$\{\text{Modules over } R\} = \mathcal{A}$ ⇝ complexes $M_{i+1} \to M_i \xrightarrow{d_i} M_{i-1}$ ⇝ $H_i(M.)$

functorial:

$$\begin{array}{ccccccc} M_{i+1} & \to & M_i & \xrightarrow{d_i} & M_{i-1} & \rightsquigarrow & H_i(M.) \\ \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} & & \downarrow H_i(f) \\ N_{i+1} & \to & N_i & \to & N_{i-1} & \rightsquigarrow & H_i(N.) \end{array}$$

· $0 \to K. \to M. \xrightarrow{f} N. \to 0 \Rightarrow \dots \to H_i(K) \to H_i(M) \to H_i(N) \xrightarrow{\partial_i} H_{i-1}(K) \to \dots$ l.e.s.h

· $f: M. \to N.$ is a qis $:\Longleftrightarrow \forall i: \boxed{H_i(f) = \text{isom}}$

Problem: F = functor (not exact) $\Rightarrow F(\text{qis})$ might fail to be q.s !!

Def. $f \sim 0 \Longleftrightarrow \exists$ homotopy h: $\boxed{hd + dh = f}$ $\Rightarrow F(f \sim 0) = [F(f) \sim F(0) = 0]$

Prop: $f \sim 0 \Rightarrow H_i(f) = 0$, more general: $f, g: M. \to N.$, $f \sim g$ (i.e. $f - g \sim 0$) $\Rightarrow H_i(f) = H_i(g)$

$M., N.$ complexes (recall: modules K, L ⇝ $\mathrm{Hom}_R(K,L)$ = R-module)

Goal: $\mathrm{Hom}(M., N.)$: $\mathrm{Hom}_{..}(M., N.)$ = double complex

$\mathrm{Hom}_{ij}(M., N.) := \mathrm{Hom}_R(M_{-i}, N_j)$

$$\begin{array}{ccc} \mathrm{Hom}(M_{-i}, N_j) & \xrightarrow{d_N} & \mathrm{Hom}(M_{-i}, N_{j-1}) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(M_{-(i+1)}, N_j) & \to & \mathrm{Hom}(M_{-(i+1)}, N_{j-1}) \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}_{i+1,j}(M., N.) & \to & \mathrm{Hom}_{i+1,j-1}(M., N.) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{i,j}(M., N.) & \to & \mathrm{Hom}_{i,j-1}(M., N.) \end{array}$$

$M_{-(i+1)} = M_{-i-1}$, M_{-i}

switch signs in every other column!

$\mathrm{Tot}\,\mathrm{Hom}_{..}(M., N.) = \boxed{\mathrm{Hom}_.(M., N.)}$

3rd possibility: $\boxed{\{f: M. \to N. \mid \text{complex map}\}}$ = Set

$\mathrm{Hom}_K(M., N.) = \bigoplus_{i+j=K} \mathrm{Hom}_{ij}(M., N.) = \bigoplus_{i+j=K} \mathrm{Hom}(M_{-i}, N_j) = \bigoplus_v \mathrm{Hom}(M_v, N_{v+K})$

$\mathrm{Hom}_0(M., N.) = \bigoplus_v \mathrm{Hom}(M_v, N_v)$

$d: \mathrm{Hom}_0(M., N.) \to \mathrm{Hom}_{-1}(M., N.)$

$\mathrm{Hom}(M_v, N_v) \to \mathrm{Hom}(M_v, N_{v-1}) \oplus \mathrm{Hom}(M_{v+1}, N_v)$

$\varphi \longmapsto (d_N \circ \varphi, \varphi \circ d_M)$

$\Rightarrow d(\varphi) = d_N \varphi - \varphi d_M$

$Z_0(\mathrm{Hom}_.(M., N.)) = \boxed{\{\varphi \mid d(\varphi) = 0\}}$

$B_0(\mathrm{Hom}(M., N.)) = \mathrm{Im}(\mathrm{Hom}_1(M., N.) \to \mathrm{Hom}_0(M., N.))$

