$\mathbb{P}^n \xhookrightarrow{v_d} \mathbb{P}^N$ d-Veronese

$(z_0 : \dots : z_n) \longmapsto (\dots : [\text{all degree-}d\text{ monomials}] : \dots)$

hom. coord. rings: $\mathbb{P}^n \hat{=} k[z_0, \dots, z_n] =: S$

$v_d(\mathbb{P}^n) \hat{=} \bigoplus_{v=0}^{\infty} S_{d \cdot v}$

$v_2 : \mathbb{P}^1 \to \mathbb{P}^2$

$(z_0 : z_1) \longmapsto (\underbrace{z_0^2}_{w_0} : \underbrace{z_0 z_1}_{w_1} : \underbrace{z_1^2}_{w_2})$

$w_0 w_2 - w_1^2 = 0$

$v_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$

$(z_0 : z_1 : z_2) \mapsto (z_0^2 : z_1^2 : z_2^2 : z_0 z_1 : z_0 z_2 : z_1 z_2)$, $w_{22}, w_{00}, w_{01}, w_{11}, w_{12}$

$w_{22} \cdot w_{00} = w_{02}^2$

Segre embedd.: $\mathbb{P}^a \times \mathbb{P}^b \xhookrightarrow{s} \mathbb{P}^{(a+1)(b+1)-1}$

$\underline{z}, \underline{w} \longmapsto (\dots : z_i w_j : \dots)$

$S(\mathbb{P}^a \times \mathbb{P}^b)$

---

dual description: $N \xrightarrow{\iota} \mathbb{Z}^H = \mathbb{Z}^k$ ($\mathbb{Z}$-linear), $(\sigma \subseteq N_\mathbb{R}) \xrightarrow{\iota_\mathbb{R}} \mathbb{R}^k_{\geq 0}$

$a \longmapsto (\langle a, h_1 \rangle, \dots, \langle a, h_k \rangle)$

$(N, \sigma) \to (\mathbb{Z}^k, \mathbb{R}^k_{\geq 0})$

$\Downarrow$

$TV(\dots) \to TV(\dots)$

$\boxed{TV(\sigma) \hookrightarrow \mathbb{A}^k}$

$\left.\begin{matrix} TV(\tau) \hookrightarrow TV(\sigma_1) \\ \searrow TV(\sigma_2) \end{matrix}\right\}$ ⇝ gluing ⇝ Σ = fan in $N_\mathbb{R}$

⇝ $\underline{TV(\Sigma)}$:= gluing of all $TV(\sigma) \mid \sigma \in \Sigma$

eg: $\mathbb{P}^2$

Projective toric varieties

$N = \mathbb{Z}^n$ $M = (\mathbb{Z}^n)^*$ $M \times N \to \mathbb{Z}$, $\langle m, n \rangle \in \mathbb{Z}$

$\sigma \subseteq N_\mathbb{R} = \mathbb{R}^n$ polyh. cone $\longmapsto \sigma^\vee \subseteq M_\mathbb{R} \longmapsto$

- $\boxed{\sigma^\vee \cap M = \text{s.g.}}$
- $\mathbb{C}[\sigma^\vee \cap M]$
- $\operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M] = TV(\sigma)$

$\sigma^\vee \cap M \supseteq H = $ „Hilbert basis" $\{h_1, \dots, h_k\}$ (finite!!)

(unique set of min. gens)

$\Downarrow$ $\{r \in \sigma^\vee \cap M \mid r \neq r_1 + r_2,\ r_i \in (\sigma^\vee \cap M) \setminus 0\}$

$\mathbb{N}^e \twoheadrightarrow \sigma^\vee \cap M \Rightarrow \mathbb{A}^e \hookleftarrow TV(\sigma)$

$e_i \longmapsto h_i$

$\underline{M}$ Let $\Delta \subseteq M_\mathbb{R}$ be a lattice polytope, i.e. polyhedron with:

