Projective spaces

k = field, $k = \bar{k}$ (e.g. $k = \mathbb{C}$)   Std.: $V = (n+1)$-dim. vector space

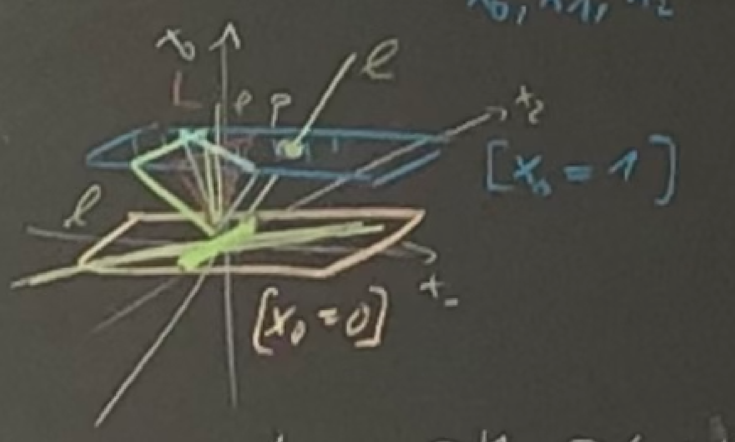
$$\mathbb{P}(V) := V \setminus \{0\} / k^* \text{, i.e. } v, v' \in V \setminus \{0\},\ v \sim v' \Longleftrightarrow \exists \lambda \in k^* : v' = \lambda \cdot v$$

$V = \mathbb{R}^2$

$$= \{\text{lines } \ell \subseteq V \text{ through zero}\}$$

$$\boxed{\left(\underline{\text{or}}:\ \mathbb{P}(V) := V^* \setminus \{0\} / k^* \quad \text{„Grothendieck notation"}\right) \quad \left(\mathbb{P}_{\text{usual}}(V) = \mathbb{P}_{\text{Groth}}(V^*)\right)}$$

$V = \mathbb{R}^3$



$$[x_0 = 1] \xhookrightarrow{\iota} \mathbb{P}^n = \mathbb{P}(k^{n+1})$$
$\cong$
$\mathbb{A}^n$
$$p \longmapsto k \cdot p = \text{line}$$

$$\mathbb{P}^n \setminus \mathbb{A}^n = \{\text{„new points"}\} = \{\text{lines } \ell \subseteq k^{n+1} \text{ s.t. } \ell \subseteq [x_0 = 0]\}$$
($k^{n+1} = V$)

Fact: any 2 lines L, L' in $\mathbb{P}^2$ intersect: $L \subseteq \mathbb{P}^2 \ \hat{=}\ $ plane $L \subseteq k^3$, $L' \ \hat{=}\ L' \subseteq k^3$ $\Big\}$ $L \cap L' \subseteq k^3$ line $\ell \subseteq k^3$

$\Downarrow$

$\ell \in \mathbb{P}^2$

Coordinates: $z_0, z_1, \dots, z_n$ = coordinates for k^{n+1} ($= V$) „homogeneous"

(!!) $z_i \neq$ coord. fct. on $\mathbb{P}^n$! They are still called „homog. coord.".

$\sim\ \hat{=}\ (z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ class $= (z_0 : z_1 : \dots : z_n)$

„old points": $\left((z_0 \neq 0) : z_1 : \dots : z_n\right) = \left(1 : \underbrace{\frac{z_1}{z_0} : \dots : \frac{z_n}{z_0}}\right)$ [not all are $= 0$]

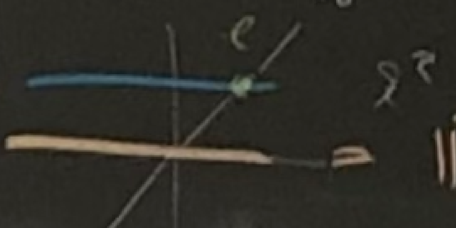
$\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)$

„new points": $\{(0 : z_1 : \dots : z_n)\} = \mathbb{P}^{n-1}$

$U_0 = [z_0 \neq 0] = \mathbb{A}^n \hookrightarrow \mathbb{P}^n$ open embed.

$$\mathbb{P}^n = \underset{[\text{old}]}{\mathbb{A}^n} \sqcup \underset{[\text{new}]}{\mathbb{P}^{n-1}}$$
infinite hyperplane