$\bigoplus \mathrm{Hom}(M_v, N_{v+1}) \to \dots$

$h \longmapsto f \Longleftrightarrow h: f \sim 0$

$f \Longleftrightarrow f \sim 0$

$H_0(\mathrm{Hom}_.(M., N.)) = Z/B = \{\text{complex maps}\}/\sim 0 =: \mathrm{Hom}_{K(\mathcal{A})}(M., N.)$ $K(\mathcal{A})$ = category

Prop: Let's · $P., C.$ = complexes $\in K^-(\mathcal{A}) = K_+(\mathcal{A})$, $f: P. \to C.$

reminder: $f: M. \to N.$ $(f \sim 0) \Rightarrow \{H_i(f) = 0\}$

i.e. $P.: \dots \to P_n \to P_{n-1} \to \dots \to P_{n_0} \to 0 \to 0 \dots$

$C.$: similarly

such that: · P_i = projective R-modules $\forall i$

· $C.$ = exact

$\Big\} \Rightarrow H_i(f) = 0$, $H_i(f): H_i(P) \to H_i(C) = 0$

$\Rightarrow f \sim 0$.

Proof:

$$\begin{array}{ccccccc} \dots \to & P_{i+1} & \to & P_i & \xrightarrow{d^P} & P_{i-1} & \xrightarrow{d_P} \dots \\ & \downarrow f & & \downarrow f_i & & \downarrow f & \\ \dots \to & C_{i+1} & \xrightarrow{d^C_{i+1}} & C_i & \xrightarrow{d^C} & C_{i-1} & \xrightarrow{d_c} \end{array}$$

$f_i = h d_P + d_C h$

$f_i - h_{i-1} d^P \overset{(2)}{=} d^C_{i+1}(h_i)$ [goal]

$\psi: P_i \to C_i$, $d^C_i \circ \psi = (?)$

$d^C_i \psi = d^C f_i - d^C h_{i-1} d^P$

$d^C h_{i-1} = f_{i-1} - h d^P$

$d^C_i \psi = \underbrace{d^C_i f_i - f_{i-1} d^P_i}_{0 \ (f = \text{complex map})} + \cancel{h d^P d^P} \, 0 \Rightarrow d^C_i \psi = 0$

$\Rightarrow \psi: P_i \to Z_i(C.) \subseteq C_i$

$\Rightarrow \psi: P_i \to B_i(C.) = \mathrm{Im}(C_{i+1} \xrightarrow{d^C_{i+1}} C_i)$

$\psi: \boxed{P_i \xrightarrow{h_i} C_{i+1}} \xrightarrow{d_{i+1}} B_i(C) \subseteq C_i$

$\uparrow d^C_{i+1}$ C_{i+1} □

Prop: (1) $P. \in K^-(\mathcal{A})$; $A. \xrightarrow{f} B.$ s.t. $f = \text{qis} \Rightarrow \mathrm{Hom}_{K(\mathcal{A})}(P., A.) \xrightarrow{\sim} \mathrm{Hom}_{K(\mathcal{A})}(P., B.)$

(2) $M. \in K^-(\mathcal{A}) \Rightarrow \exists P. \in K^-(\mathcal{A})$, $(\varphi: P. \to M.)$, $\varphi = \text{qis}$ ("projective resolution" of $M.$)

and $P. \in K^-(\mathcal{A})$ is unique (up to isos) and: $P. \xrightarrow{\text{qis}} M.$, $\downarrow f$, $P' \to N.$ $\Rightarrow \exists! F: P. \to P'$

if $f = \text{qis} \Rightarrow F$ = homotopy equivalence

Proof (1) $\mathrm{Cone}(f)$ ⇝ $0 \to B. \to \mathrm{Cone} f \to A.[1] \to 0$ LES

$B_i \hookrightarrow B_i \oplus A_{i-1} \twoheadrightarrow A_{i-1}$

$\Rightarrow \mathrm{Cone}(f)$ = exact complex.