- vertices $\in M = \mathbb{Z}^n$
- bounded, i.e. compact

Example: $\mathbb{P}^1$; $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ (A, B, C) (1); $\mathbb{P}^2$; $\mathbb{P}^2 \xhookrightarrow{v_2} \mathbb{P}^5$ (00, 01, 11, 02, 22); $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ (A, B, C, D) (2); surface $\subseteq \mathbb{P}^6$

$(z_A : z_B : z_C : z_D)$, $(z_{00} : z_{01} : z_{11} : z_{10})$

$0 + Y = 2X = X + X$

- $\mathbb{P}(\Delta) \hookrightarrow \mathbb{P}(\Delta \cap M) = \mathbb{P}^{\#(\Delta \cap M) - 1}$

hom. coord. of this space

$z_A \hat{=} z^{(A,1)}$

- equations: $a_i \in \Delta \cap M$ ⇝ $(a_i, 1) \in M \oplus \mathbb{Z}$

1

0

⇝ $z_i = z_{a_i}$ = corresp. hom. coord

ex $A + C = 2B$ (1) ⇝ $z_A \cdot z_C = z_B z_B = z_B^2$

or $A + C = B + D$ (2) ⇝ $z_A z_C = z_B z_D$

formally: If $\sum_{i \in I} (a_i, 1) = \sum_{j \in J} (a_j, 1) \Rightarrow \prod_{i \in I} z_i = \prod_{j \in J} z_j$

$(a_i, a_j \in \Delta \cap M)$

Recall $\mathbb{N}^e \twoheadrightarrow \sigma^\vee \cap M \to \mathbb{C}[\mathbb{N}^e] = \mathbb{C}[z_1, \dots, z_e] \xrightarrow{\pi} \mathbb{C}[\sigma^\vee \cap M]$

$H = \{h_1, \dots, h_e\}$

$\mathbb{A}^e \hookleftarrow TV(\sigma)$

„polyh. cone": $C = \sum_{i=1}^{N} \mathbb{R}_{\geq 0} \cdot v_i$, $v_i \in M, N$

$\operatorname{Ker} \pi \hat{=}$ linear relations among the $h_i \in H$

ex $\subseteq M_\mathbb{R}$, 0, A, B, C, $\sigma^\vee$

$A + C = 2B$ $\quad \underline{A + C - 2B = 0}$

$\Downarrow$

$z_A z_C = z_B^2$

Mixing is allowed

eg $\subseteq \mathbb{R}^2_{\geq 0}$

$\nabla = \Delta^c + (\text{cone } C)$

Δ^c = (compact) polytope

$\Delta^c \cap M$ ⇝ hom. coord.: z_A, z_B

$H \subseteq C$ Hilbert basis, $H = \{a, b\}$

$\mathbb{P}(\nabla) \overset{\text{equations}}{=} V(z_A x_b - z_B x_a) \subseteq \mathbb{P}^1 \times \mathbb{A}^2 = \{(c_0 : c_1), (d_0, d_1)\}$

$(z_0 : z_1), (x_0, x_1)$

$(z_A : z_B), (x_a, x_b)$

mixed relations

$A \in \Delta^c \cap M$ ⇝ $(A, 1)$

$a \in H \subseteq C$ ⇝ $(a, 0)$

⇝ look for relations among them

$A = [1, 0]$ ⇝ $[1 0 1]$

$B = [0, 1]$ ⇝ $[0 1 1]$

$a = [1, 0]$ ⇝ $[1 0 0]$

$b = [0, 1]$ ⇝ $[0 1 0]$

$[1 1 1]$, $[1 1 1]$

$A + b = B + a \Longrightarrow \underline{z_A x_b = z_B x_a}$

- Class of examples: Δ^n = n-dim unit simplex = conv $\{0, e_1, \dots, e_n\} \subseteq \mathbb{R}^n$

$n = 1$ ⇝

$n = 2$ ⇝

$\cong$ conv $\{e_0, e_1, \dots, e_n\} \subseteq \mathbb{R}^{n+1}$

- Segre: $\mathbb{P}(\Delta_1) \times \mathbb{P}(\Delta_2) = \mathbb{P}(\Delta_1 \times \Delta_2)$

$\mathbb{P}^a \times \mathbb{P}^b$

- Veronese: $\mathbb{P}(\Delta)$ ⇝ Veronese v_d (of degree d) ⇝ $\mathbb{P}(d \cdot \Delta)$

$X \hookrightarrow \mathbb{P}^N$

$\| $

$V_+(J)$ ⇝ $S(X) := k[z_0, \dots, z_N] / (J : \mathfrak{m}^\infty)$ hom. coord ring.

