alg. geom. / k : $k = \bar{k}$ (e.g.: $k = \mathbb{C}$)

- $\mathbb{C}^n \supseteq \underline{V(J)} = \{ \underline{c} \in \mathbb{C}^n \mid f(\underline{c}) = 0 \ \forall f \in J \}$ ⇝ declare them as „closed subsets"
  $(J \subseteq \mathbb{C}[x_1, \dots, x_n])$ ⇝ „Zariski topology"
  $f \in \mathbb{C}[\underline{x}]$ ⇝ $D(f) := \mathbb{C}^n \setminus V(f)$ (open)
- Spec $k[x_1, \dots, x_n]$, more gen.: R = ring ⇝ $\operatorname{Spec} R := \{ P \subsetneq R \ \ PI \}$

$$\begin{array}{ccc} R = \text{ring} & \rightsquigarrow & \operatorname{Spec} R \\ \cup & & \cup \\ J \text{ ideal} & \rightsquigarrow & V(J) = \{ P \in \operatorname{Spec} R \mid P \supset J \} \end{array}$$

Relation:

$R = \mathbb{C}[x_1, \dots, x_n] \overset{\text{HNS}}{\Longrightarrow}$ all max ideals are: $\mathfrak{m}_{\underline{c}} := (x_1 - c_1, \dots, x_n - c_n)$ $(\underline{c} \in \mathbb{C}^n)$

$\Rightarrow \mathbb{C}^n \longleftrightarrow \operatorname{Max\,Spec} \mathbb{C}[\underline{x}] \subseteq \operatorname{Spec} \mathbb{C}[\underline{x}]$

$\underline{c} \longmapsto \mathfrak{m}_{\underline{c}}$

$\mathbb{C}^1$ $\mathbb{C}[x]$ $c \doteq (x - c)$ $(x-c), (0)$

$\mathbb{C}^2$: $R = \mathbb{C}[x, y]$ $(x - c, y - d)$ PI: $(y^2 - x^3)$

$C := V(y^2 - x^3) = \{ (c, d) \mid d^2 = c^3 \}$ ↳ added point on C

Ex: $R \xrightarrow{\varphi} S \Rightarrow Q \in \operatorname{Spec} S \rightsquigarrow \varphi^{-1}(Q) \in \operatorname{Spec} R$

$\Downarrow$

$$\begin{array}{ccc} \operatorname{Spec} S & \longrightarrow & \operatorname{Spec} R \quad \text{cont.} \\ \cup & & \cup \\ \operatorname{Max\,Spec} S & \not\dashrightarrow & \operatorname{Max\,Spec} R \end{array}$$

Flatness **Def.** R = ring, M = R-module. $M = (R\text{-})\text{flat} :\Leftrightarrow (\otimes_R M)$ = exact functor

$\Leftrightarrow \forall N_1 \hookrightarrow N_2$: $N_1 \otimes_R M \longrightarrow N_2 \otimes_R M$ stays injective.

Example: $\mathbb{Z}/2\mathbb{Z}$ is not flat as a $\mathbb{Z}$-module.

If $0 \to A \to B \to C \to 0$ is exact (R-modules)

$\dots \to \operatorname{Tor}_2^R(C, M) \to \operatorname{Tor}_1^R(A, M) \longrightarrow \operatorname{Tor}_1^R(B, M) \longrightarrow \operatorname{Tor}_1^R(C, M) \to A \otimes M \longrightarrow \underset{\operatorname{Tor}_0^R(B, M)}{\underset{\shortparallel}{B \otimes M}} \longrightarrow C \otimes M \longrightarrow 0$

- M = flat (over R) $\Longleftrightarrow \forall$ R-modules N : $\operatorname{Tor}_1^R(M, N) = 0$
- IF M has finite presentation $\Longrightarrow \big[M = \text{flat} \Longleftrightarrow \forall \mathfrak{m} \in \operatorname{Max\,Spec} R : \operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0 \big]$
  should/will be supplemented by the „local criterion of flatness" $\Leftrightarrow \forall P \in \operatorname{Spec} R : \operatorname{Tor}_1^R(M, R/P) = 0$

Example: ① $\mathbb{C}[t] \xrightarrow{\varphi_1} \mathbb{C}[x, t]/(x^2 - t)$, $t \longmapsto t$ $\mathbb{C}^2 \supseteq V(x^2 - t) \xrightarrow{\pi_1} \mathbb{A}^1 = \mathbb{C}^1$

② $\mathbb{C}[t] \xrightarrow{\varphi_2} \mathbb{C}[x, t]/(xt - t)$ $\mathbb{C}^2 \supseteq V(t(x-1)) \xrightarrow{\pi_2}$