$\mathrm{Hom}_.(P., A.) \xrightarrow{F} \mathrm{Hom}_.(P., B.)$ $F = \mathrm{Hom}(P., f) \Rightarrow \mathrm{Cone}(F) = \mathrm{Hom}(P., \mathrm{Cone}(f))$

$\mathrm{Hom}(P_{i+1}, A_i) \oplus \mathrm{Hom}(P_i, B_{i+1})$ qis

$A_i \oplus B_{i+1} \cong \mathrm{Cone}(f)$

$f: A \to B \Rightarrow \mathrm{Cone}(f) = \text{exact}$; $\mathrm{Cone}(F) = \mathrm{Hom}_\bullet(P_\bullet, \mathrm{Cone}(f))$ Lemma before

$H_0(\mathrm{Cone}(F)) = H_0(\mathrm{Hom}_\bullet(P_\bullet, \mathrm{Cone}(f))) = \mathrm{Hom}_{K(A)}(P_\bullet, \underset{C_\bullet}{\mathrm{Cone}(f)}) = 0$ } $\mathrm{Cone}(F) = \text{exact}$

$H_i \ \forall i$:

$F: \mathrm{Hom}_\bullet(P_\bullet, A_\bullet) \to \mathrm{Hom}_\bullet(P_\bullet, B_\bullet)$ is a $\underline{\text{q.i.s}}$

$\Rightarrow H_0 \mathrm{Hom}_\bullet(P_\bullet, A_\bullet) \xrightarrow{\sim} H_0 \mathrm{Hom}_\bullet(P_\bullet, B_\bullet)$

$\mathrm{Hom}_{K(A)}(P_\bullet, A_\bullet) \xrightarrow{\sim} \mathrm{Hom}_{K(A)}(P_\bullet, B_\bullet)$ (1) ✓

(2) existence of $P_\bullet \to M_\bullet$:

P'_{i+1}

$P'_{i+1} \oplus$, P''_{i+1} ??

already constructed

$$P_i \to P_{i-1} \to \dots$$
$$\downarrow f_i \qquad \downarrow f_{i-1}$$
$$Z_{i+1}(M) \subseteq M_{i+1} \to M_i \to M_{i-1} \to \dots$$

- $H_\nu(P_\bullet) \xrightarrow{f} H_\nu(M_\bullet)$ is an isom for $\nu \le i-1$
- $f_i: Z_i(P_\bullet) \xrightarrow{\text{surjective}} Z_i(M_\bullet)$

Induction hypothesis

$f_i^{-1}(B_i(M_\bullet)) \twoheadrightarrow B_i(M_\bullet)$, $f_i^{-1}(B_i(M_\bullet)) \subseteq P_i$, $P'_{i+1} \to f_i^{-1}(B_i(M_\bullet))$

need the hypothesis:

$\boxed{\text{Every } L \in \mathrm{Ob}(A) \text{ has a surj. } P \twoheadrightarrow L \text{ with } P = \text{proj} \in \mathrm{Ob}(A)}$ =: „A has enough projectives"

$A = \mathrm{Mod}_R \Rightarrow \underline{\text{OK}}$

Uniqueness in (2)

$$P_\bullet \xrightarrow[q]{\text{q.i.s}} M_\bullet$$
$$F \downarrow \qquad \downarrow f$$
$$P'_\bullet \xrightarrow[q']{\text{q.i.s}} M'_\bullet$$

(i) $\exists$ unique $F: P_\bullet \to P'_\bullet$ in $K(A)$ such that the square commutes

Proof: $P_\bullet \xrightarrow{f \circ q} M'_\bullet$, $F: P_\bullet \to P'_\bullet$, $P'_\bullet \xrightarrow[q']{\text{q.i.s}} M'_\bullet$

$\mathrm{Hom}_{K(A)}(P_\bullet, P'_\bullet) \xrightarrow{q'} \mathrm{Hom}_{K(A)}(P_\bullet, M'_\bullet)$ bijection by (1) ✓