Q: What is the defining ideal of $\mathbb{P}(\Delta)$? ⇝ $S(\Delta) := S(\mathbb{P}(\Delta))$

$\boxed{k[z_p \mid p \in \Delta \cap M]}$ $\longrightarrow$ $k[M] \underset{(?)}{=} k[z_1^{\pm 1}, \dots, z_n^{\pm 1}, t]$

$(\prod z_p - \prod z_q)$

$z_p \longmapsto x^{[p, 1]}$

$= \boxed{k[M \oplus \mathbb{N}]}$

$k[z_P \mid P \in \Delta\cap M] \xrightarrow{\ \Phi\ } k[M\oplus \mathbb{N}]$

$z_P \longmapsto \chi^{[P,1]} \in k[M\oplus\mathbb{Z}]$

Example $\Delta = \square$ (D, C, A, B)

$k[z_A, z_B, z_C, z_D] \longrightarrow k[\mathbb{Z}^3] \supseteq k[x^{\pm1}, y^{\pm1}, t]$ (Φ)

$z_A \longmapsto \chi^{[0,0,1]} = t$

$z_B \longmapsto \chi^{[1,0,1]} = xt$

$z_C \longmapsto \chi^{[1,1,1]} = xyt$

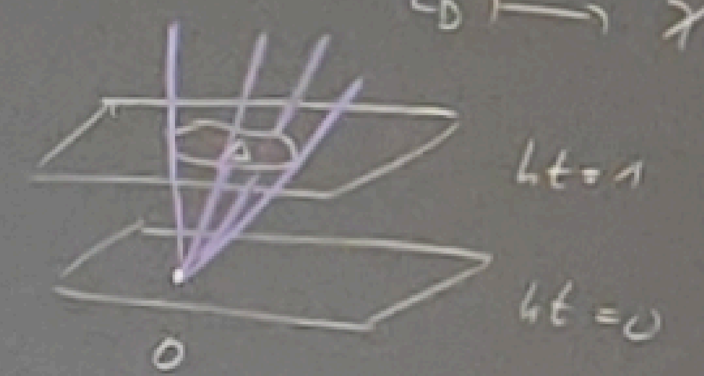
$z_D \longmapsto \chi^{[0,1,1]} = yt$

$\mathrm{im}(\Phi) = k[t, xt, yt, xyt] \subseteq k[x,y,t]$

$\cong k[z_P \mid \ldots] / [\mathrm{Ker}\,\Phi = \text{defi. ideal}] = S(\mathbb{P}(\Delta))$

$(\mathrm{Ker}\,\Phi \supseteq (z_A z_C - z_B z_D))$

Q What is the image of Φ ?



A. $\mathrm{Im}\,\Phi$ is generated, as a k-vs by all monomials $\chi^{[m,k]}$: $\cdot\ m\in M,\ k\in\mathbb{N}$

$\cdot\ \exists$ representation $m = (P_1,1)+(P_2,1)+\ldots+(P_k,1) \mid P_i \in \Delta\cap M$

$\uparrow z_{P_1}, \quad \uparrow z_{P_2}, \quad \ldots \quad \uparrow z_{P_k}$

$= k[\mathbb{N}\cdot(\Delta\cap M, 1)]$, $\cap$

$\mathbb{N}$ - linear combinations $\subset M\oplus\mathbb{Z}$

$C(\Delta) := \mathbb{R}_{\geq 0}\cdot(\Delta,1) = \{\mathbb{R}_{\geq0}\text{-linear combinations of } (\Delta,1)\}$

$=$ cone over $(\Delta,1)$ inside $M_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$.

