$$\begin{array}{ccc} P_\bullet & \xrightarrow{q.i.s} & M_\bullet \\ {\scriptstyle F}\big\downarrow{\scriptstyle q.i.s} & & {\scriptstyle f}\big\downarrow{\scriptstyle q.i.s} \\ P'_\bullet & \xrightarrow{q.i.s} & M'_\bullet \end{array}$$

Q Construct an inverse $G : P' \to P$ of F (within $K^-(\mathcal{A})$)

$\hookrightarrow \mathrm{Hom}_K(P', P) \xrightarrow[\sim]{F} \mathrm{Hom}(P', P')$

$\eta \longmapsto F \circ \eta$

$G \longmapsto \mathrm{id}$ $\Big\}$ $\boxed{FG = \mathrm{id}_{P'}}$

$\boxed{\text{Lemma: } A \overset{F}{\underset{G}{\rightleftarrows}} B \;:\; \begin{array}{l}\bullet\ F = \text{isom} \\ \bullet\ FG = \mathrm{id}\end{array} \Big\} \Rightarrow G = \text{isom},\ G = F^{-1}}$ $\rightsquigarrow$ apply to the homology level of the diagram

$\mathrm{Hom}_K(P, P) \xrightarrow{F} \mathrm{Hom}_K(P, P') \xrightarrow[\sim]{G} \mathrm{Hom}_K(P, P) \xrightarrow[\sim]{F} \mathrm{Hom}_K(P, P')$

$\rightsquigarrow \bullet\ G = q.i.s.$

$\bullet\ H(G) = H(F)^{-1}$

id, $FG = \mathrm{id}$

$\mathrm{id}_P \longmapsto \mathrm{id}_P$

$\mathrm{id}_P \longmapsto GF$ $\Big\}$ $GF = \mathrm{id}_P$ as maps $P \to P$ in the K-category

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- $\mathcal{A}$ = category (abelian) with "enough projectives" ($\Leftrightarrow \forall M \in \mathcal{A}\ \exists P \twoheadrightarrow M$); eg: $\mathcal{A} = \mathrm{mod}_R$
- $F : \mathcal{A} \to \mathcal{B}$ right exact $\rightsquigarrow$ define $L_i F : \mathcal{A} \to \mathcal{B}$, $M \longmapsto (?)$
  - $M \xleftarrow{q.i.s} P_\bullet$ ("proj resol")
  - $F(P_\bullet)$ = complex in $\mathcal{B}$
  - $L_i F(M) = H_i(F(P_\bullet))$
- Claim: $\{L_i F \mid i \geq 0\}$ are characterized by:

① $L_0 F = F$ $\quad$ ($P_1 \to P_0 \to M \to 0$ $\circlearrowright F$

$FP_1 \to FP_0 \to FM \to 0$ remains exact $\Rightarrow \mathrm{Ker}(FP_0 \to FM) = \mathrm{Im}(FP_1 \to FP_0)$

$\Rightarrow H_0(P) = FP_0 / \mathrm{Im}(FP_1 \to FP_0) = FM$)

② $M = P$ - projective $\Longrightarrow L_{\geq 1}(P) = 0$.

③ $0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow 0 \leftarrow FM'' \leftarrow FM \leftarrow FM'$

$0 \leftarrow L_0 FM'' \leftarrow L_0 FM \leftarrow L_0 FM' \leftarrow L_1 FM'' \leftarrow L_1 FM \leftarrow$

$\exists$ "natural" $L_i FM' \leftarrow L_{i+1} FM''$

i.e. $L_{i+1} FM'' \xrightarrow[\partial_i]{} L_i FM'$ s.t. is exact !!

Proof of ③. $0 \to M' \xrightarrow{g} M \xrightarrow{h} M'' \to 0$ $\rightsquigarrow$ choose $P'_\bullet \to M'$, $P_\bullet \to M$

$q.i.s$, $q.i.s$, $q.i.s$

$\boxed{P'_\bullet \xrightarrow{G} P_\bullet}$ $\rightsquigarrow$ $\boxed{\mathrm{cone}(G)}$ = projective complex

$\hookrightarrow 0 \to P_\bullet \to \mathrm{cone}(G) \to P'_\bullet[1] \to 0$

$\Rightarrow 0 \to F(P_\bullet) \to F(\mathrm{cone}(G)) \to F(P'_\bullet[1]) \to 0$

take the long ex. seq of homology

($0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$

$\Rightarrow H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet) \xrightarrow{\partial} H_{i-1}(A_\bullet)$ ... ex)