$F \longmapsto (f \circ q)$

F = unique preimage

(ii) If $f = \text{q.i.s}$

$$P_\bullet \xrightarrow[\text{q.i.s}]{q} M_\bullet$$
$$F \downarrow \qquad \text{q.i.s} \downarrow f$$
$$P'_\bullet \xrightarrow[q']{\text{q.i.s}} M'_\bullet$$

$\Rightarrow F: P_\bullet \to P'_\bullet$ is a $\boxed{\text{q.i.s}}$

(1) again $\Rightarrow$ this map $\mathrm{Hom}_{K(A)}(P'_\bullet, P_\bullet) \xrightarrow{F} \mathrm{Hom}_{K(A)}(P'_\bullet, P'_\bullet)$ is a bijection

$\hookrightarrow \exists G \longmapsto \mathrm{id}$

$G: P'_\bullet \to P'$; G = inverse of F (inside $K(A)$) □

So far: $M_\bullet \in K^-(A) \Longrightarrow \exists!\, P_\bullet \xrightarrow[q]{\text{q.i.s}} M_\bullet$ („projective resolution of $M_\bullet$")

Special case: $M \in A \Longrightarrow \underbrace{\exists!}_{\text{unique in } K(A)} P_\bullet \xrightarrow{\text{q.i.s}} M := [0 \to 0 \to \dots \to 0 \to M \to 0 \to \dots]$

$[1] \quad [0] \quad [-1]$

$$H_i(\dots) = \begin{cases} 0 & \text{if } i \neq 0 \\ M & \text{if } i = 0 \end{cases}$$

$\Updownarrow$

$$\dots \to P_2 \to P_1 \to P_0 \to 0 \dots$$
$$q \downarrow \quad q \downarrow \quad q \downarrow \quad \downarrow$$
$$0 \to 0 \to M \to 0 \to$$

$q = \text{q.i.s} \Longleftrightarrow P_\bullet = \text{exact at the places } i \ge 1$

$\underline{i = 0}$: $P_0 / \mathrm{im}(P_1 \to P_0) \xrightarrow{\sim} M$

$\Longleftrightarrow$ $\underbrace{\dots \to P_2 \to P_1 \to}_{\text{exact}} P_0 \xrightarrow{q_0} M \to 0$ exact, too; exact

= exact complex

Example: $R = \mathbb{Z}$

- $M = \mathbb{Z} \rightsquigarrow P_\bullet \xrightarrow{\text{q.i.s}} M$, $P_\bullet = \mathbb{Z}$

i.e. $(P_2 = 0) \to (P_1 = 0) \to \boxed{P_0 = \mathbb{Z} \xrightarrow{\sim} M = \mathbb{Z}} \to 0$

- $M = \mathbb{Z}/2\mathbb{Z}$: $0 \to \boxed{\mathbb{Z}^1 \xrightarrow{\cdot 2} \mathbb{Z}^1} \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} \to 0$, $\mathbb{Z}/2\mathbb{Z} = M$

$1 \longmapsto \bar{1}$

$\mathrm{Ker}\, q = 2\mathbb{Z} \cong \mathbb{Z}$

- $R = k[x, y]$; $M = k := k[x,y]/(x,y)$

$$0 \leftarrow k \xleftarrow{q_0} k[x,y]^1 \xleftarrow{q_1} k[x,y]^2 \xleftarrow{q_2} k[x,y] \leftarrow 0$$

$\bar{1} \longleftarrow 1$; $x \longleftarrow e_1$, $y \longleftarrow e_2$; $ye_1 - xe_2 \longleftarrow 1$

$n = 2$

$\mathrm{Ker}\, q_0 = (x, y)$; $\mathrm{Ker}\, q_1 = \langle ye_1 - xe_2 \rangle$

$k \oplus k\varepsilon,\ \varepsilon^2 = 0$

- $R = k[\varepsilon]/\varepsilon^2$; $M = k = k[\varepsilon]/(\varepsilon) = R/\varepsilon \cdot R$.

$$0 \leftarrow k \xleftarrow{q_0} R \xleftarrow{\cdot\varepsilon} R \xleftarrow{\cdot\varepsilon} R \leftarrow \dots$$

$\bar{1} \longleftarrow 1$

$0 \longleftarrow \varepsilon$

$\boxed{\mathrm{Ker}\, q_0} = \varepsilon \cdot k \xleftarrow[\cdot\varepsilon]{\sim} R/\varepsilon = \boxed{M}$

$\varepsilon \longleftarrow 1$

Theorem (Hilbert's syzygy th.)