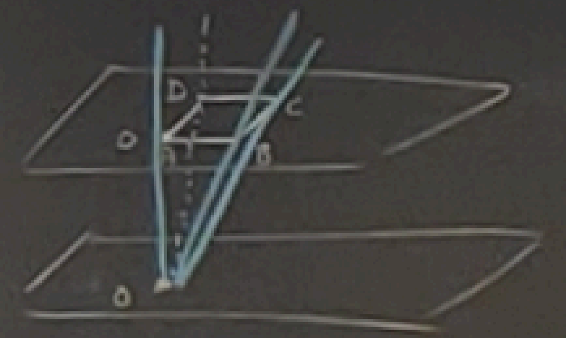
$k[C(\Delta)\cap(M\times\mathbb{Z})] \ni \chi^{[P,1]} \mid P\in\Delta\cap M$.

$\cup$

$k[\mathbb{N}\cdot(\Delta\cap M, 1)] = S(\mathbb{P}(\Delta))$

Sometimes an equality, sometimes not.

Example: $\square$ (D, C, A, B)



Remark If $S(\mathbb{P}(\Delta)) = k[C(\Delta)\cap(M\times\mathbb{Z})] \Rightarrow \mathrm{Spec}\, S(\mathbb{P}(\Delta)) = TV(C(\Delta)^\vee)$

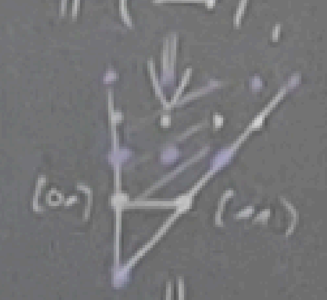
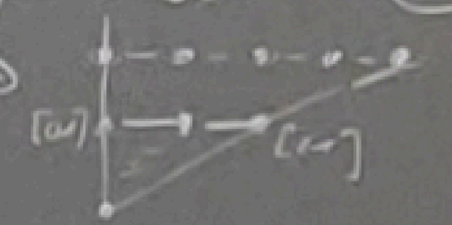
$\|$

affine cone over $\mathbb{P}(\Delta)$ $\quad (\mathbb{A}^{n+1}\setminus 0 \longrightarrow \mathbb{P}^n)$

$(c_0,\ldots,c_n) \longmapsto (c_0:\ldots:c_n)$

$\Delta \subseteq M_{\mathbb{R}}$ lattice polytope $\square$ (D, C, A, B) $\rightsquigarrow \mathbb{P}(\Delta) \hookrightarrow \mathbb{P}^3$

(ex: $\mathbb{P}(\text{—})$, $\mathbb{P}(\text{—·—})$ both $\cong \mathbb{P}^1$, different $S(\ldots)$)

$\bigoplus_{v\in\mathbb{N}} S_v =$  $\supsetneq$  $= S = \bigoplus_{v\in\mathbb{N}} S_v$

$k[\chi^{[0,1]}, \chi^{[1,1]}] \cong k[x,y]$

$k[\chi^{[0,1]}, \chi^{[1,1]}, \chi^{[2,1]}] \cong k[a,b,c]/(ac-b^2)$

$S(\Delta)$ - quotient of $k[z_P \mid P\in\Delta\cap M]$

$= k[\mathbb{N}\cdot(\Delta,1)]$

affine charts (of $\mathbb{P}(\Delta)\subseteq\mathbb{P}^N$)

- look at affine charts of $\mathbb{P}^N$
- choose a variable out of ~~$z_0, \ldots, z_N$~~

① out of $z_P \mid P\in\Delta\cap M$

$\hat{=}$ choose a $P\in\Delta\cap M$

e.g. Choose a vertex of Δ

② z_P - chart of $\mathbb{P}^N$: $D_+(z_P)$ = affine space, coord z_Q/z_P $(Q\in\Delta\cap M)$

$= \mathrm{Spec}\, k[\frac{z_Q}{z_P} \mid Q\in\Delta\cap M] = \mathrm{Spec}\, \boxed{k[z_Q \mid Q\in\Delta\cap M]}_{(z_P)}$

