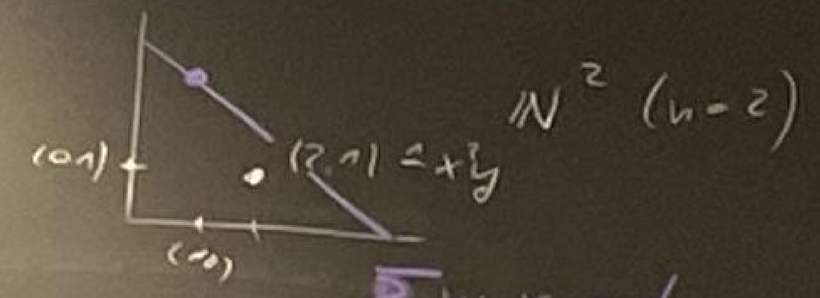
Grading 1

$\mathbb{C}[x_1,\dots,x_n] = R$, "$\mathbb{N}$" $\mathbb{Z}$-grading , fix $\boxed{w \in \mathbb{N}^n}$ ($\mathbb{R}^n_{\geq 0}$), i.e. $w=(w_1,\dots,w_n)$

$\rightsquigarrow$ R becomes graded via $\boxed{\deg x_i := w_i}$

Fix $d \in \mathbb{N} \rightsquigarrow R_d$



① $I \subseteq R$ ideal (not homogeneous yet)

Goal: Make I homogeneous: $I \rightsquigarrow I^h$

before: make $f \in R$ homogeneous: $f \rightsquigarrow f^h$

Build new ring $\hat{R} = \mathbb{C}[x_1,\dots,x_n,t]$, $\deg t = 1$ ($\deg x_i := w_i$)

$f(\underline{x}) \rightsquigarrow f^h(\underline{x},t)$ homog.

monomial $x^\alpha \rightsquigarrow \deg x^\alpha = \sum \alpha_i w_i$ $\rightsquigarrow \deg f^h = \deg f$ (goal)

$f \in R \rightsquigarrow \deg f := \max(\deg \text{summands})$

Example: • $w = (1,\dots,1)$
• $R = \mathbb{C}[x,y]$
• $f(x,y) = x^2 + y^4$ non-homog.! $t^4\left(\frac{x^2}{t^2} + \frac{y^4}{t^4}\right)$
• $I = (\dots)$ $\quad x^2t^2 + y^4 \; " \; t^4\left(\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^4\right)$

$f(x,y) = x^2 + y^4 \rightsquigarrow x^2t^2 + y^4$

$x + xy^3 + y \rightsquigarrow xt^3 + xy^3 + yt^3$

• $f^h(\underline{x},t) = t^{\deg f} \cdot f\left(\frac{x_1}{t^{w_1}},\dots,\frac{x_n}{t^{w_n}}\right)$ "homogenize f".

• $f^h(\underline{x},1) = f(\underline{x})$ "dehomogenize f^h" ($t=1$)

$f \longmapsto f^h \longmapsto f$

$G \longmapsto G(\underline{x},1) \longmapsto G(\underline{x},1)^h$ (homog. $\in \hat{R}$)

Example: • $G(x,y,t) = x^2y^3 + ty^4 \rightsquigarrow G(x,y,1) = x^2y^3 + y^4 \rightsquigarrow x^2y^3 + ty^4$

• $tx^2y^3 + t^2y^4 \rightsquigarrow$ —"— $\rightsquigarrow x^2y^3 + ty^4$

In general: • $G(\underline{x},t)$ = homog $\rightsquigarrow G(\underline{x},1)^h \cdot t^? = G(\underline{x},t)$

• $f,g \in R \rightsquigarrow f^h + g^h \longleftrightarrow (f+g)^h$

If $\deg f + a = \deg g$ ($a \in \mathbb{N}$) $\Rightarrow \boxed{t^a f^h + g^h = t^? (f+g)^h}$

Ex $f(x,y) = x + y^2 \Rightarrow tx + y^2$
$g(x,y) = y^5 \Rightarrow \boxed{y^5}$
$+ \quad x + y^2 + y^5 \to xt^4 + y^2t^3 \; \boxed{y^5}$

$f \in R \rightsquigarrow \boxed{in_w(f)} = \sum$ terms of highest degree ($in_w(x^3 + y^2 + xy^2) = x^3 + xy^2$)
$= f^h(\underline{x},t)|_{t=0}$

$I = (f_1,\dots,f_k) \rightsquigarrow I^h = \cancel{(f_1^h,\dots,f_k^h)}$

$I^h = \langle f^h \mid f \in I \rangle$

Problem: How to calculate I^h?

