$X \subseteq \mathbb{P}^n_k$ projective $\rightsquigarrow$ $\Gamma(X, \mathcal{O}_X) := \{(\text{global}) \text{ regular fcts on } X\} = k$

$z_0, \dots, z_n$ = homog. coord. $\qquad$ $p \in X \rightsquigarrow z_i(p)$ does not make sense.

$f : \mathbb{P}^n \longrightarrow k$ function $\quad \hat{=} \quad \boxed{\mathbb{P}^n \times (\mathbb{A}^1 = k)}$ $\qquad (\mathbb{P}^n = \{\ell \subseteq k^{n+1} \mid \text{lines through } 0\})$

$\downarrow \pi$

$\mathbb{P}^n$ $\qquad f \hat{=} F$ = section of π.

Look for other fibrations:

$$\underline{\mathcal{O}(-1)} := \{(\ell, c) \mid \ell \in \mathbb{P}^n \ (\ell \subseteq k^{n+1}),\ c \in k^{n+1},\ \underline{c \in \ell}\} \subseteq \mathbb{P}^n \times k^{n+1}$$

$(\ell, ?)$

"tautological bundle" $\qquad \downarrow \pi$ $\qquad$ F - section (?)

$$\pi^{-1}(\{\ell\}) = \{(\ell, c) \mid c \in \ell\} = \{\ell\} \times \ell \qquad \ell \in \mathbb{P}^n$$

Q: Are $z_0, \dots, z_n$ sections?

$$\mathcal{O}(+1) = \{(\ell, c) \mid \ell \in \mathbb{P}^n,\ c : \ell \longrightarrow k\}$$

$(\ell, \underline{c = ?})$

$\pi \downarrow \quad \nearrow F$

$\mathbb{P}^n \ni \ell$

$c \in \ell^*$

$(c_0, \dots, c_n) \overset{z_0}{\longmapsto} c_0$

$\ell^{\otimes d} = \underbrace{\ell \otimes_k \dots \otimes_k \ell}_{d\text{-times}} \cong k^1 \quad (d \geq 0)$

$\ell^{\otimes(-d)} = \mathrm{Hom}(\ell^{\otimes d}, k) = (\ell^{\otimes d})^*$

THUS: $\boxed{z_0, \dots, z_n}$ are $\boxed{\text{sections}}$ of the line bundle $\underline{\mathcal{O}(+1)}$ over $\mathbb{P}^n$. $\quad \Big\|$ $z_1 \cdot z_3$ = section of $\underline{\mathcal{O}(2)}$ (global)

- naive affine geometry over $k = \bar{k}$ $\rightsquigarrow$ $\boxed{\mathrm{Spec}\, A}$

$$\mathcal{O}(d \in \mathbb{Z}) := \{(\ell, c) \mid \ell \in \mathbb{P}^n,\ c \in \ell^{\otimes(-d)} = \mathrm{Hom}(\ell^{\otimes d}, k)\}$$

$\mathcal{O}(0) = \mathbb{P}^n \times k$

$\mathcal{O}(-1)$: $\frac{z_1}{z_2^2}$ is a section, but not everywhere defined!

- naive proj. geometry $(X \subseteq \mathbb{P}^n)$ $\rightsquigarrow$ $\underline{\mathrm{Proj}\, S}$ for S - graded (k-algebra)
  $X = V_+(J) \dots$ $\qquad$ next blackboard $\qquad$ over $\mathbb{N}$ $\qquad$ ring

  With: $S_0 = k$; S_1 = f. dim. k-vs; $\langle S_1 \rangle = S$

$$\underline{\mathrm{Proj}\, S} := \{P \subseteq S \text{ homog. PI such that } P \not\supseteq S_1\}$$

ex: $S = k[z_0, \dots, z_n]$, $S_1 = \mathrm{span}\{z_0, \dots, z_n\}$

$P \supseteq S_1 \iff (z_0, \dots, z_n) \subseteq P \iff (z_0, \dots, z_n) = P$ $\qquad V_+(z_0, \dots, z_n) = \emptyset$ in $\mathbb{P}^n$

$1 \in (P : \underline{z}^\infty)$!! $\qquad V_+(1)$

Zariski topology on Proj S

Recall: $A \to B \implies \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ $\quad\Big|\quad$ Now: $S \xrightarrow{\varphi} T$ graded ring hom.