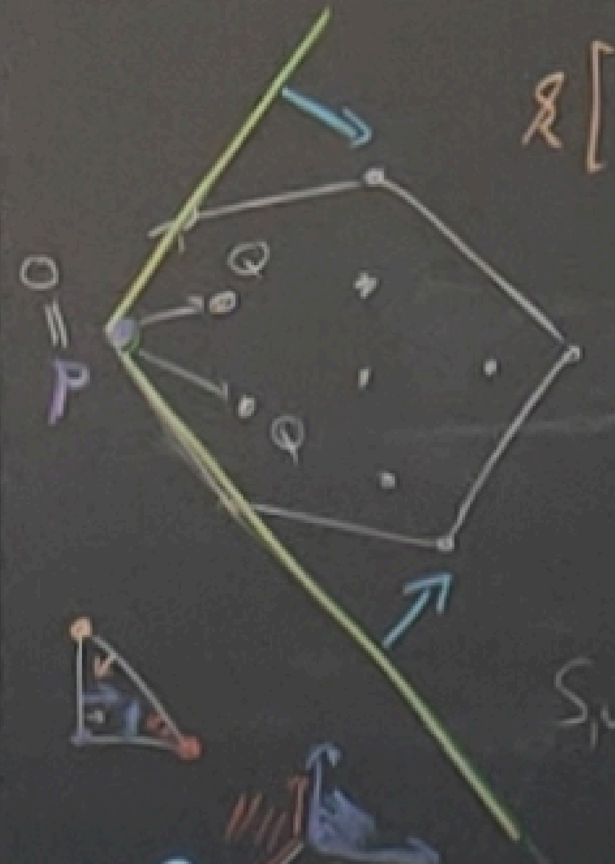
$\downarrow \Phi$ $\quad z_Q \mapsto \chi^{[Q,1]}$

$S(\Delta)$

$D_+(z_P)\cap\mathbb{P}(\Delta) = \mathrm{Spec}\, S(\Delta)_{(z_P)} = \mathrm{Spec}\, S(\Delta)_{(\chi^{[P,1]})}$

$\|$

$k[\mathbb{N}\cdot(\Delta\cap M, 1)]$



$k[\chi^Q/\chi^P \mid \ldots] = k[\chi^{[Q,1]}/\chi^{[P,1]} \mid Q\in\Delta\cap M]$

$\|$ χ^{Q-P}

$k[\Delta^{\mathbb{Z}} - P]$ $\qquad \Delta^{\mathbb{Z}} - P \subseteq \mathbb{R}_{\geq0}\cdot(\Delta-P)$

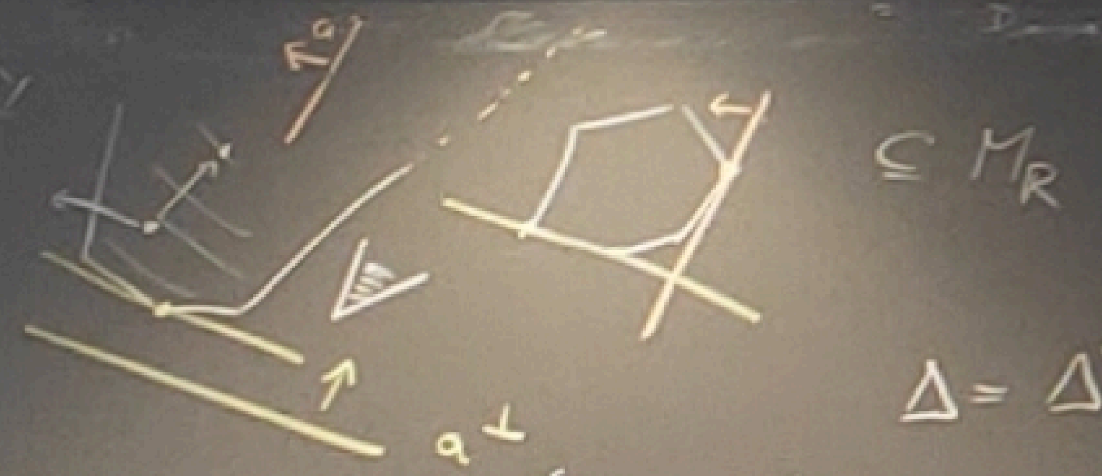
$\|$ $\qquad$ Pretend for a moment that $\Delta^{\mathbb{Z}} - P$ generates