① fiber $\pi_1^{-1}(0) = \operatorname{Spec} \mathbb{C}[x, t]/\underset{\underset{(x^2 - t, t)}{\shortparallel}}{(x^2 - t) + (t)} = \operatorname{Spec} \mathbb{C}[x]/(x^2)$ ‖ $\pi_1^{-1}(1) = \operatorname{Spec} \boxed{\mathbb{C}[x]/\underset{(x-1)(x+1)}{(x^2 - 1)}}$ $= \mathbb{C}[x]/(x-1) \times \mathbb{C}[x]/(x+1)$

② fiber $\pi_2^{-1}(0) = \operatorname{Spec} \mathbb{C}[x, t]/(xt - t, t) = \operatorname{Spec} \mathbb{C}[x] = \mathbb{C}^1$

$R \rightsquigarrow (\operatorname{Spec} R, R)$

$\mathbb{C}[x]/(x) \rightsquigarrow (pt, \mathbb{C}[x]/(x) = \mathbb{C})$

$\mathbb{C}[x]/(x^2) \rightsquigarrow (pt, \mathbb{C}[x]/(x^2) = \mathbb{C} \oplus \mathbb{C}x)$

$X \xrightarrow{\pi} Y$ ≙ $(y \in Y) \longmapsto \pi^{-1}(y) \longmapsto v(\pi^{-1}(y)) \in \mathbb{Z}$ locally const.

φ_1 = flat : $\mathbb{C}[t] \longrightarrow \mathbb{C}[x] = \underset{\underset{x^2}{\shortparallel}}{\mathbb{C}[t]} \oplus \underset{\underset{x^2}{\shortparallel}}{\mathbb{C}[t]} \cdot x$ ⇝ free $\mathbb{C}[t]$-module!
$t \longmapsto x^2$

$\varphi_2 \neq$ flat:

$\operatorname{Tor}_1^{\mathbb{C}[t]}\Big(\underset{\underset{R}{\shortparallel}}{\mathbb{C}[x, t]/(xt - t)}, \mathbb{C}[t]/(t) \Big)$: $\mathbb{C}[t]/(t) \longleftarrow \boxed{\mathbb{C}[t] \xleftarrow{\cdot t} \mathbb{C}[t]} \longleftarrow 0$

$\otimes_{\mathbb{C}[t]} R \Longrightarrow R \xleftarrow{\cdot t} R$ Q: $\operatorname{Tor}_1 = \operatorname{Ker} = 0$?

$\Big(\mathbb{C}[t] \xrightarrow{\cdot t} \mathbb{C}[t]$ inj. $\underset{\otimes R}{\overset{?}{\Longrightarrow}}$ does $R \xrightarrow{\cdot t} R$ stay inj.? $\Big)$

$\mathbb{C}[x, t]/t(x-1) \xrightarrow{\cdot t} \mathbb{C}[x, t]/t(x-1)$

$(x - 1) \longmapsto 0$

Prop: N = R-module. N is flat $\Longleftrightarrow \operatorname{Tor}_1^R(R/I, N) = 0$ $\forall$ fin. gen. ideals I.

Proof: $(\Rightarrow)$ ✓

$(\Leftarrow)$ Let $M' \overset{f}{\hookrightarrow} M$ two R-modules, f = injective $\overset{(?)}{\Longrightarrow} M' \otimes_R N \xrightarrow{f \otimes \mathrm{id}_N} M \otimes_R N$ injective?

$M \otimes_R N = R^{(M \times N)} / L$, $L = \langle (m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n, \dots \rangle$

If $f \otimes \mathrm{id}$ is not injective $\Rightarrow \exists\, x = \sum_i m_i' \otimes n_i \longmapsto \sum_i f(m_i') \otimes n_i = 0$ in $M \otimes_R N$

i.e.: $\sum_i [f(m_i'), n_i] \in L$

$N = R$-module, $\mathrm{Tor}_1^R(M, R/I) = 0$ $\forall$ fin. gen. ideals $\overset{(?)}{\Longrightarrow}$ N = flat.

__Assume__ NOT $\Rightarrow$ $\exists\ M' \hookrightarrow M$: $M' \otimes N \longrightarrow M \otimes N$, $x \longmapsto 0$ } witnessed by using finitely many elements of M', M, N

$\rightsquigarrow$ find submodules $\overline{M}' \subseteq M'$, $\overline{M} \subseteq M$: $\overline{M}' \otimes N \longrightarrow \overline{M} \otimes N$, $x \longmapsto 0$

$\overline{M}' \hookrightarrow \overline{M}$ f.g. R-modules!

