(12) "Projective" Varieties

$Y \subseteq \mathbb{A}^n = k^n \rightsquigarrow k[x_1, \dots, x_n] \supseteq I(Y)$

$(k = \bar{k}) \rightsquigarrow A(Y) := k[\underline{x}]/I(Y)$

$V(J) \leftarrow\!\sim\!\sim \; J \subseteq k[\underline{x}]$

$k[x_1, \dots, x_n] \twoheadrightarrow A$

1-1: alg. subsets in $\mathbb{A}^n \longleftrightarrow$ reduced ideals in $k[x_1, \dots, x_n]$

$\longleftrightarrow$ $\boxed{\text{reduced algebras } A}$ with explicit n generators

$\sqrt{0} = 0$ f.g. over k

(alg. sets, reg. fct.) $\longleftrightarrow$ f.g. k-algebras

$= A$ $\qquad = A$

Ex: $A = k[x]/(x^2)$, $B = k[x]/(x) = k \Longleftrightarrow$ points (both)

affine varieties $\longleftrightarrow$ rings

$(\mathrm{Spec}\, A, A) \longleftrightarrow A$

($\nearrow$ Problem 13)

$A \xrightarrow{\varphi} B$ ring hom. $\rightsquigarrow \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$ cond.

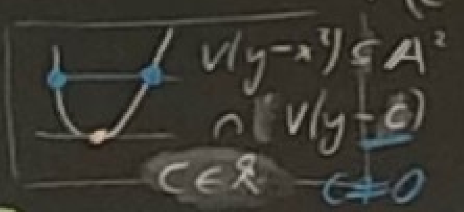
$Q \longmapsto \varphi^{-1}(Q)$

(1) $A \xrightarrow{\varphi} B$ is surjective, $I := \ker \varphi \rightsquigarrow \varphi : A/I \xrightarrow{\sim} B \rightsquigarrow \mathrm{Spec}\, B = \mathrm{Spec}\, A/I \hookrightarrow \mathrm{Spec}\, A$

$\qquad = V(I)$

$\boxed{A/I \twoheadleftarrow A}$

ex: $k[\varepsilon]/(\varepsilon^2) \longrightarrow k[\varepsilon]/(\varepsilon) = k \rightsquigarrow \mathrm{Spec}\, k \xhookrightarrow[\text{closed}]{} \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2$



$c \to 0$ $\quad c = 0$: $A = \mathbb{C}[x,y]/(y - x^2, y)$ $\quad \mathbb{C}[x]/(x^2)$

homeomorphism

$(pt, k) \hookrightarrow (pt, k[\varepsilon]/\varepsilon^2)$ is not an isomorphism.

(2) open embeddings

Def: Affine variety = pair $(\mathrm{Spec}\, A, A)$ for A = ring.

Open subset in $\mathrm{Spec}\, A$:

$\mathrm{Spec}\, A \setminus V(J) \overset{\sim}{\hookrightarrow} \mathrm{Spec}\, A$

$\mathrm{Spec}\, A \hookrightarrow \mathbb{C}^n \quad V(J)$

$\mathrm{Spec}\, A_f \hookrightarrow \mathbb{C}^{n+1} \quad V(J, f(x)t - 1)$

2 cases: (i) $J = (f) \Longrightarrow \mathrm{Spec}\, A \setminus V(f) =: D(f) = \mathrm{Spec}\, A_f$; $\;\; \hat{=} \; \boxed{A \longrightarrow A_f}$

(ii) $J \neq (f) \rightsquigarrow$ usually not affine: $(\mathbb{C}^2 \setminus 0) \hookrightarrow \mathbb{C}^2$; $\mathrm{Spec}\, \mathbb{C}[x,y] \setminus V(x,y) \hookrightarrow \mathrm{Spec}\, \mathbb{C}[x,y]$

Claim: $(\mathbb{C}^2 \setminus 0)$ $\neq$ affine, i.e. $\nexists A : (\mathbb{C}^2 \setminus 0) = \mathrm{Spec}\, A$. $\qquad \mathbb{C}^2 \leftrightarrow \mathrm{Spec}\, \mathbb{C}[x,y]$

Proof.