$x_i = \frac{z_i}{z_0}$ = coord. on $\mathbb{A}^n$



$\Rightarrow \mathbb{P}^0 = \underline{\text{new}}$ pt $\in \mathbb{P}^1$ „∞"

Remark. $\mathbb{C}^n \neq$ compact (classical topology); $\mathbb{P}^n_{\mathbb{C}}$ is compact:

$$\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^{n+1} \setminus 0 / \mathbb{C}^* = \left(\mathbb{R}^{2n+2} \setminus 0 / \mathbb{R}_{>0}\right) / S^1$$

$\mathbb{C}^* = S^1 \cdot \mathbb{R}_{>0}$, $z = \frac{z}{|z|} \cdot |z|$

$$\{v \in \mathbb{R}^{2n+2} \mid \|v\| = 1\} / S^1 = \underset{\text{compact}}{S^{2n+1}} / S^1$$

different affine charts: $U_i := [z_i \neq 0] \subseteq \mathbb{P}^n$

$\cong \mathbb{A}^n$: affine coord.: $x_\nu^{(i)} := \frac{z_\nu}{z_i} \Rightarrow x_i^{(i)} = 1$

$\hookrightarrow \mathbb{P}^n = \bigcup_{i=0}^{n} U_i$, i.e. covered by open affine subsets ($\nu = 0, \dots, n$)

$U_0 \cap U_1 = \{z \in \mathbb{P}^2 / z_0, z_1 \neq 0\}$
$[z_0 \neq 0]$ $[z_1 \neq 0]$

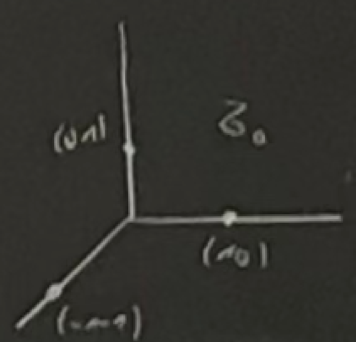
Example. $\mathbb{P}^2$:
U_0: $\mathbb{C}\left[\overset{x_1}{\frac{z_1}{z_0}}, \overset{x_2}{\frac{z_2}{z_0}}\right] \hat{=} \mathbb{A}^2$
U_1: $\mathbb{C}\left[\frac{z_0}{z_1}, \frac{z_2}{z_1}\right] = \mathbb{C}\left[\frac{1}{x_1}, \frac{x_2}{x_1}\right]$
U_2: $\mathbb{C}\left[\frac{z_0}{z_2}, \frac{z_1}{z_2}\right] = \mathbb{C}\left[\frac{1}{x_2}, \frac{x_1}{x_2}\right]$

$\cup$ W: W open $\Leftrightarrow$ $W \cap U_i \subseteq U_i$ open $\forall i$

on $\boxed{U_0 \cap U_1}$: $\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_0}{z_1}, \frac{z_2}{z_1}$ are well def. functions

$$\mathbb{C}[\ \text{-- " --}\ ] = \mathbb{C}\left[x_1, x_2, \frac{1}{x_1}, \frac{x_2}{x_1}\right] = \mathbb{C}[x_1, x_2]_{x_1} = \mathbb{C}\left[\frac{1}{x_1}, \frac{x_2}{x_1}\right]_{\frac{1}{x_1}}$$

$p = (1 : 2 : 5) \in \mathbb{P}^2$ ~~$z_0(p) = 1$~~ $z_0(p) \neq 0$ (?)
$= (2 : 4 : 10)$



Σ $TV(\Sigma) = \bigcup_{i=0}^{2} TV(\sigma_i)$

- $\sigma_0 = \langle (1,0), (0,1) \rangle \Rightarrow \sigma_0^\vee = \langle [0,1], [1,0] \rangle$

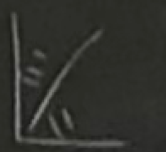
$\mathbb{C}[\sigma_0^\vee \cap \mathbb{Z}^2] = \mathbb{C}[x, y]$

- $\sigma_1 = \langle (0,1), (-1,-1) \rangle$

$\sigma_1^\vee = \langle [-1,0], [-1,1] \rangle$
$x^{-1} \cdot y$

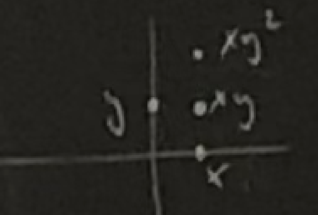
$\sigma_2 = \langle (1,0), (-1,-1) \rangle$
$\sigma_2^\vee = \langle [0,-1], [1,-1] \rangle$