Good point: $\exists f_1,\dots,f_k \in I$: • $I = (f_1,\dots,f_k)$
• $I^h = (f_1^h,\dots,f_k^h)$

Gröbner Spree

Example: $I = (y - x^2, z - x^2, y - z)$

$(yt - x^2, zt - x^2)$

$y - z \in I$

$\Downarrow$

$y - z$ ③

Lemma: Let $I = (f_1,\dots,f_k) \Rightarrow$ • $I^h = ((f_1^h,\dots,f_k^h) : t^\infty)$ $\quad (I : \mathfrak{J}^\infty)$

• $I^h = (I^h : t^\infty)$

saturate $(f_1^h,\dots,f_k^h)$ w.r.t. t

I^h is saturated.

Proof: $((I : \mathfrak{J}^\infty) : \mathfrak{J}^\infty) = (I : \mathfrak{J}^\infty)$: have to show "$\subseteq$"

$f \in LHS \rightsquigarrow f \cdot \ell \in (I : \mathfrak{J}^\infty) \; \forall \ell \in \mathfrak{J}^{\gg 0} \Rightarrow f \cdot \ell \cdot \ell' \in I \; \forall \ell, \ell' \in \mathfrak{J}^{\gg 0}$

$\Rightarrow f \in (I : \mathfrak{J}^\infty)$.

$(f_1^h,\dots,f_k^h) \subseteq I^h \Rightarrow ((f_1^h,\dots,f_k^h) : t^\infty) \subseteq I^h$. Have to show "$\supseteq$"

Start with $F(\underline{x},t) \in I^h$, suffices: $F(\underline{x},t) = g(\underline{x})^h$ for some $g \in I$

$t^? g^h = t^? g_1^h + g_2^h t^?$, $g = g_1 + g_2$

$\Rightarrow F(\underline{x},1) = g(\underline{x}) \in (f_1,\dots,f_k) \Rightarrow \exists \lambda_i(\underline{x})$: $g(\underline{x}) = \sum \lambda_i(\underline{x}) \cdot f_i(\underline{x})$

$\Rightarrow \sum t^{k_i} (\lambda_i f_i)^h = (t^k) f^h$ $\quad (\exists k_i, k)$

("GB" $\deg(\lambda_i f_i) \leq \deg f \; \forall i$)

$\sum t^{k_i} \cdot (\lambda_i^h f_i^h) \in (f_1^h,\dots,f_k^h)$

Goal: $F(\underline{x},t) = t^? \cdot f^h \in ((f_1^h,\dots,f_k^h) : t^\infty)$

$\Rightarrow t^k \cdot F = t^k \cdot t^? f^h \in \dots$ //

Second proof of "$\supseteq$": $(f_1^h,\dots,f_k^h) \subseteq I^h \Rightarrow (f_1^h,\dots,f_k^h) : t^\infty \subseteq (I^h : t^\infty) \subseteq I^h$

Claim: I^h = saturated: $F(\underline{x},t) \in (I^h : t^\infty) \rightarrow \exists \ell \; t^\ell \cdot F(\underline{x},t) \in I^h$

$\overset{t=1}{\Rightarrow} F(\underline{x},1) \in I \Rightarrow F(\underline{x},1)^h \in I^h$

$\Rightarrow F(\underline{x},t), \; t^\ell \cdot F(\underline{x},t) \xmapsto{t \mapsto 1} F(\underline{x},1)$

$\Rightarrow \exists \ell' : F(\underline{x},t) = t^{\ell'} \cdot F(\underline{x},1)^h$ //

- $(f \cdot g)^h = \underline{f^h} \cdot g^h$ $\qquad f^h := t^{\deg_w f} \cdot f\left(\frac{x_1}{t^{w_1}}, \ldots\ldots, \frac{x_n}{t^{w_n}}\right)$

$\deg_w (f \cdot g) = (\deg_w f) + (\deg_w g)$ $\qquad (fg)^h = t^{\deg_w f} \cdot t^{\deg_w g} \cdot (fg)\left(\frac{x}{t^w}\right)$

$\qquad f(x/t^w) \cdot g(x/t^w)$

- $f = \sum_{i=1}^{n} g_i \implies \exists\, k_i, k : \ t^k \cdot f^h = \sum_{i=1}^{n} t^{k_i} \cdot g_i^h$

w.l.o.g.: $n = 2$ $\qquad f = g_1 + g_2 \implies$ $\underline{Q}$: $\exists\, k, k_i : \ (g_1 + g_2)^h \cdot t^k = g_1^h \cdot t^{k_1} + g_2^h \cdot t^{k_2}$

