F}_2 = \bar{N}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{F}_2 \arrow[d, "id"', "0 \neq"] & = & H_2(\bar{M}_\cdot) \arrow[d, "= id"] \\
\mathbb{F}_2 & & H_2(\bar{N}_\cdot)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & & 0_\cdot \arrow[d, "g"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \quad M_\cdot
\end{tikzcd}$$

g = qis, $g \otimes id_{\mathbb{F}_2} = (\otimes \mathbb{F}_2)(g)$ is not a qis.

$\Longrightarrow$ $\boxed{\text{Functor (qis) might fail to be a qis}}$

Double complex

$$M_{\cdot\cdot} = \begin{tikzcd}
\arrow[r, "d'"] & \arrow[r, "d'"] & \arrow[r, "d'"] & \\
\arrow[r] \arrow[u, phantom, ""] & \arrow[r] & \arrow[r] & \\
\arrow[r] & \arrow[r] & \arrow[r] &
\end{tikzcd}$$

with vertical arrows d''

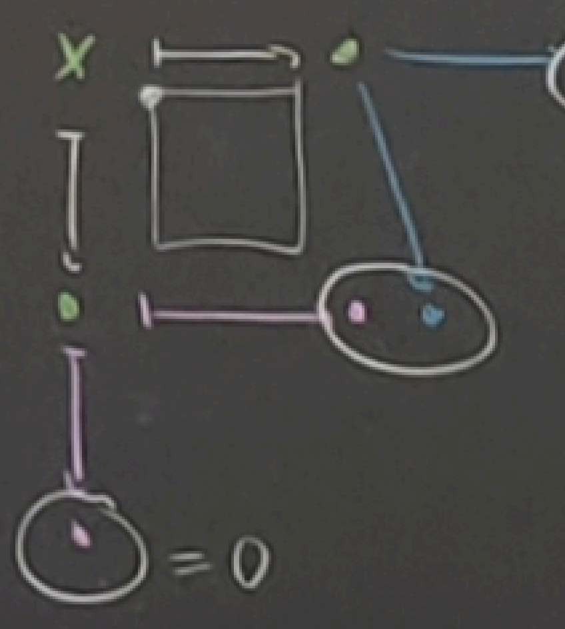
- horizontal/vertical complexes
- commuting squares ≙ $d'd'' + d''d' = 0$

$\Downarrow$

- Tot. $M_{\cdot\cdot}$ = $T_\cdot$ "total complex of $M_{\cdot\cdot}$"

$$\begin{tikzcd}
M_{i,j} \arrow[r, "d'"] \arrow[d, "d''"'] & M_{i,j-1} \arrow[d, "d''"] \\
x \in \boxed{M_{i-1,j}} \arrow[r, "d'"] \arrow[d, "d''"'] & M_{i-1,j-1} \arrow[d, "d''"] \\
M_{i-2,j} \arrow[r, "d'"] & M_{i-2,j-1}
\end{tikzcd}$$

$T_K = T_{i+j-1}$ $(K = i+j-1)$



- $T_k := \bigoplus_{i+j=k} M_{i,j}$
- $d_k: T_k \to T_{k-1}$ $d_k := d' + d''$ Claim: $d_k \circ d_{k+1} = 0 \ \forall k$ (results from !!)

$f: M_\bullet \to N_\bullet$ complex map ⇝ „**mapping cone**" $\mathrm{Cone}(f)$ – complex :

⇓

write both complexes vertically :

$$\begin{tikzcd} & \vdots & \vdots & \\ 0 \arrow[r] & M_{i+1} \arrow[r, "f"] \arrow[d, "-d_M"'] & N_{i+1} \arrow[r] \arrow[d, "d_N"] & 0 \\ 0 \arrow[r] & M_i \arrow[r, "f"] \arrow[d] & N_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_{i-1} \arrow[r, "f"] & N_{i-1} \arrow[r] & 0 \\ & \vdots & \vdots & \\ & {[1]} & {[0]} & \end{tikzcd}$$

$\Longrightarrow$ double complex $\mathrm{Cone}_{\bullet\bullet}(f)$

<u>i.e.</u>

$$\begin{tikzcd} \mathrm{Cone}(f)_i = N_i \arrow[d, "d_N"'] & \oplus & M_{i-1} \arrow[d, "-d_M"] \arrow[dll, "f"'] \\ \mathrm{Cone}(f)_{i-1} = N_{i-1} & \oplus & M_{i-2} \end{tikzcd}$$

<u>i.e</u> $(\mathrm{Cone}\, f)_i = N_{\bullet i} \oplus M[1]_{\bullet i}$ ⟸ is not true as complexes!