$\mathbb{C}\left[\frac{1}{y}, \frac{x}{y}\right]$



$\to \mathbb{C}\left[\frac{1}{x}, \frac{y}{x}\right] \hat{=} \mathbb{A}^2$

$$\mathbb{C}\left[x, y, \frac{1}{x}, \frac{y}{x}\right] = \mathbb{C}\left[x, \frac{1}{x}, y\right]$$



$\mathbb{Z}^2$ (dual)
$\| $
$M \rightsquigarrow \mathbb{C}[M]$
$\mathbb{C}[x, y, x^{-1}, y^{-1}]$

$\chi^{[1,0]} = x$, $\chi^{[0,1]} = y$

A B
$\mathbb{C}\left[x, \frac{y}{x}\right] = \mathbb{A}^2$

$\mathbb{C}[x, y]$

---

projective subsets of $\mathbb{P}^n$ $k[z_0, \dots, z_n] = S$ „homog. coord. ring of $\mathbb{P}^n$"

$f \in S$, $p \in \mathbb{P}^n \rightsquigarrow f(p) \overset{(?)}{=} 0$, e.g. $f = z_0 - z_1^2$, $(1:1) \in \mathbb{P}^1 \Rightarrow f(1,1) = 0$

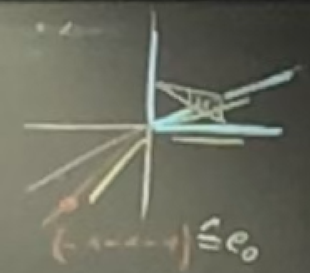
It does make sense if f = homogeneous! ($f(\lambda \cdot \underline{x}) = \lambda^d \cdot f(\underline{x})$ i.e. $f \in S_d$) $f(2,2) = -2 \neq 0$

$\mathbb{A}^2$, $\mathbb{A}^2$, $\mathbb{A}^2$ $\subseteq \mathbb{P}^2$

$\mathbb{P}^1 \times \mathbb{P}^1$

$\mathbb{P}^1$

$\mathbb{P}^3$:

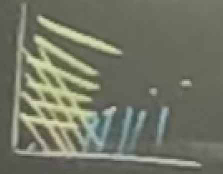


$(\dots) \subseteq e_0$

- $e_1, e_2, e_3 \in \mathbb{Z}^3$ standard basis
  $e_0 = -e_1 - e_2 - e_3$
- cones $\sigma_i = \langle e_0, e_1, e_2, e_3$ without $e_i \rangle$

$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$, $U_i \cong \mathbb{A}^n$

$\sigma =$  $= \mathbb{R}^3_{\geq 0}$ $\hat{=}\ \mathbb{C}^3$

$\partial\sigma =$  $= \bigcup [3$ 2-dim. cones$]$ = fan of 3 2-dim. cones

$\Rightarrow$ gluing of 3 $\mathbb{C}^2 \times (\mathbb{C}^*)$

π, $(1,1,1)$, 0

$\mathbb{C}^3 \setminus \{0\} = \bigcup_{i=0}^{2} [x_i \neq 0]$ $\quad\|\quad$ $\mathbb{P}^2 = \bigcup [z_i \neq 0]$

$\mathbb{P}^n$, $(z_0 : \dots : z_n)$

$f \in S_d \rightsquigarrow [f = 0] \subseteq \mathbb{P}^n$ makes sense
$V_+(f) = V_p(f)$

$f \in S_d$, $d \geq 1 \Rightarrow D_+(f) = \underline{\text{affine}}$

$\mathfrak{J} \subseteq S$ $\underline{\text{homog}}$ ideal $\rightsquigarrow V_+(\mathfrak{J}) \subseteq \mathbb{P}^n$ $\quad (S = k[x_0, \dots, x_n]$, $\deg x_i = 1)$

Zariski topology: $\cdot\ V_+(\mathfrak{J}) =$ „closed subsets"