$A = \{\text{reg. fct. on } \mathbb{C}^2 \setminus 0\}$

$\rightsquigarrow \mathrm{Spec}\, \underbrace{A_{red}} = A/\sqrt{0}$

$\mathbb{C}^2 \setminus V(x)$

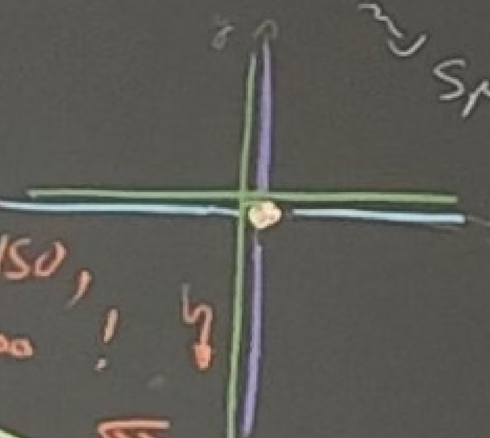
$\mathbb{C}^2 \setminus (V(x) \cup V(y))$

$= \mathbb{C}^2 \setminus V(xy)$

$\mathbb{C}^2 \setminus V(y)$

$$\boxed{\begin{matrix} (\mathbb{C}^2 \setminus 0) & \overset{\iota}{\hookrightarrow} & \mathbb{C}^2 \\ A & \hookleftarrow & \mathbb{C}[x,y] \end{matrix}}$$

ι = ISO, too!



$\mathbb{C}[x^{\pm 1}, y] = \mathbb{C}[x,y]_x$

$\mathbb{C}[x^{\pm 1}, y^{\pm 1}] = \mathbb{C}[x,y]_{xy}$

$A \hookleftarrow \mathbb{C}[x,y]$

$\mathbb{C}[x, y^{\pm 1}] = \mathbb{C}[x,y]_y$

$\rightsquigarrow A$ is in between $\mathbb{C}[x,y]_x$, $\mathbb{C}[x,y]_y$ and $\mathbb{C}[x,y]$

$\boxed{\mathbb{C}[x,y] \overset{\sim}{\hookrightarrow} A \text{ is an ISO!}}$

$\Longrightarrow \cdot\; A \subseteq \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$

$\cdot\; \forall a \in A \rightsquigarrow a = \dfrac{f(x,y)}{(xy)^n}$; $\quad \left.\begin{matrix} A \subseteq \mathbb{C}[x,y]_x : y \text{ is not a denominator} \\ A \subseteq \mathbb{C}[x,y]_y : x \quad \text{-"-} \end{matrix}\right\} \Longrightarrow a \in \mathbb{C}[x,y]$

$X = \mathrm{Spec}\, A \Longrightarrow A$ = regular functions on X

$\left(A = \mathbb{C}[x_1, \dots, x_n]/I \right.$, $X = V(I) \subseteq \mathbb{A}^n$

$\Downarrow \qquad \Downarrow$

$\boxed{A \qquad X = \mathrm{Spec}\, A}$

residue field

$\kappa(P)$

$f \in A$, $P \in \mathrm{Spec}\, A$

$f(P) := \bar{f} \in A/P \hookrightarrow \mathrm{Quot}\, A/P$

Good news: $\mathrm{Spec}\, A \setminus V(\underset{(f_1, \dots, f_n)}{J}) = \mathrm{Spec}\, A \setminus \left(\bigcap_{i=1}^{n} V(f_i) \right)$

$= \bigcup_{i=1}^{n} \mathrm{Spec}\, A \setminus V(f_i) = \bigcup_{i=1}^{n} D(f_i)$

"$\{D(f)\}$ = basis of open subsets of $\mathrm{Spec}\, A$"

Special examples for affine varieties

S = semigroup with 0, commutative (operation = "+") e.g. $S = (\mathbb{N}, +)$

Def: S has the "cancellation property" $:\Longleftrightarrow \forall a, b, c \in S : \left[a + c = b + c \Longrightarrow a = b \right]$

($\Longleftrightarrow \exists$ abelian group $G =$ "$S - S$" containing S)

S = s.g. for cancellation property

$G := \{ (s,t) \mid s \in S,\ t \in S \} / \sim \qquad (s,t) \sim (s',t') \iff s+t' = s'+t$ in S.

abelian group. $S \hookrightarrow G$, $s \mapsto (s,0)$ $\qquad \overline{(s,t)} =: s-t \rightsquigarrow G = \{ s-t \mid s,t \in S \}$ = „$S-S$"

Ex $\mathbb{R}^2 \supset$ polygons

Def. $P, Q \subseteq \mathbb{R}^n$ polyhedra $\rightsquigarrow P+Q := \{ p+q \mid p \in P,\ q \in Q \}$, „Minkowski sum"

Ex.