$\overset{(?)}{\rightsquigarrow}$ $\mathrm{Proj}\, T \dashrightarrow \mathrm{Proj}\, S$

$Q \longmapsto \varphi^{-1}(Q)$

still ex: $(0:0:1) \hat{=} (z_0, z_1) \not\supseteq (z_0, z_1, z_2)$

$\Downarrow$

$\in \mathrm{Proj}\, k[z_0, z_1, z_2]$

$\varphi^{-1}(z_0, z_1) = (w_0, w_1) \subseteq k[w_0, w_1]$

$\not\in$ Proj

ex: $k[z_0, z_1] \overset{\varphi}{\hookrightarrow} k[z_0, z_1, z_2]$ $\quad$ ($\deg z_i = 1$)

Proj $\Downarrow \qquad \Downarrow$

$\mathbb{P}^1 \overset{(?)}{\longleftarrow} \mathbb{P}^2$

$(w_0 : w_1) \longleftarrow (z_0 : z_1 : z_2)$

$\| $

$(z_0 : z_1)$

$\cancel{(0:0)} \longleftarrow (0:0:1)$

Remark: $S \twoheadrightarrow T$ surj.

$\Downarrow$

$\mathrm{Proj}\, T \hookrightarrow \mathrm{Proj}\, S$ closed emb.

Local structure: $\cdot$ $f \in S_d$, $d \geq 1$ $(f \neq 0)$ $\rightsquigarrow$ $D_+(f) := \mathrm{Proj}\, S \setminus V_+(f)$ $\qquad S_0$ = ring, S_1 = f.g. S_0-module.

$\cdot$ before: $J \subseteq S$ homog. ideal $\rightsquigarrow V_+(J) := \{P \in \mathrm{Proj}\, S \mid P \supseteq J\}$

$D_+(f) = \{P \in \mathrm{Proj}\, S \mid f \notin P\} = \{P \subseteq S \text{ homog. PI} \mid f \notin P\}$

Prop: $D_+(f) \overset{\text{as top. space}}{=} \mathrm{Spec}\, S_{(f)}$

Proof: $\cdot$ 1st case: $\deg f = 1$ $\qquad$ $\cdot$ $P \subseteq S$ PI, $f \notin P \longmapsto P_{(f)} = \left\{\frac{p}{f^{\deg p}} \;\middle|\; p \in P \text{ homog.}\right\}$

$\cdot$ $\mathfrak{p} \subseteq S_{(f)} \rightsquigarrow P := \left(p \in S \text{ homog.} \;\middle|\; \frac{p}{f^{\deg p}} \in \mathfrak{p}\right)$

$(P) \longleftarrow P$ $\qquad z_0, \dots, z_n \in S_1$

$f \rightsquigarrow z_i f, \ \underline{z}$ - set. that

$\cdot$ general case: $\deg f = d \geq 1$

$$S_{(d)} := \bigoplus_{r \in \mathbb{N}} S_{r \cdot d} \subseteq S \rightsquigarrow \cdot\ \mathrm{Proj}\, S = \mathrm{Proj}\, S_{(d)}$$

$$\cdot\ D_+(f) = D_+(f) \overset{\deg = 1}{=\!=} \mathrm{Spec}\, (S_{(d)})_{(f)} = \mathrm{Spec}\, S_{(f)}$$

$\mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n \qquad \mathrm{Spec}\, S \setminus V(S_1) \longrightarrow \mathrm{Proj}\, S$

$D(f) \longrightarrow D_+(f)$

Blowing up $\mathbb{A}^n$ in $0 \in \mathbb{A}^n$

① $\widetilde{\mathbb{A}}^n = \mathcal{O}(-1) = \{ (\ell, c) \mid \ell \in \mathbb{P}^{n-1}, \boxed{c \in \ell} \ (\subseteq k^n) \} \xrightarrow{\pi} \mathbb{A}^n \supseteq \mathbb{A}^n \setminus 0$

$\cap\!\!\mid \ \mathbb{P}^{n-1} \times \mathbb{A}^n$, $(y_1 : y_2 : \dots : y_n)(x_1, \dots, x_n)$