$\cdot\ D_+(f) := \mathbb{P}^n \setminus V_+(f)$, $f \in S_d$ $(d \geq 1)$

$D_+(z_i) = [z_i \neq 0] = U_i$

$U_0 \subset \mathbb{P}^n$

$(x_1, \dots, x_n)$ $\big|$ $(z_0 : \dots : z_n)$, $x_i = \frac{z_i}{z_0}$

- $\mathfrak{J} \subseteq S$ homog. ideal, $f \in \mathfrak{J}_d$ i.e. $f \in \mathfrak{J}$, $\deg f = d$ (homog)

$\rightsquigarrow f = f(z_0, \dots, z_n) \rightsquigarrow f(1, x_1, \dots, x_n) = \bar{f}(x_1, \dots, x_n)$ not homog anymore

$\qquad f(1, x_1, \dots, x_n) = f\left(\frac{z_0}{z_0}, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)$

$\underline{\text{example}}$ $f = z_0^2 - z_1 z_2 + z_0 z_1$

$\Rightarrow \bar{f}(x_1, x_2) = 1 - x_1 x_2 + x_1$ $\qquad V_+(f) \subseteq \mathbb{P}^n$ $\big|$ $V(\bar{f}) \subseteq \mathbb{A}^n = U_0 \subset \mathbb{P}^n$

$V(\bar{f}) = V_+(f) \cap U_0$

($\underline{\text{similarly}}$ $\mathfrak{J} \subseteq S \rightsquigarrow \bar{\mathfrak{J}} \subseteq k[x_1, \dots, x_n]$)

$\bar{g}(\underline{x}) \in k[x_1, \dots, x_n] \rightsquigarrow \bar{g}\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) \cdot z_0^{\deg \bar{g}} = g = \bar{g}^h \rightsquigarrow V(\bar{g}) \subseteq U_0 = \mathbb{A}^n$

$\Downarrow$

$V_+(g) = \overline{V(\bar{g})} \Longleftarrow V_+(g) \subseteq \mathbb{P}^n$

$I \subseteq k[x_1, \dots, x_n] \rightsquigarrow I^h \subseteq S$

$V(I^h) = \overline{V(I)}$, $V(I^h) \subseteq \mathbb{P}^n$, $V(I) \subseteq U_0$

$\underline{\text{Recall}}$: I^h is z_0-saturated, $\underline{\text{i.e.}}$ $\left(I^h : (z_0)^\infty\right) = I^h$

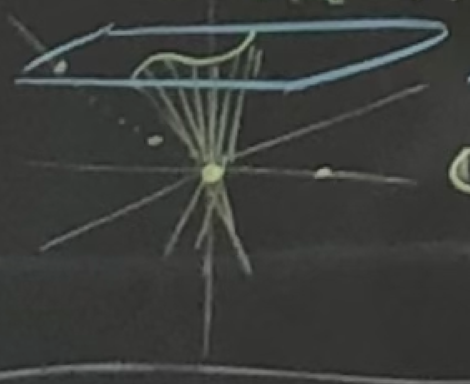
$\pi\ \mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$

$\pi^{-1}(Y) \subseteq \mathbb{A}^{n+1} \setminus 0$, $\quad Y = V_+(\mathfrak{J}) \subseteq \mathbb{P}^n$

$\Downarrow$

$V(\mathfrak{J}) = \underline{C(Y)} = \overline{\pi^{-1}(Y)} \subseteq \underline{\mathbb{A}^{n+1}}$ (aff. coord. $z_0, \dots, z_n$)
$= \pi^{-1}(Y) \cup \{0\}$

$\mathfrak{J} \subseteq S = k[z_0, \dots, z_n]$ homog. ideal



$C = V(x_2^2 - x_1^3)$

$\mathbb{A}^2 = U_0$

$C(C)$

$\underline{\text{Ex 1}}$ see directly $(*)$

$$\begin{array}{ccc} \mathbb{A}^3 \setminus 0 & \xrightarrow{\pi} & \mathbb{P}^2 \\ \cup\!| & & \cup\!| \\ \pi^{-1}(D_+(z_0)) = D(z_0) & \xrightarrow{\pi} & U_0 = \mathbb{A}^2 = [z_0 \neq 0] = D_+(z_0) \end{array}$$

$\underline{\text{Remark}}$: $\pi : \pi^{-1}(U_0) \longrightarrow U_0$ becomes a product.