$\boxed{\Delta + \nabla = - + | + /}$

Cancell prop. $\quad P+Q = P'+Q \quad$ (polyhedra) $\xrightarrow{(?)} P = P'$ (?)

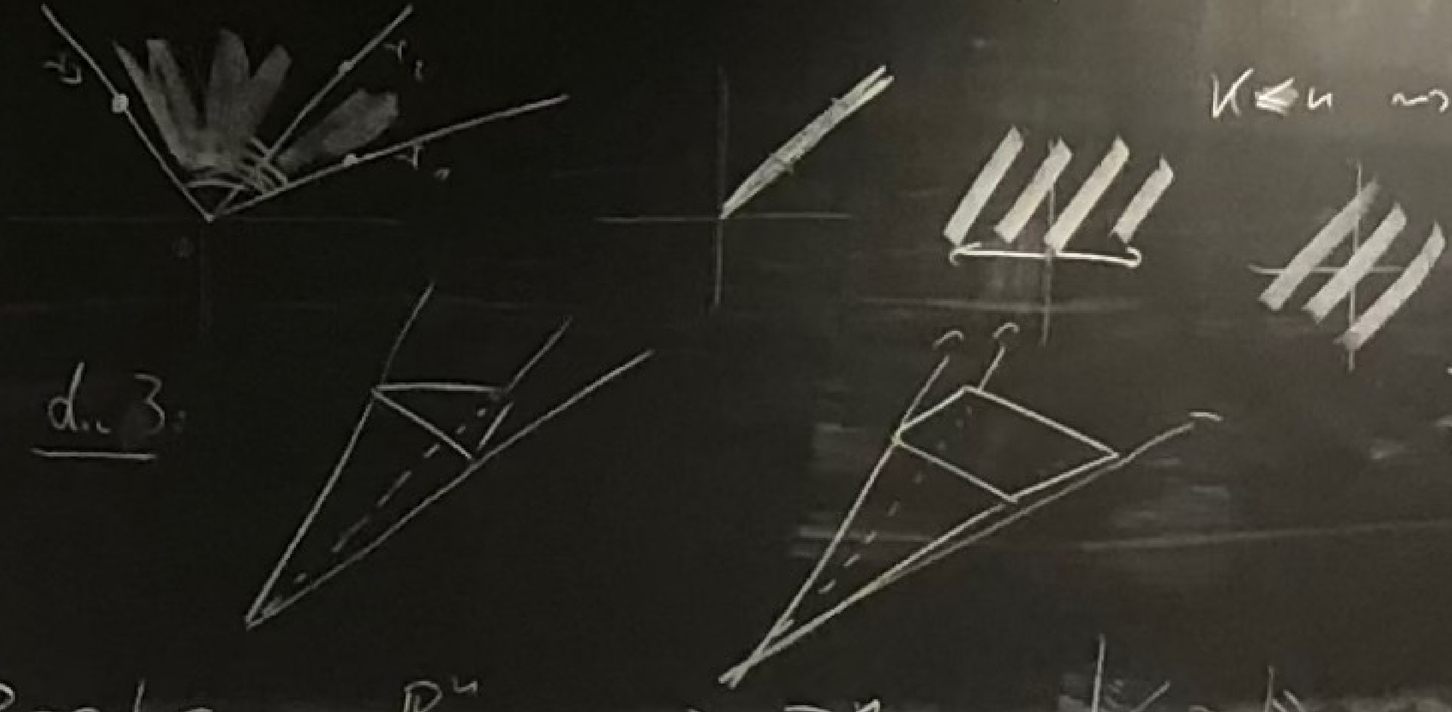
If P, Q, P' = bounded $\Rightarrow$ ok

Special ex. $\mathbb{R}^n \rightsquigarrow \sigma$ = „polyhedral cone" $:= \sum_{i=1}^{k} \mathbb{R}_{\geq 0} \cdot r_i$, $r_1, \ldots, r_k \in \mathbb{R}^n$

$k \leq n \rightsquigarrow$ „simplicial cone"

$\Rightarrow$ these σ are semigroups

dim 3:



Def. S = s.g. is finitely generated $\iff \exists\, s_1, \ldots, s_k \in S$. all s are of the form $\sum_{i=1}^{k} \lambda_i s_i$, $\lambda_i \in \mathbb{N}$.