$\Rightarrow H_i(F(P_\bullet)) \to H_i(F(\mathrm{cone}\,G)) \to H_i(P'_\bullet[1]) \to H_{i-1}(F(P_\bullet)) \to H_{i-1}(F(\mathrm{cone}\,G)) \to \cdots$

$H_i(F(P_\bullet)) = L_i F(M)$, $H_i(F(\mathrm{cone}\,G)) = L_i F(M'')$, $H_i(P'_\bullet[1]) = H_{i-1}(P'_\bullet) = L_{i-1} F(M')$, $H_{i-1}(F(P_\bullet)) = L_{i-1} F(M)$, $H_{i-1}(F(\mathrm{cone}\,G)) = L_{i-1} F(M'')$

$L_i F(M) \xrightarrow{L_i F(h)} L_i F(M'') \xrightarrow[\partial_i]{\text{new}} L_{i-1} F(M') \xrightarrow{L_{i-1} F(g)} L_{i-1} F(M) \xrightarrow{L_{i-1} F(h)} L_{i-1} F(M'')$ //

Last step Assume you have functors $F_0, F_1, \ldots$ $\mathcal{A} \to \mathcal{B}$ satisfying ①-③

($F_0 = F$, $F_{\geq 1}(P) = 0$, $\partial : F_{i+1}(M'') \to F_i(M')$ s.t. $\Rightarrow$ long ex. seq. of the $F_i(M)$'s)

$\Longrightarrow F_i = L_i F$ (functorial !)

Proof induction $i = 0$ $\quad F_0 = F \overset{①}{=} L_0 F$

$i \rightsquigarrow i+1$: Let $M \in \mathcal{A}$ $\rightsquigarrow$ compare $F_{i+1}(M)$ with $L_{i+1} F(M)$

Choose $P \overset{\pi}{\twoheadrightarrow} M$ (P = projective in $\mathcal{A}$) $\rightsquigarrow 0 \to K \xrightarrow{\iota} P \xrightarrow{\pi} M \to 0$ exact

ex $\cancel{L_{i+1} F(K)} \to L_{i+1} F(P) \to L_{i+1}(M) \xrightarrow{\sim} L_i F(K) \to L_i F(P) \to L_i F(M) \to \cdots$ exact

② : $L_{i+1} F(P) = 0$, $F_{i+1}(P) = 0$ $\quad$ ③ $\quad \wr$ $\quad \wr$ $\quad \wr$

$\cancel{F_{i+1}(K)} \to F_{i+1}(P) \to F_{i+1}(M) \xrightarrow{\sim} F_i(K) \to F_i(P) \to F_i(M) \to \cdots$

$(i \geq 0 \Rightarrow i+1 \geq 1)$

isom for $i \geq 1$

Remark: example $F(\cdot) = \cdot \otimes_R N$ $\rightsquigarrow L_i F(M) =: \mathrm{Tor}_i^R(M, N)$

$F = (\otimes_R N)$ $\quad$ ($\mathrm{Tor}_0^R(M, N) = M \otimes_R N$)

$R \to S$ algebra ; M, N = R-mod.

$\Rightarrow (M \otimes_R S) \otimes_S (N \otimes_R S) = (M \otimes_R N) \otimes_R S$

Now we claim: If $R \to S$ is flat, then $\mathrm{Tor}_i^S(M \otimes_R S, N \otimes_R S) = \mathrm{Tor}_i^R(M, N) \otimes_R S$

Proof $P_\bullet \xrightarrow{q.i.s} M$ (R-proj. resol.) $\mid \otimes_R S$

$\cdots P_3 \to P_2 \to P_1 \to P_0 \xrightarrow{\pi} M \to 0$ exact

$(P_\bullet \otimes_R S) \to M \otimes_R S$ (S-proj resol.) //

$R =$ ring, $M, N = R$-modules $\rightsquigarrow M \otimes_R N$

$\boxed{\mathrm{Tor}^{①}_{\cdot}(M,N)}$ = Left-derived functors of $M \longmapsto M \otimes_R N$, i.e. of $(\otimes_R N)$ (via $P_\cdot \to M$)

$\mathrm{Tor}^{②}_{\cdot}(M,N) :=$ — " — $N \longmapsto M \otimes_R N$, i.e. of $(M \otimes_R)$. (via $Q_\cdot \to N$)