$k[z_0, \dots, z_n]_{z_0} \longleftrightarrow k\left[\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right] = R$

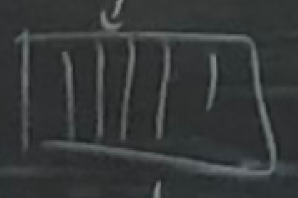
$\| \qquad\qquad \|$

$k\left[\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}, z_0, z_0^{-1}\right] \qquad k[z_0, \dots, z_n]_{(z_0)}$

$\| $

$R \otimes_k k[z_0, z_0^{-1}]$

$\operatorname{Spec} R \times \mathbb{C}^* \xrightarrow{(*)} \operatorname{Spec} R$



$\underline{\text{Def}}$ R = graded ring, M = graded R-module
$S \subseteq R$ mult. closed, homog. elements,
$\rightsquigarrow S^{-1}R \overset{\mathbb{Z}}{=}$ graded ring $\left(\deg \frac{a}{s} = \deg a - \deg s\right)$
$(S)^{-1}R = \{$degree-0-elements inside $S^{-1}R\}$

$\underline{\text{Ex}}$ $R = k[z_0, \dots, z_n]$, $S = \{z_0^k \mid k \geq 0\} \Longrightarrow (S)^{-1}R = k\left[\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right]$

2 special cases of loc

- $R \rightsquigarrow R_f$ $\quad (S = \{f^k \mid k \in \mathbb{N}\})$
  $\Rightarrow (S)^{-1}R =: R_{(f)}$
- $R \rightsquigarrow R_{\mathfrak{p}}$ $\quad (\mathfrak{p} = \text{PI})$
  $\mathfrak{p}$ = homog. PI $\rightsquigarrow S := \{$homog. el. of $S \setminus \mathfrak{p}\}$
  $\rightsquigarrow (S^{-1})R =: R_{(\mathfrak{p})}$

$\underline{\text{CAREFUL}}$ $f = p$ = irred $\Rightarrow (p) = \text{PI}$
$\Rightarrow R_{(p)}$ = lok of PI (in homog.)
$\neq$!! homog. lok of p

$\underline{\text{Example}}$ $Y = (x_2^2 - x_1^3) \rightsquigarrow \mathbb{A}^2$

$\Downarrow$

$z_0 z_2^2 - z_1^3 \rightsquigarrow \mathbb{P}^2$ $\quad \underline{\text{new points}}$ $(0 : 0 : 1)$ $\quad \|\quad U_2 \Rightarrow \frac{z_0}{z_2}, \frac{z_1}{z_2} = y_0, y_1 \Rightarrow V(y_0 - y_1^3) \subseteq \mathbb{A}^2$

$\hookrightarrow C(Y) = V(z_0 z_2^2 - z_1^3) \subseteq \mathbb{A}^3$

$\cdot\ \boxed{\mathfrak{J}} \subseteq k[z_0, \dots, z_n]$ homog. ideal for $Y = V_+(\mathfrak{J})$

$\cdot\ Y \cap U_i = V(\mathfrak{J}_i)$, $\mathfrak{J}_i \subseteq k[x_0^{(i)}, \dots, x_n^{(i)}]$

$\boxed{\mathfrak{J}_{(z_i)}} \subseteq k[z_0, \dots, z_n]_{(z_i)}$

Ex 2

$Y = V_+(\mathfrak{J}) \rightsquigarrow S(Y) =$ „homog. coord. ring" $:= k[z_0, \dots, z_n] / \mathfrak{J}$ $\quad \cancel{\mathfrak{J}}$ $\underline{(\mathfrak{J} : \underline{z}^\infty)}$

$\mathfrak{J} \rightsquigarrow (\mathfrak{J} : (z_0, \dots, z_n)^\infty) \supseteq \mathfrak{J}$

$V_+(z_0, \dots, z_n) = \emptyset$ $\quad$ ($\underline{\text{ex}}$ $((\underline{z}) : (\underline{z})^\infty) = (1)$)

„irrelevant ideal"

$\underline{\text{Claim}}$ $(\mathfrak{J} : \underline{z}^\infty)$ has the same local behavior as $\mathfrak{J}$

$\underline{\text{Remark}}$ $C(Y) = V(\mathfrak{J} : \underline{z}^\infty)$

Sept 22 Bedlewo, close to Poznan Summer school „Liquid tensor project"
Peter Scholze: Condensed mathematics