$(\iff \exists\ \mathbb{N}^k \twoheadrightarrow S$, $e_i \mapsto s_i)$

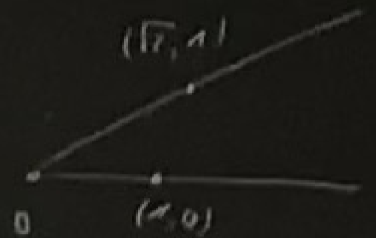
σ = pol. cone in $\mathbb{R}^n \rightsquigarrow \sigma \cap \mathbb{Z}^n$ = s.g. $\qquad$ over $\mathbb{R}$

Prop. σ = pol. cone in $\mathbb{R}^n$ (rational, $\mathbb{Q}$) $\rightsquigarrow S(\sigma) = \sigma \cap \mathbb{Z}^n$ is f.g. s.g..

Proof $\sigma \ni e_2$

$e_1 \in \mathbb{Z}^n$

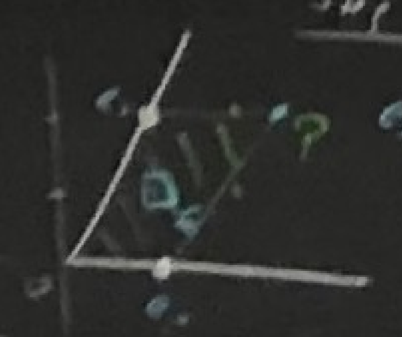
Counter example



Step 1. Choose $e_1, \ldots, e_k$ = (first) integral pts on the rays of σ

Step 2. $\langle e_1, \ldots, e_k \rangle = \mathbb{N}e_1 + \ldots + \mathbb{N}e_k \subseteq \sigma \cap \mathbb{Z}^n$

Ex. for „$\neq$".



$e_1 + e_2 = 2 \cdot$? $\qquad$ Q. „=" ?

Step 3. $P = \underbrace{\underbrace{\{ \sum_{i=1}^{k} \lambda_i e_i \mid \lambda_i \in [0,1] \subseteq \mathbb{R} \}}_{\text{compact}} \cap \underbrace{\mathbb{Z}^n}_{\text{discrete}}}_{\text{finite}}$

$\Downarrow$

$S(\sigma) = \langle e_1, \ldots, e_k, P \rangle$ $\qquad \square$

Stronger. If σ = pol, rat cone with vertex 0.

$H := \{ h \in S(\sigma) \mid \nexists\, a, b \in S(\sigma) \setminus \{0\} : h = a+b \}$ = set of non-decomposable elements ($S(\sigma) = \sigma \cap \mathbb{Z}^n$)

$\Rightarrow S(\sigma) = \langle H \rangle$; $\#H < \infty \rightsquigarrow H$ = THE minimal generating system: „Hilbert basis"

normaliz: input σ, output H of $S(\sigma)$.

Ex.

$(1,0) = (2,0) + (-1,0)$

$\sigma \subseteq \mathbb{R}^n \rightsquigarrow S(\sigma) = \sigma \cap \mathbb{Z}^n$, $\underset{\text{Hilbert basis}}{H(\sigma)} \subseteq S(\sigma) \rightsquigarrow \mathbb{C}[S(\sigma)]$ = „sg algebra"

Def. S = s.g. $\cdot\ \mathbb{C}[S] := \bigoplus_{s \in S} \mathbb{C} \cdot \chi^s$ ($S \subseteq \mathbb{C}[S]$ is the canonical basis as a $\mathbb{C}$-vs)

$\cdot\ (\lambda_s \cdot \underset{\chi^s}{\cancel{s}}) \cdot (\lambda_t \cdot \underset{\chi^t}{\cancel{t}}) = (\lambda_s \lambda_t) \cdot (\cancel{st})$ $\qquad \cancel{st = s+t}$ $\quad \chi^s \cdot \chi^t = \chi^{s+t}$