$(M_{i-1} = M[1]_i)$

$\mathrm{Cone}(f) := \mathrm{Tot}_\bullet\left(\mathrm{Cone}_{\bullet\bullet}(f)\right)$

they are complex maps (respect the differentials)

$$\boxed{0 \to N_\bullet \hookrightarrow \mathrm{Cone}(f) \longrightarrow M_\bullet[1] \to 0}$$

~~$0 \to M_\bullet[1] \to \mathrm{Cone}(f) \to N_\bullet \to 0$~~

$$\Longrightarrow \exists\ \partial_i : H_i(M_\bullet[1]) \longrightarrow H_{i-1}(N_\bullet)$$

$$H_i(M_\bullet[1]) = H_{i-1}(M_\bullet)$$

Claim: $\partial_i = H_{i-1}(f)$ //

$$\dots \to H_i(\mathrm{cone}) \longrightarrow \boxed{H_{i-1}(M_\bullet) \xrightarrow{f} H_{i-1}(N_\bullet)} \to H_{i-1}(\mathrm{cone}) \to \boxed{H_{i-2}(M_\bullet)} \xrightarrow{f} \dots$$

∂_i

Corollary $f: M_\bullet \to N_\bullet$ is a <u>qis</u> $\Longleftrightarrow$ $\mathrm{Cone}(f) =$ <u>exact</u> complex.

Def.

$$\begin{tikzcd} M_{i+1} \arrow[r] \arrow[d, "f_{i+1}"'] & M_i \arrow[r, "d_i^M"] \arrow[d, "f_i"] \arrow[dl, "h_i"'] & M_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] \arrow[dl, "h_{i-1}"'] & \dots \\ N_{i+1} \arrow[r, "d^N_{i+1}"'] & N_i \arrow[r] & N_{i-1} \arrow[r] & \dots \end{tikzcd}$$

<u>i.e.</u> $f: M_\bullet \to N_\bullet$ is a map of complexes

$\boxed{h}$: $M_\bullet \longrightarrow N_\bullet[-1]$ (<u>not</u> respecting the differentials), i.e

$h_i : M_i \longrightarrow N_{i+1}$ $(i \in \mathbb{Z})$ is called a <u>homotopy</u> $f \sim 0$

(„between f and 0")

$:\Longleftrightarrow f_i = d^N h + h d^M = d^N_{i+1} h_i + h_{i-1} d_i^M$,

($X, Y =$ top. spaces, $f, g: X \to Y$ are homotopic $:\Longleftrightarrow \exists\ F: X \times [0,1] \to Y$: $F(.,0) = f$, $F(.,1) = g$)

Prop: $f \overset{h}{\sim} 0 \Longrightarrow H_\bullet(f) = 0$

Proof.

$$\begin{tikzcd} M_{i+1} \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] \arrow[dl, "h"'] & M_{i-1} \arrow[d, "f"] \arrow[dl] \\ N_{i+1} \arrow[r] & N_i \arrow[r] & N_{i-1} \end{tikzcd}$$

$x \longmapsto 0$, $f(x) \in N_i$

$x \in H_i(M_\bullet)$ ⇝ lift this to $x \in Z_i(M_\bullet)$ $\left(\xrightarrow{f} Z_i(N_\bullet) \supseteq B_i(N_\bullet)\right)$

$\downarrow$

$H_i(N_\bullet) = Z_i(N_\bullet)/B_i(N_\bullet)$

<u>goal</u>: ⊛ → $B_i(N_\bullet)$

$h(x) = y \mapsto f(x)$

$\Longrightarrow f(x) = d^N(y) \in B_i(N)$ ⟹ <u>ok</u>!