$X \subseteq \mathbb{A}^n \longleftrightarrow A = A(X)$ $(\operatorname{Spm} A, A) \longleftrightarrow$	$Y \subseteq \mathbb{P}^n \rightsquigarrow S(Y)$

① Veronese embedding

$v_2 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

$(z_0 : z_1) \longmapsto (\underset{w_0}{z_0^2} : \underset{w_1}{z_0 z_1} : \underset{w_2}{z_1^2})$

„2-nd Veronese of $\mathbb{P}^1$"

General version: $v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{d+n}{n}-1}$

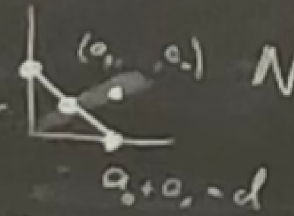
i.e. homog. coord. of $\mathbb{P}^{\binom{d+n}{n}-1}$ $\hat{=} \{m \in \nabla\}$ $m \hat{=}$ monomial $\underline{z}^m$

$m \longleftrightarrow w_m$

$(z_0 : \ldots : z_n) \longmapsto (\ldots, w_m = \underline{z}^m, \ldots)$

$\binom{d+n}{n} = \binom{d+n}{d} = \#[$degree d monomials in $z_0, \ldots, z_n]$

ex: $n = 1$, $d = 2$ $\binom{2+1}{2} = 3$



(a_0, a_1) $\mathbb{N}^{n+1}$ – $(a_0, \ldots, a_n) = z_0^{a_0} \cdots z_n^{a_n}$

$a_0 + a_1 = d$ – $\deg = d \iff \sum a_i = d$

$\nabla \,\|\, \#(d \cdot \Delta^n \cap \mathbb{Z}^{n+1}) = \binom{d+n}{n}$

$\mathbb{P}^1 \longleftarrow \mathbb{P}^2 \setminus \{(0:0:1)\}$

$(w_0 : w_1) \longleftarrow\!\shortmid (w_0 : w_1 : w_2)$

$\cancel{(0:0)} \longleftarrow\!\shortmid (0:0:1)$

$(z_0^2 : z_0 z_1) \longleftarrow\!\shortmid (z_0^2 : z_0 z_1 : z_1^2)$

$z_0 \neq 0$ $\| $ $(z_0 : z_1)$

$\Longrightarrow \mathbb{P}^1 = v_2(\mathbb{P}^1) \subset \mathbb{P}^2$

$v_2 : \mathbb{P}^1 \xrightarrow{\sim} v_2(\mathbb{P}^1)$

$\left.\begin{array}{l} \mathbb{P}^1 \xleftarrow{\varphi_1} \mathbb{P}^2 \setminus (0:0:1) \\ (w_0 : w_1) \longleftarrow\!\shortmid (w_0 : w_1 : w_2) \end{array}\right\}$ does not work at $(0:0:1)$

fix this problem. $\mathbb{P}^1 \xleftarrow{\varphi_2} \mathbb{P}^2 \setminus (1:0:0)$

$(w_1 : w_2) \longleftarrow\!\shortmid (w_0 : w_1 : w_2)$

$(z_0 z_1 : z_1^2) \longleftarrow\!\shortmid (z_0^2 : z_0 z_1 : z_1^2)$

$\| $ $(z_0 : z_1)$

$\boxed{I(Y) = (f \in S \mid f_{|Y} = 0)}$

$\mathbb{P}^1 \xrightarrow{\sim} v_2(\mathbb{P}^1)$

$\wr\!\!\downarrow$ $\qquad$ $\wr\!\!\downarrow$

$\mathbb{P}^1 \qquad \mathbb{P}^2$

ideal $I(v_2(\mathbb{P}^1))$ $\underset{\mathbb{P}^2}{\cap}$ $\longleftrightarrow$ $\operatorname{Ker} \Phi = \{ f(\underline{w}) \mid f(v_2(\mathbb{P}^1)) = 0 \}$

which $f(w_0, w_1, w_2)$ vanish on $v_2(\mathbb{P}^1)$?