$\cdot\ \mathbb{C} \hookrightarrow \mathbb{C}[S]$, $1 \mapsto \chi^0$

$\sigma = \langle [1,0], [1,2] \rangle$

$S(\sigma) = \langle [1,0], [1,1], [1,2] \rangle$

$\mathbb{C}[S(\sigma)] = \langle \underset{A}{\chi^{[1,0]}}, \underset{B}{\chi^{[1,1]}}, \underset{C}{\chi^{[1,2]}} \rangle$ $\mathbb{C}$-alg algebra

$= \mathbb{C}[A,B,C] / (AC - B^2)$

Example. $\mathbb{N} \rightsquigarrow \mathbb{C}[\mathbb{N}] = \bigoplus_{g \in \mathbb{N}} \mathbb{C} \cdot \chi^g = \mathbb{C}[x] \Leftarrow \sigma = \mathbb{R}_{\geq 0} \subseteq \mathbb{R}^1$

$\mathbb{Z} \rightsquigarrow \mathbb{C}[\mathbb{Z}] = \bigoplus_{g \in \mathbb{Z}} \mathbb{C} \cdot \chi^g = \mathbb{C}[x, x^{-1}] = \mathbb{C}[x]_x$

$\sigma = \mathbb{R}^2_{\geq 0}$

$S(\sigma) = \mathbb{N}^2$

$\Rightarrow \mathbb{C}[S(\sigma)] = \mathbb{C}[x, y]$

$N = \mathbb{Z}^n$, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ $\rightsquigarrow$ $M \times N \longrightarrow \mathbb{Z}$, $a, b \longmapsto \langle a, b \rangle$

$\downarrow$

$N_{\mathbb{R}} = \mathbb{R}^n$, $M_{\mathbb{R}} = (\mathbb{R}^n)^* \cong \mathbb{R}^n$

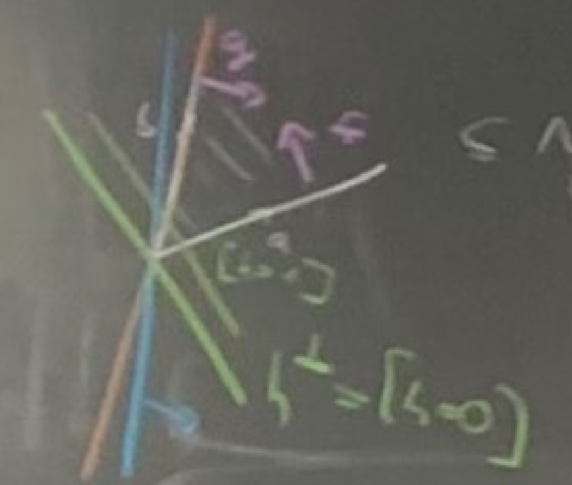
$(a_1, \dots, a_n)$ $[r_1, \dots, r_n]$

$\sigma \subseteq \mathbb{R}^n$ cone.......

$\sigma \subseteq N_{\mathbb{R}}$ cone $\rightsquigarrow$ „dual cone" $\sigma^\vee \subseteq M_{\mathbb{R}}$

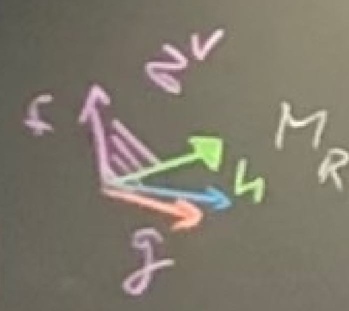
$\sigma^\vee := \{ r \in M_{\mathbb{R}} \mid \langle a, r \rangle \geq 0 \ \forall a \in \sigma \}$

$= \{ r \in M_{\mathbb{R}} \mid r \geq 0 \text{ on } \sigma \}$



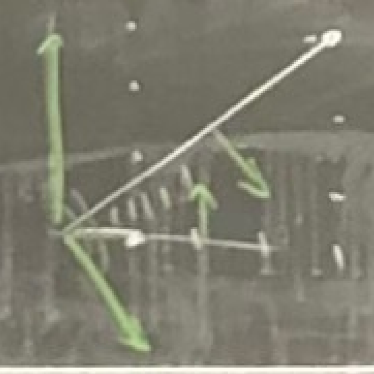
$\subseteq N_{\mathbb{R}}$

$b^\perp = [b = 0]$



$M_{\mathbb{R}}$

ex $\sigma = \langle (1,0), (4,3) \rangle$; $\sigma^\vee = \langle [0\,1], [3, -4] \rangle$



$\sigma \rightsquigarrow \sigma^\vee \rightsquigarrow \sigma^\vee \cap M \rightsquigarrow \mathbb{C}[\sigma^\vee \cap M] \rightsquigarrow \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M] = TV(\sigma)$ („toric variety")

– normal – polynomial

$\Rightarrow$ $=$ ss.