$\mathrm{Spec}\, k[\mathbb{R}_{\geq0}(\Delta-P)\cap M]$ $\qquad \mathbb{R}_{\geq0}(\Delta-P)\cap M$ (i.e. Hilbert basis)

$\|$

$TV(\sigma)$ with $\sigma = (\mathbb{R}_{\geq0}(\Delta-P)\cap M)^\vee$

Cor. Schench. Toric Geometry



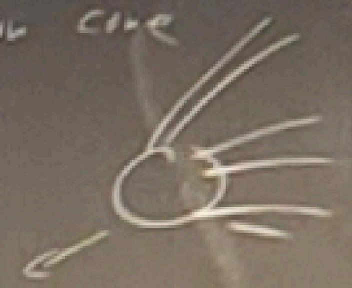
$\Delta \subseteq M_{\mathbb{R}}$ ~~polytope~~ polyhedron $\subseteq M_{\mathbb{R}}$

$a \in N_{\mathbb{R}} \longmapsto \min_{p \in \Delta} \langle p, a \rangle = \min \langle \Delta, a \rangle$

$\Delta = \Delta^c + \text{cone}$, $\text{cone} = \text{tail}(\Delta) = \{ r \in M_{\mathbb{R}} \mid r + \Delta \subseteq \Delta \}$ recession cone

$N_{\mathbb{R}} \xrightarrow{\psi} \mathbb{R} \cup \{-\infty\}$

$(\Delta = \text{compact} \iff \text{tail}(\Delta) = 0)$



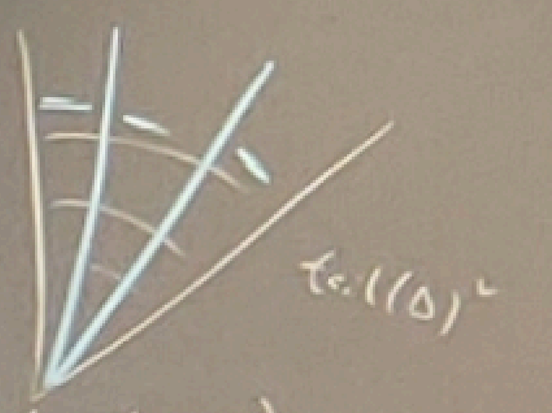
2 properties: (1) ψ is really defined $(\psi(..) \neq -\infty) \iff a \geq 0$ on $\text{tail}(\Delta)$ $\iff a \in \text{tail}(\Delta)^\vee$

(2) ψ is ~~piecewise~~ fanwise linear

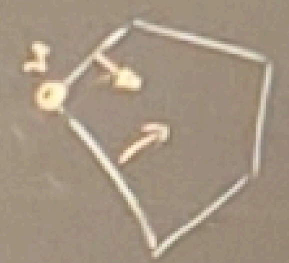
i.e. $\text{tail}(\Delta)^\vee$ is covered by pol. cones σ such that

$\psi_{|\sigma} = \text{linear}$. $\rightsquigarrow$ fan (subdividing $\text{tail}^\vee$)

$\rightsquigarrow$ „normal fan of Δ": $\mathcal{N}(\Delta)$



$\text{tail}(\Delta)^\vee$



$\min \langle \Delta, a \rangle = \langle r, a \rangle$ linear

for all $a \in \sigma = \mathcal{N}(r, \Delta)$

$\Leftarrow = \mathcal{N}(r, \Delta)$, $\mathcal{N}(r,\Delta) \subseteq \mathcal{N}(\Delta)$, dual to $\mathbb{R}_{\geq 0} \cdot (\Delta - r)$

$\Rightarrow \mathbb{P}(\Delta) = TV(\mathcal{N}(\Delta))$ (if our „product" from the 2nd stock found was allowed)

Δ

$\mathbb{P}^n \supseteq Z = V_+(J)$

$k[z_0, \dots, z_n]$, $D_+(z_i)$: $k[z_0, \dots, z_n]_{(z_i)} = $ reg. fct. on $D_+(z_i)$, $\ni \frac{z_j}{z_i}$

Spec $A \rightsquigarrow A = $ set of reg. fct. on X (Spec $A = X$)

$k[z] \hookrightarrow k[z]_{(z_i)} \hookrightarrow k[z]_{(z_i z_j)}$, $k[z] \hookrightarrow k[z]_{(z_j)} \hookrightarrow k[z]_{(z_i z_j)}$

Q What are the regular fct. on $\mathbb{P}^n$?