$\downarrow h$

$(y_1 : \dots : y_n) \in \mathbb{P}^{n-1}$

② equation of $\widehat{\mathbb{A}}^n \subseteq \mathbb{P}^{n-1} \times \mathbb{A}^n$: $\left(\boxed{x_i y_j - x_j y_i} \mid i, j = 1, \dots, n \right)$

On $D_+(y_i)$ (i = fixed): $\forall j: \ x_i y_j = x_j y_i \iff x_i \cdot \boxed{\frac{y_j}{y_i}} = x_j \implies (x_1, \dots, x_n) = x_i \cdot \left(\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i} \right) = x_i \cdot \left(\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i} \right)$

- $x_i = 0 \implies x_j = 0 \ \forall j$
- $x_i \neq 0$

i.e.: $(x_1, \dots, x_n) \in \ell \triangleq (y_1 : \dots : y_n)$

$$\boxed{\begin{array}{ccc} \widehat{\mathbb{A}}^n & \xrightarrow{\pi} & \mathbb{A}^n \\ h \downarrow & & \\ \mathbb{P}^{n-1} & & \end{array}}$$

what happens over

- $\mathbb{A}^n \setminus 0 \implies (x_1, \dots, x_n) \in \mathbb{A}^n \setminus 0$, $\exists i: x_i \neq 0$ $\Downarrow$ $y_j = x_j / x_i \cdot y_i \ \forall j$ $\implies y_i \neq 0$ $\implies (y_1 : \dots : y_n) = \left(\frac{x_1}{x_i} : \dots : \frac{x_n}{x_i} \right)$ $(\neq 0)$
- $0 \in \mathbb{A}^n \implies \pi^{-1}(0) \subseteq \widehat{\mathbb{A}}^n$?

$\Downarrow$ over $\mathbb{A}^n \setminus 0$: $\underline{y}$ is determined by $\underline{x}$:

$\hookrightarrow \pi|_{\pi^{-1}(\mathbb{A}^n \setminus 0)} : \pi^{-1}(\mathbb{A}^n \setminus 0) \xrightarrow{\sim} \mathbb{A}^n \setminus 0$

restricted to $\mathbb{A}^n \setminus 0$: $\hookrightarrow$ becomes

$$\begin{array}{ccc} \mathbb{A}^n \setminus 0 & \xrightarrow{\sim} & \mathbb{A}^n \setminus 0 \\ \downarrow h & & \| \\ \mathbb{P}^n & & \end{array}$$

$\widehat{\mathbb{A}}^n \ni (y_1 : \dots : y_n)$, $(x_1, \dots, x_n)$, $\downarrow \mathbb{P}^{n-1}$

$\rightsquigarrow$ in $\widehat{\mathbb{A}}^n$: $0 \in \mathbb{A}^n$ was replaced by $\pi^{-1}(0) = \mathbb{P}^{n-1} \times \{0\} \cong \mathbb{P}^{n-1} =: \underline{\underline{E}}$ („exceptional divisor")

$\mathbb{A}^2$

$\triangleq$ polar coordinates, $\triangleq$ $(x, y) \longleftrightarrow (r, \varphi)$

$h: \widehat{\mathbb{A}}^n \longrightarrow \mathbb{P}^{n-1}$; $h^{-1}(\ell) = \{ \{\ell\} \times \ell \}$, $\ell \in \mathbb{P}^{n-1}$, $\{\ell\} \in \mathbb{P}^{n-1}$, $\ell \subseteq \mathbb{A}^n$

③ Local description

$$\boxed{\begin{array}{ccc} \widehat{\mathbb{A}}^n & \xrightarrow{\pi} & \mathbb{A}^n \\ \downarrow h & \subset \mathbb{P}^{n-1} \times \mathbb{A}^n & \\ \mathbb{P}^{n-1} & & \end{array}} \rightsquigarrow \boxed{\begin{array}{ccc} h^{-1}(D_+(y_i)) & \xrightarrow{\pi} & \mathbb{A}^n \\ \downarrow h & & \\ D_+(y_i) \subset \mathbb{P}^{n-1} & & \end{array}} \quad \boxed{\begin{array}{ccc} \widehat{\mathbb{A}}^n_{(i)} & \longrightarrow & \mathbb{A}^n \\ \downarrow & \subset D_+(y_i) \times \mathbb{A}^n & \\ D_+(y_i) & & \end{array}}$$