$F(x,t)$

Choose k_1, k_2: $\ F(x,t) =$ homogeneous

$F(x,1) = g_1^h(x,1) + g_2^h(x,1) = g_1(x) + g_2(x) = g_1 + g_2$

$\implies \exists\, k: \ F(x,t) = (g_1 + g_2)^h \cdot t^k$ $\qquad$ ($\underline{\text{recall}}$: $F(x,t) = \text{hom.} \implies \exists\, k \ \ F(x,t) = F(x,1)^h \cdot t^k$

- $I^h = t$-saturated
- $I^h = t$-sat. hull of $(f_1^h, \ldots, f_n^h)$

$\mathbb{C}[t] \hookrightarrow \mathbb{C}[x, t] \supseteq I^h \implies \tilde{X} := V(I^h) \subseteq \mathbb{A}^n \times \mathbb{A}^1$

$X := V(I) \subseteq \mathbb{A}^n$

$$\begin{tikzcd} \mathbb{C}[x]/\mathrm{in}\, I & p^{-1}(0) \arrow[rr, hook] \arrow[d] \arrow[dr, hook] & & \tilde{X} \arrow[d, "p"] \arrow[dr, hook] & \mathbb{C}[x,t]/I^h \\ & & \mathbb{A}^n & & \mathbb{A}^n \times \mathbb{A}^1 \arrow[dl] \\ \mathbb{C} \arrow[uu] & \{0\} \arrow[rr, hook] \arrow[ur, leftarrow] & & \mathbb{A}^1 & \mathbb{C}[t] \arrow[uu] \end{tikzcd}$$

$p: \tilde{X} \to \mathbb{A}^1 \ \hat{=}\ p^*: \mathbb{C}[t] \hookrightarrow \boxed{\mathbb{C}[x,t]/I^h} = T$

$I^h = (I^h : t^\infty) \implies I^h = (I^h : (t))$, i.e. $f \in \mathbb{C}[x,t]$, $f(x,t) \cdot t \in I^h \implies f \in I^h$

$\Updownarrow$

$f \in T$, $\ f \cdot t = 0 \implies f = 0$ in T

$\iff T \xrightarrow{\cdot t} T$ is injective

$\iff \left[\mathbb{C}[t] \xrightarrow{\cdot t} \mathbb{C}[t]\right] \otimes \mathrm{id}_T$ = injective

$\underline{\text{i.e.}}$: $\mathrm{Tor}_1^{\mathbb{C}[t]}\left(\mathbb{C}[t]/(t), T\right) = 0 \implies \boxed{T \text{ is flat over } \mathbb{C}[t]}$

$\underline{\text{i.e.}}$: $\boxed{p: \tilde{X} \to \mathbb{A}^1 \text{ is flat}}$

2 types of fibers:
- $p^{-1}(0)$
- $p^{-1}(c)$, $c \neq 0$

$p^{-1}(0) \hat{=} \operatorname{Spec} \boxed{\left(\mathbb{C}[x,t]/I^h\right) / (t)} = \mathbb{C}[x] / (f^h(x,0)) = \mathbb{C}[x] / (\mathrm{in}\, I)$

(I^h, t)

$f^h \in I^h$: $\ f^h(x,t) = (\mathrm{in}\, f)(x) + t \cdot (\text{lower terms})$

$f^h(x,0) = (\mathrm{in}\, f)(x)$

$f = x^2 + \boxed{y^3}$

$f^h = x^2 t + y^3$

"general fiber" $\ p^{-1}(c)$, $c \neq 0$

$\underline{\text{instead}}$ $\ p^{-1}(\mathbb{A}^1 \setminus \{0\}) = \operatorname{Spec} \boxed{\mathbb{C}[x, t^{\pm 1}]/I^h}$

$\operatorname{Spec} \mathbb{C}[t]_t = \operatorname{Spec} \mathbb{C}[t, t^{-1}]$

$\operatorname{Spec} \left(\mathbb{C}[x]/I\right)[t^{\pm 1}] = \operatorname{Spec} \mathbb{C}[x, t^{\pm 1}] / I = (f_1(x), \ldots, f_n(x))$