Prop: $\mathrm{Tor}^{②} = \mathrm{Tor}^{①}$

Proof: $F = (\otimes_R N) \rightsquigarrow L_\cdot F(M) = \mathrm{Tor}^{①}_{\cdot}(M,N)$

$F_i(M) := \mathrm{Tor}^{②}_i(M,N)$ (for a fixed N)

task: check properties ① - ③ for $F_\cdot$

- $F_0(M) = \mathrm{Tor}^{②}_0(M,N) = M \otimes_R N$
- $F_{i \geq 1}(\underset{\text{proj}}{P}) = 0$?

to show: $\mathrm{Tor}^{②}_{i\geq 1}(P,N) = 0$. Take $\underset{\text{proj.}}{Q_\cdot} \xrightarrow{\text{q.i.s}} N$ $(\dots Q_2 \to Q_1 \to Q_0 \to N \to 0$ exact$)$

then: consider $\dots \to P\otimes Q_{i+1} \to P \otimes Q_i \to P \otimes Q_{i-1} \to \dots$ stays exact ? ok, because projective P are flat!

homology ??

③ Let $\boxed{0 \to M' \to M \to M'' \to 0}$ (*) exact $\otimes_R N \rightsquigarrow$ long exact sequ. for $\mathrm{Tor}^{①}$

Claim: $\mathrm{Tor}^{②}$ does also do this! Choose $Q_\cdot \to N$ q.i.s, $Q_i =$ proj.

$$\begin{array}{l} (*) \otimes_R Q_\cdot \Rightarrow \quad 0 \to M' \otimes_R Q_i \to M \otimes_R Q_i \to M'' \otimes_R Q_i \to 0 \\ \qquad\qquad\qquad\quad \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\quad \downarrow \\ \qquad\qquad\qquad 0 \to M' \otimes_R Q_{i-1} \to M \otimes_R Q_{i-1} \to M'' \otimes Q_{i-1} \to 0 \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\quad M' \otimes_R Q_{i-2} \end{array}$$

stay exact

homology: $\mathrm{Tor}^{②}_{i-1}(M',N)$

i.e. $\boxed{0 \to M' \otimes_R Q_\cdot \to M \otimes_R Q_\cdot \to M'' \otimes_R Q_\cdot \to 0}$ short ex. seq. of complexes

Recall: $M =$ projective $:\iff \mathrm{Hom}_R(M, \cdot) =$ exact

$M =$ flat $:\iff (M \otimes_R \cdot) =$ exact $\overset{\text{claim}}{\iff} \mathrm{Tor}^R_1(M,N) = 0 \quad \forall N$

Proof: $(\Leftarrow)$ $0 \to N' \to N \to N'' \to 0 \quad / M\otimes$

$\to \underset{=0}{\mathrm{Tor}^R_1(M,N'')} \to M\otimes N' \to M \otimes N \to M \otimes N'' \to 0$

$(\Rightarrow)$ $F =$ exact $\Rightarrow L_{\geq 1}F = 0$; need only $L_1 F$

$M \rightsquigarrow \exists\, P \twoheadrightarrow M \Rightarrow \boxed{0 \to K \to P \to M \to 0}$ exact

$\underset{0}{L_1F(P)} \to \underset{0}{L_1F(M)} \to F(K) \to F(P) \to F(M) \to 0$ ∎

Prop: Let $N = R$-module of finite presentation

$N =$ projective $\iff N$ is flat $\iff \forall\, \mathfrak{m} \in \mathrm{Max\,Spec}\, R: \mathrm{Tor}^R_1(R/\mathfrak{m}, N) = 0$.

Proof: $(\Rightarrow)(\Rightarrow)$ ✓

remains to show: $\forall\, \mathfrak{m} \in \mathrm{Max\,Spec}\,R: \mathrm{Tor}_1(R/\mathfrak{m}, N) = 0 \overset{?}{\Longrightarrow} N =$ projective.

We already know: "projective" is a local property $(N =$ proj $\iff \forall \mathfrak{m}: N_{\mathfrak{m}} =$ proj. $R_{\mathfrak{m}}$-module $\mathrm{Hom}(N, \cdot)_{\mathfrak{m}} = \mathrm{Hom}(N_{\mathfrak{m}}, \cdot))$

$\rightsquigarrow$ w.l.o.g.: $R =$ local ring, $\mathfrak{m} \in R$ only max ideal.