$k[\underline{y}/y_i, \underline{x}] \big/ (x_i \cdot \frac{y_j}{y_i} = x_j)_{(j \neq i)} = k[\underline{y}/y_i, \underline{x}] \big/ (x_i y_j - x_j y_i) \longleftarrow k[x_1, \dots, x_n] = \boxed{k[\underline{x}]}$

$\dots \longmapsto x_j$

$\dots \longmapsto x_j$

$\uparrow$

$k\left[\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}\right] = \boxed{k[\underline{y}/y_i]}$ $= k[\underline{x}/x_i]$

$\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}, x_i$ $\boxed{k[\underline{y}/y_i, x_i]}$

$\downarrow$ $x_i \cdot \frac{y_j}{y_i}$ $\| \ \subset$

$\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}, x_i$

$\frac{x_1}{x_i}$ x_i $k[\underline{x}/x_i, x_i]$

$x_j = x_i \cdot \frac{x_j}{x_i} \longleftarrow\!\mid x_j$

$k(\underline{x}, y)$

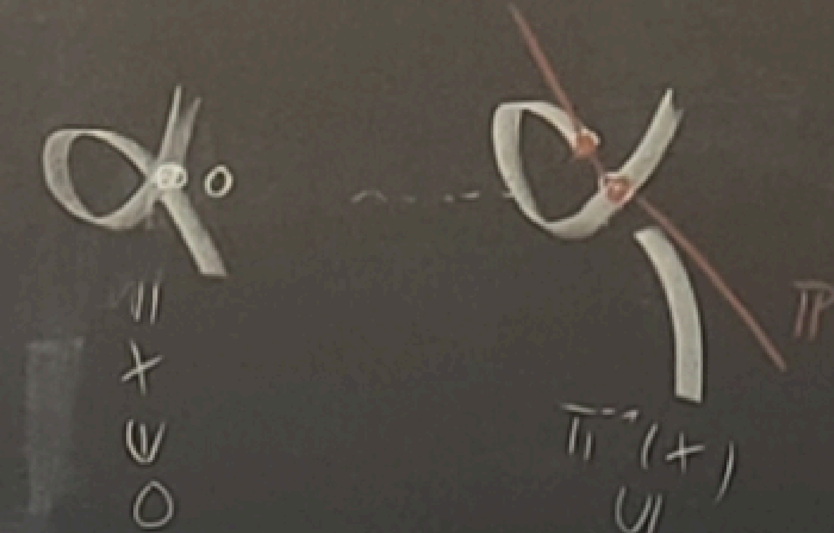
$x_1 + y_1$

$$\boxed{\begin{array}{ccc} k[\underline{x}/x_i, x_i] & \xleftarrow{\pi^*} & k[\underline{x}] \\ \uparrow h^* & \widehat{\mathbb{A}}^n \text{ - box over } D_+(y_i) & \\ k[\underline{x}/x_i] & & \end{array}}$$

- $k[\frac{x}{x_i}, x_i] \neq$ localization of $k[\underline{x}]$!!
- $k[\underline{x}] \hookrightarrow k[\underline{x}/x_i, x_i]$ is $\underline{\text{not}}$ flat !!

$\underline{\text{example}}$: $k[x, y] \hookrightarrow k[\frac{x}{y}, y]$ (chart of the blow up of $\mathbb{A}^2$)

Strict transforms (in blowing ups) $\widehat{\mathbb{A}}^n \xrightarrow{\pi} \mathbb{A}^n$ (blow up $0 \in \mathbb{A}^n$) $\Big\{ \widetilde{\mathbb{A}}^n_{(i)}$



$\mathbb{P}^{n-1} = E$

$X \supseteq 0$, $\pi^{-1}(X) \supseteq \pi^{-1}(0) = E \cong \mathbb{P}^{n-1}$

$E \subseteq \pi^{-1}(X) \longrightarrow X = V(\mathfrak{J})$ ($\mathfrak{J} \subseteq k[x_1, \dots, x_n]$) eg $\mathbb{A}^2 \supset V(y^2 - x^3)$

total transform

$\pi^{-1}(X \setminus 0) \xrightarrow{\sim} (X \setminus 0)$

$E \nsubseteq \pi^{\#}(X) = \overline{\pi^{-1}(X \setminus 0)}$

strict transform

$\mathfrak{J} \subseteq \boxed{k[x_1, \dots, x_n] \hookrightarrow k[\underline{x}, \underline{x}/x_i]}$ i-chart of $\widetilde{\mathbb{A}}^n$