$\mathbb{C}[x]/I \cong X$

$\boxed{X \times (\mathbb{A}^1 \setminus 0)}$

$f^h(x,t) = t^{\deg_w f} \cdot f\left(\frac{x_1}{t^{w_1}}, \ldots, \frac{x_n}{t^{w_n}}\right)$

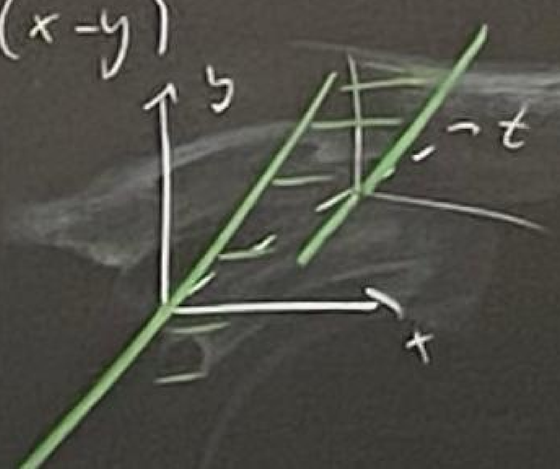
$$\begin{tikzcd} \boxed{\mathbb{C}[x, t^{\pm 1}]/I} \arrow[rr, "\sim"] & & \boxed{\mathbb{C}[x, t^{\pm 1}]/I^h} \\ & \mathbb{C}[t] \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$x_i \longmapsto t^{-w_i} \cdot x_i$

$I \ni f(x) \longmapsto f\left(\frac{x_1}{t^{w_1}}, \ldots, \frac{x_n}{t^{w_n}}\right) = t^{-\deg_w f} \cdot f^h(x,t)$

$\underline{\text{Ex}}$: $\ \operatorname{Spec} \mathbb{C}[x,y,t]/(x-y) \subseteq \mathbb{A}^3$

$= V(x-y)$



$$\boxed{\begin{tikzcd} p^{-1}(\mathbb{A}^1 \setminus 0) \arrow[r, hook] \arrow[d] & \mathbb{A}^n \times (\mathbb{A}^1 \setminus 0) \arrow[dl] \\ \mathbb{A}^1 \setminus 0 & \end{tikzcd}} \cong \boxed{\begin{tikzcd} X \times (\mathbb{A}^1 \setminus 0) \arrow[r, hook] \arrow[d] & \mathbb{A}^n \times (\mathbb{A}^1 \setminus 0) \arrow[dl] \\ (\mathbb{A}^1 \setminus 0) & \end{tikzcd}}$$

$\tilde{X}$ over $\mathbb{A}^1 \setminus 0$: $\ \cong X = V(I)$

over 0: $\ V(\mathrm{in}\, I)$

$V(I^h) \supseteq p^{-1}(\mathbb{A}^1 \setminus 0) = V(I^h) \setminus V(t) \rightsquigarrow V(I^h) = \overline{V(I^h) \setminus V(t)} = V(I^h : t^\infty)$

before

Artin–Rees

$I \subseteq A$ = noeth. ring, M = f.s. A-module.

• "filtrations" of M: $M_\nu \supseteq M_{\nu+1} \supseteq \dots$ (decreasing), $M_\nu \subseteq M$

• ex. $M_\nu = I^\nu \cdot M$, $M_{\nu+1} = I \cdot M_\nu$. $\quad M \supseteq IM \supseteq I^2M \supseteq \dots$

Q. $N \subseteq M$ submodule. $\quad$ Q $\bigcap_\nu I^\nu M \overset{??}{=} 0$

• take a filtration of M, e.g.: $M_k = I^k M$

• $N_k = M_k \cap N$ "induced filtration", e.g. $N_k = N \cap I^k M$

contained: $N'_k := I^k \cdot N \qquad N'_k \subseteq N_k \subseteq N'_{k-5}$

Def. • $\{M_k\}$ is called an I-filtration $\iff I M_k \subseteq M_{k+1} \subseteq M_k$

• this is called "I-stable" $\iff \exists k: I M_\ell = M_{\ell+1}$ for all $\ell \geq k$.

$\tilde A = \bigoplus_{n\geq 0} I^n \cong \bigoplus_{n\geq 0} I^n \cdot t^n$ (i.e. $\deg t = 1$)

graded ring, A-Algebra: $\tilde A_0 = A$ $\qquad f \in I$

$\tilde A$ = f.s. A-algebra:

$I = (f_1, \dots, f_r)$

$\Rightarrow \tilde A$ is generated, as an A-algebra, by $f_1 t, \dots, f_r t$.