M = f.g. $k[x_1, \dots, x_n]$-module

$\Rightarrow \exists\ 0 \leftarrow M \leftarrow \boxed{P_0 \leftarrow P_1 \leftarrow \dots \leftarrow P_n} \leftarrow 0$

proj. $k[\underline{x}]$-modules.

finite proj. resol.

Construction: $F: A \to B$ additive functor, $A = \mathrm{Mod}_R$, $B = \mathrm{Mod}_R$ (e.g. $F = \otimes_R N$)

Let F be right exact.

New: $F: K(A) \to K(B)$

$M_\bullet \longmapsto [\dots \to F(M_i) \to F(M_{i-1}) \to \dots]$

Very New: $\mathbb{L}F: K(A) \to K(B)$ after the break

$\mathrm{Hom}_K(X,Y) = H_0(\mathrm{Hom}_\bullet(X,Y))$

$\mathrm{Hom}_{K(\mathcal{A})}(X, Y[n]) = H_0(\mathrm{Hom}_\bullet(X, Y[n])) = H_n(\mathrm{Hom}_\bullet(X,Y))$

$F: \mathcal{A} \to \mathcal{B}$ right exact (ex: $F(M) := M \otimes_R N$)

Mod_R, Mod_R

$F: K^-(\mathcal{A}) \to K^-(\mathcal{B}) \rightsquigarrow \mathbb{L}F: K^-(\mathcal{A}) \to K^-(\mathcal{B})$ $\quad \mathbb{L}F = F \circ$ [proj resolution]

$M_\bullet \longmapsto F(M_\bullet)$

$M_\bullet \xleftarrow{g \text{ q.i.s}} P_\bullet \longmapsto F(P_\bullet)$ functor.

Most important feature:

$\boxed{M_\bullet \xrightarrow{f, g} N_\bullet} \Rightarrow \boxed{\mathbb{L}F(M_\bullet) \xrightarrow{\mathbb{L}f} \mathbb{L}F(N_\bullet)}$

$P_\bullet \xrightarrow{\exists f, G} P'_\bullet \Rightarrow F(P_\bullet) \xrightarrow{F(G)} F(P'_\bullet)$ $\quad \boxed{\mathbb{L}F(g) = \text{q.i.s}}$

Now: $\boxed{gf = \text{q.i.s}} \Rightarrow f = \text{homotopy eq.} \Rightarrow F(G) = \text{homotopy equiv.} \Rightarrow F(G) = \text{q.i.s}$

left derived functor of F

$\mathbb{L}F(M_\bullet) \in K^-(\mathcal{B})$

$H_i(\mathbb{L}F(M_\bullet)) = L_iF(M_\bullet)$

Special situation: $F: \mathcal{A} \to \mathcal{B}$; $M \in \mathcal{A}$ (e.g. $M = R$-module)

$P_\bullet \xrightarrow[\text{q.i.s}]{g} M \iff \dots \to P_2 \to P_1 \to P_0 \to M \to 0$ exact

$(L_iF)(M) := H_i(F(P_\bullet))$

Example: $F: (R\text{-mod}) \to (R\text{-mod})$: $F(M) := M \otimes_R N$, i.e. $F = \otimes_R N$

$L_i(\otimes_R N)(M) = H_i(P_\bullet \otimes_R N) = \mathrm{Tor}_i^R(M,N)$

Ex: $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, $i = 0, 1, \dots$

$F = (\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z})$, $\quad M = \mathbb{Z}/2\mathbb{Z}$