□

Application: $C_\bullet =$ complex, <u>if</u> $\exists\ h$ being a homotopy $id_{C_\bullet} \sim 0$

↳ <u>then</u> $C_\bullet =$ <u>exact</u>.

$$\begin{tikzcd} C_{i+1} \arrow[r] \arrow[d, equal] & C_i \arrow[r] \arrow[d, equal] \arrow[dl, "h"'] & C_{i-1} \arrow[d, equal] \arrow[dl, "h"'] \\ C_{i+1} \arrow[r] & C_i \arrow[r] & C_{i-1} \end{tikzcd}$$

Proof. $h: id \sim 0 \Longrightarrow H_i(id) = 0 \quad \forall i$, $H_i(id) = id_{H_i(C)}$ $\Big\}$ $\Longrightarrow id: H_i(C_\bullet) \xrightarrow{\sim} H_i(C_\bullet)$ is zero $\Longrightarrow H_i(C_\bullet) = 0$.

Example. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$ <u>Q</u>: $\exists\ h =$ homotopy : $id \sim 0$? NO

$\otimes_{\mathbb{Z}} \mathbb{F}_2$ ⟶ <u>not</u> exact

$f =$ hom. equiv. $\Longrightarrow$ $f =$ qis $(\forall i: H_i(M) \xrightarrow{\sim} H_i(N))$

⇑

Def. $f: M_\bullet \longrightarrow N_\bullet$ is called a homotopy equivalence

$:\Longleftrightarrow \exists\ g: N_\bullet \to M_\bullet$: $(f \circ g) \sim id_{N_\bullet}$, $(g \circ f) \sim id_{M_\bullet}$

(<u>i.e</u>: $(fg - id_N) \sim 0$, $(gf - id_M) \sim 0$)

(<u>recall</u>: f is called an <u>isomorphism</u> of complexes $\Longleftrightarrow \exists\ g: N_\bullet \to M_\bullet$: $f \circ g = id_{N_\bullet}$, $g \circ f = id_{M_\bullet}$)

<u>Proof of</u> „homotopy equiv." $\Longrightarrow$ „<u>qis</u>"

$f \circ g \sim id_N \Longrightarrow H_\bullet(fg - id_N) = 0 \Longrightarrow H_\bullet(fg) = H_\bullet(id_N) = id_H$

$H_\bullet(fg) = H_\bullet(f) \circ H_\bullet(g)$

D

$\mathcal{A} =$ abelian category (<u>e.g</u>: R-modules) ⇝ $K(\mathcal{A}) =$ „homotopy category"

<u>objects</u>: complexes $M_\bullet$ over $\mathcal{A}$.

<u>morphisms</u>: $\mathrm{Hom}_{K(\mathcal{A})}(M_\bullet, N_\bullet) = \{\text{complex maps } f: M_\bullet \to N_\bullet\} / \text{homotopy}$

<u>i.e.</u> $f, g: M_\bullet \to N_\bullet$, $f \sim g \Longrightarrow \bar{f} = \bar{g}$ as elements of $K(\mathcal{A})$.

<u>Advantage</u>:
- $f =$ homotopy equivalence $M_\bullet \to N_\bullet \Longleftrightarrow \bar{f} =$ isom in $K(\mathcal{A})$
- $\bar{f} \in \mathrm{Hom}_K(M_\bullet, N_\bullet)$ (⇝ might be <u>represented by</u> an $f: M_\bullet \to N_\bullet$)

  ⇝ $H_i(f): H_i(M) \to H_i(N)$ does <u>not</u> depend on <u>this choice</u> $\Big\}$ functors $H_i: K(\mathcal{A}) \to \mathcal{A}$, $X_\bullet \longmapsto H_i(X_\bullet)$

$K^b(\mathcal{A}), K^+(\mathcal{A}), K^-(\mathcal{A}) \subseteq K(\mathcal{A})$ <u>ex</u>: $X_\bullet \in K^-(\mathcal{A})$ $(= K_-(\mathcal{A}))$: $X_\bullet \to X_{i+1} \to \dots \to X_{i_0} \to 0 \to 0 \dots$ //