$\longleftrightarrow k[w_0, w_1, w_2] \xrightarrow{\Phi} k[z_0, z_1]$ $\qquad \ni (w_0 w_2 - w_1^2)$

$w_0 \longmapsto z_0^2$

$w_1 \longmapsto z_0 z_1$ $\qquad w_2 \longmapsto z_1^2$

$S(v_2(\mathbb{P}^1)) = k[w_0, w_1, w_2] / I(v_2(\mathbb{P}^1)) = k[\;] / \operatorname{Ker} \Phi \xrightarrow{\sim} \operatorname{Im} \Phi = \bigoplus_{2 \mid v} S_v \subset \bigoplus_{v \in \mathbb{N}} S_v = S$ $\qquad k[w_0, w_1, w_2]$

$S(\mathbb{P}^n) = S = k[z_0, \ldots, z_n]$

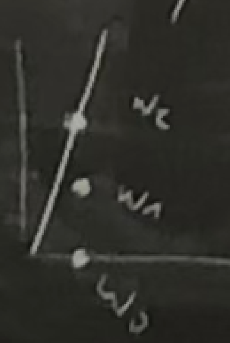
$S(v_d(\mathbb{P}^n)) = \bigoplus_{v=0}^{\infty} S_{d \cdot v}$

$\left(\begin{array}{l} \cdot\ Y = V_+(\mathfrak{z}) \rightsquigarrow S(Y) = S / (\mathfrak{z} : \mathfrak{z}^\infty) \quad \sqrt{\mathfrak{z}} \\ \cdot\ \underline{\text{today}}:\ Y = v_2(\mathbb{P}^1) \underset{\mathbb{P}^2}{\cap} \end{array} \right)$

ex: $k[z_0, z_1] \longleftrightarrow k[z_0^2, z_0 z_1, z_1^2] = k[w_0, w_1, w_2]$ $\qquad w_0 w_2 - w_1^2$

$\updownarrow$ $\qquad \updownarrow$

$\mathbb{N}^3 / e_0 + e_2 - 2e_1$



$Y \hookrightarrow \mathbb{P}^n \rightsquigarrow S(Y)$

$\Downarrow$

$C(Y) \hookrightarrow \mathbb{A}^{n+1}$

$A(C(Y)) = S(Y)$

② Segre embedding

$\mathbb{P}^m \times \mathbb{P}^n = \cancel{\mathbb{P}^{m+n+1}}$

$\cancel{(z_0 : \ldots : z_m), (w_0 : \ldots : w_n) \longmapsto (z_0 : \ldots : z_m : w_0 : \ldots : w_n)}$

$\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\Phi} \mathbb{P}^{(m+1)(n+1)-1}$ $\quad (m+1)(n+1)-1 = mn+m+n$

$(z_0 : \ldots : z_m), (w_0 : \ldots : w_n) \longmapsto (z_i w_j \mid i = 0 \ldots m,\ j = 0 \ldots n)$

$z_0 \neq 0$, $w_0 \neq 0$ $\qquad z_0 w_0 \neq 0$

$\operatorname{Im} \Phi \subseteq \mathbb{P}^N$ closed algebraic subset.

coord. on p_{ij} $(i = 0, \ldots, m,\ j = 0, \ldots, n)$. $\qquad k[p_{ij}] \xrightarrow{\psi} k[z_0, \ldots, z_m, w_0, \ldots, w_n]$

$\operatorname{Ker} \psi = (p_{ij} p_{kl} - p_{il} p_{kj} \mid i, j, k, l)$ $\qquad p_{ij} \longmapsto z_i w_j$

$z_i w_j\, z_k w_l - z_i w_l\, z_k w_j$

$\deg p_{ij} = e_i + f_j$ $\qquad \mathbb{Z}^{m+n+1}$ base vector e_i, f_j

$\deg z_i = e_i$, $\deg w_j = f_j$

$S(\mathbb{P}^m \times \mathbb{P}^n) = k[z_i w_j \mid\;]$

Ex: $\mathbb{P}^1 \times \mathbb{P}^1 \underset{\mathbb{P}^3}{\cap} \hat{=} k[z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1] = k[A, B, C, D] / AD - BC$

$\mathbb{P}^1$; $\mathbb{P}^2 \supseteq V(F)$ $\deg F = 2 \rightsquigarrow$ w.l.o.g. $F = z_0 z_2 - z_1^2$

$\hookrightarrow V(F) = v_2(\mathbb{P}^1)$

$\mathbb{P}^3 \supseteq V(F)$ $\deg F = 2 \rightsquigarrow$ w.l.o.g. $F = z_0 z_2 - z_1 z_3$

$\rightsquigarrow V(F) = \mathbb{P}^1 \times \mathbb{P}^1$