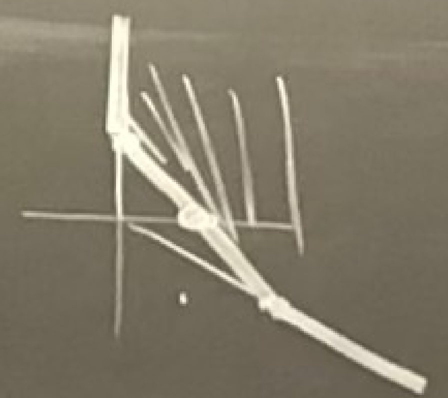
Ex $\sigma = \mathbb{R}^2_{\geq 0} \rightsquigarrow \sigma^\vee = \mathbb{R}^2_{\geq 0} \rightsquigarrow \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2] = \mathbb{C}[\mathbb{N}^2] \rightsquigarrow \mathbb{C}[x, y] \rightsquigarrow \mathbb{C}^2 = \mathrm{Spec}\, \mathbb{C}[x, y]$

$\sigma = \langle (1\,0), (1\,2) \rangle \rightsquigarrow \sigma^\vee = \langle [0\,1], [2, -1] \rangle$ Hilbert basis $= [0\,1], [1\,0], [2\, -1]$ $\rightsquigarrow \mathbb{C}[A, B, C] / AC - B^2$

$\Downarrow$

$TV(\sigma) = V(AC - B^2) \subseteq \mathbb{C}^3$

$\mathrm{conv}\,(\sigma^\vee \cap M) \setminus \{0\}$



Continued fraction

$\sigma \longmapsto TV(\sigma) =$ affine variety || always assume: σ has 0 as a vertex

~~$N_{\mathbb{R}}$~~ Stat $N \cong \mathbb{Z}^n$

$\sigma^\vee$ has 0 as a vertex $\iff \dim \sigma = n$.

ex $\sigma = \{0\} \subseteq \mathbb{R}^2 \rightsquigarrow \sigma^\vee = M_{\mathbb{R}} \rightsquigarrow \sigma^\vee \cap M = M \ (\cong \mathbb{Z}^n)$

$TV(0) = \mathrm{Spec}\, \mathbb{C}[\mathbb{Z}^n] = \mathrm{Spec}\, \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = D(x_1 \cdots x_n) \subseteq \mathbb{C}^n \Rightarrow \{ c \in \mathbb{C}^n \mid c_i \neq 0 \ \forall i \}$

$\mathbb{C}[x_1, \dots, x_n]_{x_1 \cdots x_n}$

$= (\mathbb{C}^*)^n \sim (S^1)^n$ „alg. torus"

$\mathbb{C}^* = \boxed{\cdot} \sim S^1 = \bigcirc$ „n-dim group"

functoriality $(\sigma, N) \xrightarrow{\gamma} (\sigma', N')$

$(\sigma \subseteq N_{\mathbb{R}}$ rat. cone$)$

• $\gamma: N \longrightarrow N'$ group hom.

$(\rightsquigarrow \gamma_{\mathbb{R}}: N_{\mathbb{R}} \longrightarrow N'_{\mathbb{R}}$ linear map among $\mathbb{R}$-v.s.$)$

• $\sigma \subseteq N_{\mathbb{R}}$, $\sigma' \subseteq N'_{\mathbb{R}}$: $\gamma_{\mathbb{R}}(\sigma) \subseteq \sigma'$.

Ex $\gamma = \mathrm{id}_N$: $N \longrightarrow N = N'$, $a \longmapsto a$

$\sigma = \,$, $\sigma' = \,$

ex $\sigma = \longrightarrow$ $\sigma' = \,$

$\gamma: N \to N' \rightsquigarrow \gamma^*: \underset{M'}{(N')^*} \longrightarrow \underset{M}{N^*}$, $\gamma^*((\sigma')^\vee) \subseteq \sigma^\vee$

$\Big\} \gamma^*: \mathbb{C}[(\sigma')^\vee \cap M'] \longrightarrow \mathbb{C}[\sigma^\vee \cap M]$ s.g. hom.