Task. $\mathrm{Tor}^R_1(R/\mathfrak{m}, N) = 0 \overset{?}{\Longrightarrow} N =$ proj. $\rightsquigarrow$ choose $R^n \twoheadrightarrow N$ minimal

$\Downarrow$

$\boxed{(R/\mathfrak{m})^n \xrightarrow{\sim} N \otimes R/\mathfrak{m}}$

$\Rightarrow 0 \to K \to R^n \to N \to 0 \quad | \otimes_R R/\mathfrak{m} = k$

$\underset{=0}{\mathrm{Tor}^R_1(N,k)} \to K \otimes k \to k^n \xrightarrow{\sim} N \otimes k \to 0$ exact

$\Longrightarrow K \otimes_R k = 0 \Rightarrow K = 0 \Rightarrow N =$ free ∎

Consequence: $M = R$-module f.g., $R =$ noetherian

$F =$ right exact functor

$\Longrightarrow L_{n+i}F(M) = L_i F(K)$ $(i \geq 1)$

Proof: $i = 1$

$0 \to K \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0$ (start of a proj. resolution)

$\to P_{n+1} \to P_n \to K$ proj resol for K

lhs: • apply F

• take homology of $F(P_\cdot)$ at $n+1$ $(n+i)$

rhs: • apply $F \Rightarrow$ take hom of $F(P_\cdot)$ at $n+1$ //

Corollary: $0 \to K \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0$, If $\mathrm{Tor}^R_{n+1}(M, R/\mathfrak{m}) = 0 \ \forall\, \mathfrak{m} \in \mathrm{Max\,Spec}\,R$

$\Longrightarrow K$ is projective!

Proof: $\boxed{\mathrm{Tor}^R_{n+1}(M, R/\mathfrak{m}) = 0 \ \forall \mathfrak{m}} \Longrightarrow \mathrm{Tor}^R_1(K, R/\mathfrak{m}) = 0 \ \forall \mathfrak{m} \Longrightarrow K =$ projective! ∎

Now: $K =: P_n \Longrightarrow 0 \to P_n \to P_{n-1} \to \dots \to P_0 \to M \to 0$ is a proj resol. of length $\leq n$!

Theorem (Hilbert's syzygy theorem) $R = \mathbb{C}[x_1, \dots, x_n]$, M = f.g. R-module

$\Rightarrow$ $\underbrace{\exists \text{ proj. resol. of length} \le n}_{\text{"}pd(M) \le n\text{"}}$: $0 \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0$.

(ex: $R = \mathbb{C}[\varepsilon]/\varepsilon^2 \Rightarrow pd(\mathbb{C} = R/(\varepsilon)) = \infty$)

Proof it suffices to show that $\mathrm{Tor}^R_{n+1}(M, \underset{(*)}{\mathbb{C}[x]/m}) = 0 \quad \forall\, m \subseteq \mathbb{C}[x]$

- idea: calculate $(*)$ by projectively resolving $\mathbb{C}[x]/m$! (i.e. show that $\exists$: $0 \to Q_n \to Q_{n-1} \to \dots \to Q_0 \to \mathbb{C}[x]/m \to 0$)
- $m = m_c = (x_1 - c_1, \dots, x_n - c_n)$ for some $c \in \mathbb{C}^n$ (HNS)
- w.l.o.g.: $m = m_0 = (x_1, \dots, x_n)$ ⇝ look for $0 \to \underbrace{Q_n \to \dots \to Q_0}_{\text{projective}} \to \mathbb{C}[x]/(x) \to 0$
- $n=1$ $R = \mathbb{C}[x]$ $0 \to \mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x] \to \mathbb{C} \to 0$, $x \mapsto 0$
- $n=2$ $R = \mathbb{C}[x,y]$ $0 \to \mathbb{C}[x,y] \to \mathbb{C}[x,y]^2 \to \mathbb{C}[x,y] \to \mathbb{C} \to 0$, $\mathbb{C}[x,y]/(x,y)$; $1 \mapsto (y, -x)$, $e_1 \mapsto x$, $e_2 \mapsto y$, $1 \mapsto \bar{1}$

in general: $0 \leftarrow \mathbb{C} \leftarrow \underset{0}{\mathbb{C}[x]} \leftarrow \dots \leftarrow \underset{p}{\mathbb{C}[x]^{\binom{n}{p}}} \leftarrow \dots \leftarrow \underset{n}{\mathbb{C}[x]^{\binom{n}{n}}} \leftarrow 0$