$\rightsquigarrow$ w.l.o.g. M, M' = f.g.

$M' \hookrightarrow M = \langle m_1, \dots, m_k \rangle$ generators over R.

$M' = M_0 \hookrightarrow M_1 := M_0 + \langle m_1 \rangle \hookrightarrow M_2 := M_0 + \langle m_1, m_2 \rangle \hookrightarrow \dots \hookrightarrow M_k = M$

$M_0 \hookrightarrow M_1 \hookrightarrow \dots \hookrightarrow M_k$, $M_0 = M'$, $M_k = M$

$M_0 \otimes N \longrightarrow M_1 \otimes N \longrightarrow \dots \longrightarrow M_k \otimes N$ not injective

$\Rightarrow \exists\, i$: $\boxed{M_i} \otimes N \to \boxed{M_{i+1}} \otimes N$ is not injective

$\boxed{M_{i+1} = M_i + \langle m_{i+1} \rangle}$

$0 \to M_i \to M_{i+1} \to M_{i+1}/M_i \to 0$

$M_{i+1}/M_i = \langle \overline{m}_{i+1} \rangle \xleftarrow{\sim} R/I$, $\overline{m}_{i+1} \longleftarrow 1$

$\Rightarrow$ $\boxed{\underset{0^*}{\mathrm{Tor}_1^R}(\underset{R/I}{M_{i+1}/M_i}, N)} \to M_i \otimes N \to M_{i+1} \otimes N$

$\mathrm{Tor}_1(R/I, N) \neq 0 \Rightarrow$ [$I \subseteq R \rightsquigarrow I \otimes N \longrightarrow R \otimes N$ does not stay injective]

$0 \to I \to R \to R/I \to 0 \Rightarrow \underset{=0}{\mathrm{Tor}_1^R(R, N)} \to \mathrm{Tor}_1^R(R/I, N) \to \boxed{I \otimes N \to R \otimes N} \to R/I \otimes N \to 0$

$\rightsquigarrow$ repeat the 1st step $\rightsquigarrow$ w.l.o.g.: I = f.g. //

① $R = \mathbb{C}[\varepsilon]/\varepsilon^2$; $N = R$-module $\Longleftrightarrow$ $N = \mathbb{C}$-vs, with $\varphi : N \longrightarrow N$, i.e. $\varphi \in \mathrm{End}_{\mathbb{C}}(N)$ and $\varphi^2 = 0$.

($\rightsquigarrow$ $\mathrm{Im}\,\varphi \subseteq \mathrm{Ker}\,\varphi$)

$H(N) := \mathrm{Ker}\,\varphi / \mathrm{Im}\,\varphi$

N = flat $\Longleftrightarrow$ $H(N) = 0$.

Q. N is flat $\Longleftrightarrow$ (??)

$\Updownarrow$

$\mathrm{Tor}_1^R(N, R/I) = 0$ $\forall I$

$I \hookrightarrow R$

$0 \to (\varepsilon) \longrightarrow R \to R/(\varepsilon) \to 0$, $(\varepsilon) = \varepsilon \cdot R = \varepsilon \cdot \mathbb{C}$, $R/(\varepsilon) = \mathbb{C}$

ideals in $R = \mathbb{C}[\varepsilon]/\varepsilon^2$: $(0), (1), (\varepsilon)$

$\boxed{\mathrm{Ker}\,\varphi = \mathrm{Im}\,\varphi}$

N is flat $\Longleftrightarrow$ $I \otimes N \longrightarrow R \otimes N$ stays flat

$\mathbb{C} = \mathbb{C}[\varepsilon]/(\varepsilon)$; $\mathbb{C}[\varepsilon]/(\varepsilon) \otimes N = \mathbb{C} \otimes_R N \xrightarrow{\cdot \varepsilon} N \Rightarrow$ $\boxed{N/\varepsilon \cdot N \xrightarrow{\cdot \varepsilon} N \text{ is inj.}}$ $\Longleftrightarrow$ $N/\varepsilon N \underset{\cdot \varepsilon}{\xrightarrow{\sim}} \varepsilon \cdot N$

- K-modules $\hat{=}$ K-vs
- $\mathbb{Z}$-module $\hat{=}$ abelian grps.
- $R[x]$-module $\hat{=}$ R-modules together with $\varphi \in \mathrm{End}_R(M)$ ($\hat{=}$ (M, φ) $M = R$-module...)
- R/I-modules $\hat{=}$ R-modules s.t. $I \cdot$ module $= 0$
- $S^{-1}R$-module $\hat{=}$ R-modules s.t. $s \in S \rightsquigarrow (\cdot s) : M \xrightarrow{\sim} M$