$TV(\sigma') \xleftarrow{TV(\gamma)} TV(\sigma)$

Example: $N = \mathbb{Z}$, $\sigma = \{0\}$, $\sigma' = [0, \infty) \subseteq \mathbb{R}^1$ $\Rightarrow$ $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[\mathbb{Z}] \supseteq \mathbb{C}[\mathbb{N}] = \mathbb{C}[(\sigma')^\vee \cap M]$

$\mathbb{C}[x]_x = \mathbb{C}[x, x^{-1}] \supseteq \mathbb{C}[x] \Rightarrow \mathrm{Spec}: D(x) \hookrightarrow \mathbb{A}^1$, $D(x) = \mathbb{A}^1 \setminus 0$

Special case: $N = N' \rightsquigarrow$ $\boxed{\sigma \subseteq \sigma' \Rightarrow TV(\sigma) \to TV(\sigma')}$

Special instance of this: $\tau \leq \sigma$, i.e. $\tau = \mathrm{face}(\sigma)$

Fact: Take $r \in \sigma^\vee \rightsquigarrow \sigma \cap [r = 0] =$ face of σ, and all faces arise this way.

$(\sigma \subseteq [r \geq 0])$ $\qquad \tau := \mathrm{face}(\sigma, r) < \sigma$

$[r \geq 0]$ $r \in \sigma^\vee$

$\Downarrow$

$\tau := \mathrm{face}(\sigma, r) \rightsquigarrow \tau^\vee = ?$

$\tau = \sigma \cap (r^\perp) \Rightarrow \tau^\vee = \sigma^\vee + (r^\perp)^\perp = \sigma^\vee + \mathbb{R} \cdot r = \sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-r)$

$(\tau^\vee \cap M) = \sigma^\vee - \mathbb{N} \cdot r \rightsquigarrow \mathbb{C}[\tau^\vee \cap M] = \mathbb{C}[\sigma^\vee \cap M]_{\chi^r}$ $\Rightarrow$ $\boxed{TV(\mathrm{face}(\sigma, r)) = D(\chi^r) \subseteq TV(\sigma)}$

$r \in \sigma^\vee \rightsquigarrow \chi^r$

$\sigma^\vee$, $M_{\mathbb{R}}$, $\tau^\vee$, r, $-r$

Toolset (for pol. cones)

• $(\sigma^\vee)^\vee = \sigma$

• $(\sigma \cap \tau)^\vee = \sigma^\vee + \tau^\vee$

$(\sigma + \tau)^\vee = \sigma^\vee \cap \tau^\vee$

• $\sigma = V$ subspace $(\subseteq N_{\mathbb{R}})$

$\sigma^\vee = V^\perp$ $(\subseteq M_{\mathbb{R}})$

Example: „fan" Σ: finite collection of cones

s.t. • $\sigma \in \Sigma$, $\tau \leq \sigma \Rightarrow \tau \in \Sigma$

• $\sigma, \sigma' \in \Sigma \Rightarrow (\sigma \cap \sigma') < \sigma, \sigma'$

Ex $\xleftarrow{\sigma_-} \underset{0 = \tau}{\cdot} \xrightarrow{\sigma_+}$ $\Sigma = \{\sigma_+, \sigma_-, \tau\}$

$TV(\tau) \hookrightarrow TV(\sigma) = \mathbb{C}^1$, coordinate $= x$

$TV(\tau) \hookrightarrow TV(\sigma') = \mathbb{C}^1$, coord $= y$

$D(x) = D(y)$

$\boxed{\mathbb{C}[x, x^{-1}]}$ $\boxed{\mathbb{C}[x]}$ $\mathbb{C}[y, y^{-1}]$ $\boxed{\mathbb{C}[y]}$ $x^{-1} \mapsto y$

$\mathbb{P}^1$ $x = y^{-1}$

$\mathbb{C}[x, \frac{1}{x}] \supseteq \mathbb{C}[x]$ $\mathbb{C}^1$

$\mathbb{C}^* \supseteq \mathbb{C}[\frac{1}{x}]$ $\mathbb{C}^1$