Def: $f : \mathbb{P}^n \to k$ is regular $:\iff f_{|D_+(z_i)} \in S(\mathbb{P}^n)_{(z_i)} = k[z_0, \dots, z_n]_{(z_i)} \subseteq k(z_0, \dots, z_n)$ = Quot $k[\underline{z}]$

$\Gamma(\mathbb{P}^n, \mathcal{O}) := \{ \text{regular } f : \mathbb{P}^n \to k \} = \{ f \in k(z_0, \dots, z_n) \mid \forall i : f \in k[\underline{z}]_{(z_i)} \} = \bigcap_{i=0}^{n} k[\underline{z}]_{(z_i)}$ $\left(\bigcap^{n} \text{ in } k[z_0^{\pm 1}, \dots, z_n^{\pm 1}] \right)$

$k[\underline{z}]$: factorial $\Rightarrow f \in \Gamma$: denominator = constant, also: $\deg f = 0 \Rightarrow \Gamma(\mathbb{P}^n, \mathcal{O}) = k$!!

$Z \subseteq \mathbb{P}^n_k$, $Z = V_+(J) \rightsquigarrow$ what is $\Gamma(Z, \mathcal{O}) = ?$

Prop. Even then, we have $\Gamma(Z, \mathcal{O}) = k$. (If $Z = $ irred.)

Proof. Deal with $S(Z)$ instead of $k[z_0, \dots, z_n]$

$S(Z)$ does not need to be factorial.

Let $f \in \Gamma(Z, \mathcal{O}) \rightsquigarrow S := S(Z) \rightsquigarrow$ local charts: $S_{(z_i)}$

$\Rightarrow f$ can be written as $f = \frac{F}{z_i^d}$ $(\forall i)$

$\Rightarrow z_i^d \cdot f \in S_d \subseteq S$ $\forall i$ $(\exists d)$, S = quotient $k[z_0, \dots, z_n]/J$

$\underline{z}^d = (z_0, \dots, z_n)^d$

$f \in S_{(z_i)} \subseteq \text{Quot } S$

$\Rightarrow (\underline{z}^d) \cdot f^k \subseteq (\underline{z}^d)$ $\forall k \geq 0$

$\Rightarrow z_0^d f^k \in (\underline{z}^d)$ $\forall k \geq 0 \rightarrow f^k \in \frac{1}{z_0^d} \cdot S$ $\forall k \Rightarrow S[f] \subseteq \frac{1}{z_0^d} S$ (f.g. S-module) $\Rightarrow$ $\boxed{f \text{ is integral over } S}$

$S = k[z]/J \rightsquigarrow f \in \text{Quot } S$, f is integral over S. Ind.: S, f = homogeneous.

$\exists$: $z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$ $(= p(z))$ | $a_i \in S$; $p(f) = 0$ | $\deg f = 0$

$f^n + a_{n-1} f^{n-1} + \dots + a_0 = 0$

$a_i = \boxed{(a_i)_0} + (a_i)_1 + \dots$

$p(z) = p_0(z) + p_{\geq 1}(z)$

$0 = p(f) = (p(f))_0 + (p(f))_{\geq 1} \Rightarrow p(f)_0 = 0$, i.e. w.l.o.g.: $\deg a_i = 0 \Rightarrow \boxed{a_i \in k}$

$\Rightarrow f^n + a_{n-1} \dots + a_0 = 0$, $a_i \in k$, $\Rightarrow f$ = integral over $k \Rightarrow f \in k$ // $\square$

$z_i = ?$