② R = domain. __Def__ N is torsion free $\Longleftrightarrow$ $\forall\, r \in R \setminus 0$: $N \xrightarrow{\cdot r} N$ is injective

$\Uparrow \otimes_R N$

$R \overset{\cdot r}{\hookrightarrow} R$

- N = flat $\Longrightarrow$ N is torsion free
- $\not\Leftarrow$ __Ex__
  - $\mathbb{Q}$ = torsion free $\mathbb{Z}$-module; $\mathbb{Q}$ = flat $\mathbb{Z}$-module
  - $(x, y) \subseteq \mathbb{C}[x, y]$ is torsion free, but not flat. (we had $\mathrm{Tor}_1((x,y), \mathbb{C}[x,y]/(x,y))$)
- R = principal ideal domain $\Longrightarrow$ [N-flat $\Longleftrightarrow$ N = torsion free]

__Proof__: have to show: $I \subseteq R \Rightarrow I \otimes N \longrightarrow R \otimes N$ stays inj.

PID $\Longrightarrow$ $I = (r)$ $\Longrightarrow$ $(r) \otimes N \longrightarrow R \otimes N$, $(r) \otimes N \cong R \otimes N \xrightarrow{\cdot r} R \otimes N$ inj. because of „torsion free".

__Main ex__: $R = \mathbb{C}[t]$ $\rightsquigarrow$ [flat $\Longleftrightarrow$ tors. free]

⑪ Graded rings and modules

$\mathrm{Spec}\, R \supseteq U_i$, $\bigcup U_i = \mathrm{Spec}\, R$

$\Rightarrow$ $\exists$ refinement w.l.o.g.: $U_i = D(f_i) = \mathrm{Spec}\, R_{f_i}$

$\bigcup D(f_i) = \mathrm{Spec}\, R \Longleftrightarrow (f_i \mid i \in I) = (1)$

Quasi compact: $\exists$ finite $\overline{I} \subset I$: $(f_i \mid i \in \overline{I}) = (1)$

__Disadv.__ of affine varieties, e.g.: $\mathrm{Spec}\, \mathbb{C}[x] = \mathbb{C}^1$ — not compact!!

$V(I) \subseteq \mathrm{Spec}\, R \rightsquigarrow V(I) = \mathrm{Spec}\, R/I$

$D(f) \subseteq \mathrm{Spec}\, R$ is affine: $\mathrm{Spec}\, R_f$

$\mathbb{C}^2 \setminus \{(0,0)\} \neq$ affine

__Solution__: projective sets instead affine ones, e.g. $\mathbb{P}^1$ instead of $\mathbb{A}^1$

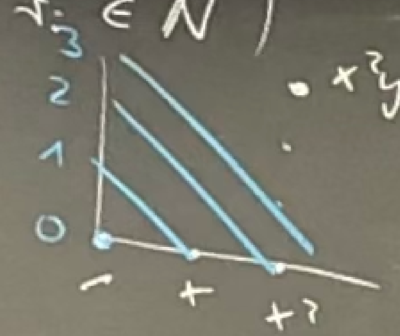
Let $A = (A, +)$ abelian group, e.g. $A = \mathbb{Z}$, $A = \mathbb{Z}^n$

Def A-graded ring is a ring R with a decomposition $R = \bigoplus_{a \in A} R_a$ (R_a = abelian groups)
such that $R_a \cdot R_b \subseteq R_{a+b}$ $(\forall a, b \in A)$

Notation $f \in R_a$ ⇝ "f is homogeneous of degree a"; $\deg(f) = a$.

Example $A = \mathbb{Z}$ ⇝ $\mathbb{C}[x_1, \dots, x_n]$ $\deg x_i := 1$ $(\Rightarrow \deg x_1^{r_1} \cdots x_n^{r_n} = \sum r_i \in \mathbb{N})$

$\mathbb{C}[x_1, \dots, x_n] = \bigoplus_{r \in \mathbb{N}^n} \mathbb{C} \cdot x^r = \bigoplus_{d=0}^{\infty} \left(\bigoplus_{r \in \mathbb{N}^n, \sum r_i = d} \mathbb{C} \cdot x^r \right)$ $\quad x^r = x_1^{r_1} \cdots x_n^{r_n}$, $r = (r_1, \dots, r_n)$

$\mathbb{C}[x_1, x_2] =$  $\cdot\, x^2 y^3 \triangleq (2,3)$ $\quad \mathbb{N}^2$