($g \in \tilde A$, w.l.o.g. $g \in \tilde A_d = I^d \cdot t^d = (f_1^d, f_1^{d-1} f_2, \dots, f_r^d) \cdot t^d = \langle$ degree-d-monomials in $f_1 t, \dots, f_r t \rangle$)

$\Downarrow$

$\tilde A$ = noetherian!

M = A-module, $M_\bullet$ = I-filtration ⇝ $\tilde M := \bigoplus_{n \geq 0} M_n t^n = M_0 \oplus M_1 \oplus \dots \qquad I \cdot M_k \subseteq M_{k+1} \subseteq M_k$

$\tilde M$ = $\tilde A$-module! (graded) $\quad$ have to show: $\tilde A_d \cdot M_e \subseteq M_{d+e}$, $\tilde A_d = I^d$

Example $A = \mathbb{C}[x,y]$, $I = (x,y)$

$\tilde A_0 = \mathbb{C}[x,y]$

$\tilde A_1 = (x,y)$

$\tilde A_2 = (x,y)^2$

$f = f_0 + f_1 t$, e.g. $x - xt \neq 0 \quad (x, -x, 0)$; $x \in A$, $x \in I$

$x - xt \overset{?}{=} x(1-t)$??

Prop: $M_\bullet$ as before. Then $M_\bullet$ = I-stable $\iff \tilde M$ = noeth. $\tilde A$-module $\iff$ f.s. $\tilde A$-module.

Proof ($\Rightarrow$) stable $\Rightarrow \tilde M = \underbrace{M_0 \oplus \dots \oplus M_n}_{\text{finitely many submod. } M_i \subseteq M} \oplus I M_n \oplus I^2 M_n \oplus \dots$

w.l.o.g. $M_k = I^k M$

$M = \langle m_1, \dots, m_k \rangle$ ⇝ $\tilde M = \langle m_1 t, \dots, m_k t \rangle$

$\boxed{A \supseteq I \leadsto \tilde A := \bigoplus_{k\geq 0} I^k \cdot t^k \quad \tilde A = \text{"blow up of } A \text{ in } I\text{"}}$

($\Leftarrow$) Let's assume not stable

$\tilde M = M_0 \oplus M_1 \oplus M_2 \oplus \dots$

$\tilde M(0) := M_0 \oplus I M_0 \oplus I^2 M_0 \oplus \dots$

$\tilde M(1) := M_0 \oplus M_1 \oplus I M_1 \oplus I^2 M_1 \oplus \dots$

$\tilde M(2) := M_0 \oplus M_1 \oplus M_2 \oplus I M_2 \oplus \dots$

$\subseteq \tilde M$

$\Rightarrow$ terminates $\Rightarrow \exists k: I M_k = M_{k+1}$. //

Corollary (Artin-Rees-Lemma). ① $N \subseteq M \Rightarrow \exists \mu: I(I^\nu M \cap N) = I^{\nu+1} M \cap N \quad \forall \nu \geq \mu$

Proof. • M ⇝ $M_\bullet := I^\bullet M$ = I-stable ⇝ $\tilde M$ = $\tilde A$-module, $\tilde M = \bigoplus I^k M$, $\tilde A = \bigoplus I^k$

• $\tilde N := \bigoplus_{k \geq 0} (I^k M \cap N) \subseteq \tilde M \Rightarrow \tilde N$ = noetherian $\tilde A$-module (submodule of a noeth. one); $\tilde A$-module

$\Rightarrow k \mapsto I^k M \cap N$ is I-stable $\Rightarrow \exists k\ \forall \ell \geq k$: $I(I^k M \cap N) = I^{k+1} M \cap N$

② $\forall \nu \geq \mu$: $I^\ell (I^\mu M \cap N) = I^{\mu+\ell} M \cap N \quad (\forall \ell \geq 0)$

$I^\ell N \quad \boxed{I^{\mu+\ell} M \cap N} \quad I^{\mu+\ell} N \qquad \mu = \text{fix}, \ell \geq 1$

③ Special setup: • $N := \bigcap_{\nu=0}^{\infty} I^\nu M$ $\quad$ ①: $I \cdot (I^\nu M \cap \bigcap_{i=0}^{\infty} I^i M) = I^{\nu+1} M \cap \bigcap_{i=0}^{\infty} I^i M$, $\quad = I \cdot N$

• eg: $I = \mathfrak{m}$ in a local ring

$\Rightarrow N = 0$