- Koszul-complex
- $\sum_{p=0}^{n} \binom{n}{p} (-1)^p = (1-1)^n = 0$

$\mathrm{Tor}_i^R(M,N)$: $P_\bullet \to M$ ⇝ apply $P_\bullet \otimes N$ ⇝ homology (left derived fct. of $\otimes$)

- $M \longmapsto \mathrm{Hom}_R(M,N)$ $\quad \mathrm{Hom}(\cdot, N) : \mathrm{Mod}_R \xrightarrow{F} \mathrm{Mod}_R^{opp}$ right exact: $\boxed{M' \to M \to M'' \to 0 \Rightarrow FM' \to FM \to FM'' \to 0}$ ex. $|\ \mathrm{Mod}^{opp}$

$L_i F(M) = \mathrm{Ext}_R^i(M,N)$

i.e. $P_\bullet \to M \Rightarrow \mathrm{Hom}(P_{i-1}, N) \to \mathrm{Hom}(P_i, N) \to \mathrm{Hom}(P_{i+1}, N)$ $\quad \mathrm{Hom}(M',N) \leftarrow \mathrm{Hom}(M,N) \leftarrow \mathrm{Hom}(M'',N) \leftarrow 0 \ |\ \mathrm{Mod}$

$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \ |\ \mathrm{Hom}(\cdot, \mathbb{Z})$

$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$

Example: $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ $\quad \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ $\quad H_i = \mathrm{Ext}^i(M,N)$ $\quad 0 \leftarrow \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \leftarrow 0$ $\quad \mathrm{Ext}^1 = \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Ext}^0 = 0$

$i=0 \Rightarrow 0$; $\quad i=1$: $0 \to \boxed{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z} \to 0 \Rightarrow \mathrm{Hom}(\mathbb{Z},N) \xleftarrow{\cdot 2} \mathrm{Hom}(\mathbb{Z},N) \leftarrow 0$

Def M, N = R-modules $\quad \mathrm{Ex}_R(M,N) := \{0 \to N \to X \to M \to 0\}/\cong$

$0 \to N \to X \downarrow Y \to M \to 0$

Theorem $\mathrm{Ext}^1_R(M,N) \xrightarrow[\text{bijective}]{} \mathrm{Ex}_R(M,N)$

Before: $\mathrm{Ex}(\cdot,\cdot)$ is a bi-functor, i.e. $M \to M' \Rightarrow \mathrm{Ex}(M,N) \leftarrow \mathrm{Ex}(M',N)$

$N \to N' \Rightarrow \mathrm{Ex}(M,N) \to \mathrm{Ex}(M,N')$

$\boxed{0 \to N \to X \xrightarrow{\alpha} M' \to 0}$, $\beta: M \to M'$, $0 \to N \to \tilde{X} \to M \to 0$

$\tilde{X} = \{(x,m) \in X \oplus M \mid \alpha(x) = \beta(m)\}$

$\boxed{0 \to N \xrightarrow{\alpha} X \to M \to 0}$ $\downarrow$ $\boxed{0 \to N' \to \tilde{X} \to M \to 0}$

$\tilde{X} := X \oplus N' / (\alpha(u), -\beta(u)) \mid u \in N$

$+ : \{0 \to N \to X \to M \to 0,\ 0 \to N \to Y \to M \to 0\}$ ⇝ $0 \to N \oplus N \to X \oplus Y \to M \oplus M \to 0$

$\mathrm{Ext}^1_R(M,N) \longrightarrow \mathrm{Ex}(M,N)$, $\xi \in \mathrm{Ext}^1_R(M,N)$ $\quad$ choose $P \to M \Rightarrow 0 \to K \to P \to M \to 0$

$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(P,N) \to \mathrm{Hom}(K,N) \to \mathrm{Ext}^1(M,N) \to \cancel{\mathrm{Ext}^1(P,N)} = 0$

$\exists\, \tilde{\xi} \longmapsto \xi$

$\Rightarrow K \xrightarrow{\tilde{\xi}} N \Rightarrow$ $\boxed{0 \to K \to P \to M \to 0} \in \mathrm{Ex}(M,K)$ $\downarrow \tilde{\xi}$ $\boxed{0 \to N \to \tilde{K} \to M \to 0} \in \mathrm{Ex}(M,N)$

$\mathrm{Ex}(M,N) \to \mathrm{Ext}^1_R(M,N)$: $0 \to N \to X \to M \to 0$, $0 \to K \to P \to M \to 0$

Example: $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ ⇝ $\mathrm{Ex}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$

- $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$
- $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$