M = R-module ⇝ M is "graded R-module" :⟺ $M = \bigoplus_{a \in A} M_a$; $R_a \cdot M_b \subseteq M_{a+b}$
(R = graded ring) (same grading group A)

Ex · R = graded R-module

· M = graded R-module ⇝ $M(k) := M$ as an abelian group, $M(k)_a = M_{a+k}$ ($k \in A$)

Remark Every M_a is a R_0-module ; R_0 = ring,
↳ $R_0 \cdot M_b \subseteq M_b$

Ex · $R(k)$: $R(k)_a := R_{a+k}$ ⇝ $\forall k \in A$ $\exists$ module $R(k)$ ($= R$ as an abelian group!)

· $R = \mathbb{C}[x]$; $\deg x = 1$ $\quad R(3) = \mathbb{C}[x]$ as $\mathbb{C}[x]$-module

$\boxed{R(3)}_k = R_{k+3}$ $\quad k = 0$ $\quad$ $\deg x^3 = 0$

$\deg x = \cancel{1}\, {-2}$ $\quad \deg x = 1$

$\deg x^2 = \cancel{2}\, {-1}$

$B \xrightarrow{\varphi} A$ hom. of abelian groups

R = B-graded ring
$f \in R_b$, i.e. $\deg f = b$ $\Big\}$ ⟶ R becomes also an A-graded ring: $f \in R_b \Rightarrow \deg_{\text{new}}(f) := \varphi(b)$.

Ex. · $\mathbb{C}[x, y]$ is $\mathbb{Z}^2$-graded $\quad \deg(x^a y^b) := (a, b) \in \mathbb{N}^2 < \mathbb{Z}^2$

· $\mathbb{Z}^2 \xrightarrow{(1,1)} \mathbb{Z}$
$(a, b) \longmapsto a + b$ $\quad$ new-$\deg(x^a y^b) = a + b$

	$x y^4$, $x^2 y^3$	
deg	$(1,4)$, $(2,3)$	$(\mathbb{Z}^2)$
$\mathbb{Z}$-deg:	5, 5	

Let R = A-graded ring $(A = \mathbb{Z})$
$I \subseteq R$ is an ideal. $\quad$ **Q**: Is I = graded module?

Issue: $I \hookrightarrow R$

– $1_R \in R$: $\deg 1_R = d$
$\Rightarrow \deg 1_R = \deg(1_R \cdot 1_R) = d + d \Rightarrow \underline{d = 0}$

– $\mathbb{C}[x]$, $\mathbb{Z}$-grading ⇝ $\deg x^3 = 3$, $\deg x = 1$ $\quad | \quad \deg(x + x^3)$

$I \hookrightarrow R$ should become graded (of degree 0)

i.e. $I = \bigoplus_{d \in A} I_d$; $I_d \subseteq R_d$ $\quad$ $I \subseteq R$

$f \Rightarrow f = \sum f_d \in \bigoplus R_d$

$\sum g_d$, $g_d \in I_d \Rightarrow g_d \in R_d$ $\Big\}$ ⟶ $f_d = g_d$

Compare: $I \subseteq R$ is graded
⟺ $\forall f \in I$: $f = \sum f_d$ $(f_d \in R_d) \Rightarrow f_d \in I$

M, N = graded R-modules
$\varphi : M \to N$ is graded hom. of degree d
⟺ $\forall a \in A$: $\varphi(M_a) \subseteq N_{a+d}$

$\varphi : M \to N$ of degree d
≙ $\varphi : M(d) \to N$ of degree 0
≙ $\varphi : M \to N(d)$ (– " –)

Def $I \subseteq R$ is "homogeneous" ⟺ it is generated by hom. elements (i.e. R_d-elements)

Recall: I is graded ⟺ $I = \bigoplus I_d$ with $I_d = I \cap R_d$
⟺ $\forall f \in I$, $f = \sum f_d$ $(f_d \in R_d) \Rightarrow f_d \in I$

Ex: $\mathbb{C}[x]$, $\mathbb{Z}$-grading ; $\deg x = 1$
$I := (x - 1)$ $\quad$ · not homog.
· not graded : $x - 1 = f \Rightarrow f_1 = x$, $f_0 = -1$ $\quad$ but: $x, -1 \notin I$!

Prop: I = homog ⟺ graded.

Proof: $(\Leftarrow)$ · take generators $f^\nu \in I$
· $f^\nu = \sum_{d \in A} f_d^\nu$ ⇝ $f_d^\nu \in I$ still gens
homog

$(\Rightarrow)$ Exercise