Resolution: $0 \to \boxed{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z} \to 0$ $\quad (P_1 \to P_0)$

$P_\bullet = [\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}] \Rightarrow (P_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) = [\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2 \,=\, 0} \mathbb{Z}/2\mathbb{Z}] \to 0$ $\quad$ [1] [0]

$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

$\mathrm{Tor}_{\geq 2} = 0$

contrast: $R = \mathbb{Z}/2\mathbb{Z}$,

$\mathrm{Tor}_i^R(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ $(i=0)$, 0 $(i \geq 1)$

$P_\bullet = \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} = M$

Properties of $L_iF: \mathcal{A} \to \mathcal{B}$ ($F: \mathcal{A} \to \mathcal{B}$ right exact):

(i) $L_0F = F$

(ii) $L_{\geq 1}F$ (projective object of $\mathcal{A}$) $= 0$

(iii) long exact homology sequence $\quad 0 \to M' \to M \to M'' \to 0$ exact $\quad 0 \to N' \to N \to N'' \to 0$

$\Rightarrow 0 \leftarrow F(M'') \leftarrow F(M) \leftarrow F(M')$

$0 \leftarrow L_0F(M'') \leftarrow L_0F(M) \leftarrow L_0F(M') \xleftarrow{\partial} L_1F(M'') \leftarrow L_1F(M) \leftarrow L_1F(M') \xleftarrow{\partial} L_2F(M'') \leftarrow \dots$

Thm: These properties characterize the functors L_iF for a given F.

$L_0F(N') \xleftarrow{\partial} L_1F(N'')$

Example: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ $\quad (M', M, M'')$

$R = \mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, $F = (\otimes_{\mathbb{Z}} N)$

$(L_1F)(\mathbb{Z}) = 0$ (proj)

$0 \leftarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \leftarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \leftarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \leftarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$

$0 \leftarrow \mathbb{Z}/2\mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z}/2\mathbb{Z} \xleftarrow{0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z}/2\mathbb{Z} \leftarrow 0$

Proof: $\dots \to P_2 \to P_1 \to P_0 \to M \to 0$ proj resolution $\rightsquigarrow P_1 \to P_0 \to M \to 0$ exact

(i)

$F(-)$: $\boxed{F(P_1) \to F(P_0)} \to F(M) \to 0$ exact $\quad (*)$

$L_0F(M) = [0\text{-th homology of } F(P_1) \to F(P_0) \to 0]$

$(*) \Rightarrow F(M) = \mathrm{coker}(F(P_1) \to F(P_0)) = F(P_0) / \mathrm{im}(F(P_1) \to F(P_0))$

(ii) $M = P$ = projective

$(P_0 = P) \to P \to 0$ is the proj resol.

$(P_{\geq 1} = 0)$ $\quad L_iF(P) = H_i(F(P_\bullet)) = 0$ for $i \geq 1$.

(iii) $0 \to M' \xrightarrow{g} M \xrightarrow{h} M'' \to 0$

$P'_\bullet \xrightarrow{G} P_\bullet$ $\rightsquigarrow \mathrm{Cone}(G) = \text{complex} \Rightarrow 0 \to P_\bullet \to \mathrm{Cone}(G) \to P'_\bullet[1] \to 0$, exact, projective

$\Rightarrow 0 \to F(P_\bullet) \to F(\mathrm{Cone}(G)) \to F(P'_\bullet[1]) \to 0$ exact

$\leftarrow H_i(F(P'_\bullet[1])) \leftarrow H_i(F(\mathrm{Cone}\,G)) \leftarrow H_i(F(P_\bullet)) \leftarrow H_{i+1}(F(P'_\bullet[1])) \leftarrow$

$H_{i-1}(F(P'_\bullet))$ $\quad$ Show $\quad H_i(F(P)) \leftarrow H_i(F(P'_\bullet))$

$(L_{i-1}F)(M') \leftarrow L_iF(M'') \leftarrow L_iF(M) \xleftarrow{g} L_iF(M')$

$F(P_\bullet) \to F(P_{\dots})$

next week