$\rightsquigarrow \pi^{-1}(X) \cong \mathfrak{J} \cdot k[\underline{x}, \frac{\underline{x}}{x_i}]$ $\cap \widehat{\mathbb{A}}^n_{(i)}$

$X \rightsquigarrow \Gamma(\mathcal{O}_X)$

$A \xrightarrow{\varphi} B$, $\operatorname{Spec} A \xleftarrow{f} \operatorname{Spec} B$ Spec

$f^{-1}(V(I)) = \operatorname{Spec} B / \varphi(I) \cdot B$, $V(I) = \operatorname{Spec} A/I$

$\hat{\mathbb{A}}^n = k[x_1, \dots, x_n] - k[\underline{x}] \hookrightarrow k[\underline{x}/x_i, x_i] \subseteq \hat{\mathbb{A}}^n_{(i)}$

$V(\mathfrak{a}) \cong \mathfrak{a} \qquad \mathfrak{a} \cdot k[\underline{x}/x_i, x_i] \qquad \pi^{-1}(V(\mathfrak{a}))$

- $E = \pi^{-1}(0) \qquad 0 = V(x_1, \dots, x_n) \subseteq \mathbb{A}^n$

$\boxed{E} = \pi^{-1}(0) = V\left((x_1, \dots, x_n) \subseteq k[\underline{x}/x_i, x_i]\right) = V(x_i)$ (codim 1)

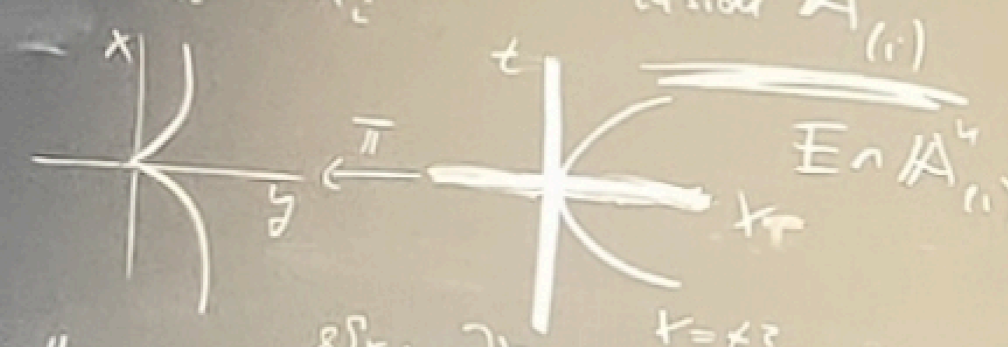
$x_j = \frac{x_j}{x_i} \cdot x_i \qquad x_i = x_i$

Ex $k[x, \frac{y}{x}] \cdot x^2(t^2 - x)$, $E \subseteq X$

- $\pi^*(X) := \overline{\pi^{-1}(X \setminus 0)} = \pi^{-1}(X \setminus \pi^{-1}(0))$

locally $\sim$ i-th chart: $k[\underline{x}/x_i, x_i]$, $t=0$; inside $k[\underline{x}/x_i, x_i]$: $\overline{V(\mathfrak{a}) \setminus V(x_i)} = V(\mathfrak{a} : (x_i)^\infty)$

$E \cap \mathbb{A}^n_{(i)}$



Example $X = V(y^2 - x^3)$, $y^2 - x^3 \in k[x,y]$

- $k[x,y] \hookrightarrow k[x, \frac{y}{x}] = k[x, t]$, $t = y/x$
  $y^2 - x^3 \mapsto (xt)^2 - x^3 = x^2(t^2 - x)$
- $k[x,y] \hookrightarrow k[y, \frac{x}{y}] = k[y, s]$, $s = x/y$
  $y^2 - x^3 \mapsto y^2 - y^3 s^3 = y^2(1 - y s^3)$

$k[\underline{x}, y]_{(y_1)} = k[\underline{x}, y/y_1]$

$k[x, s]_{(y_1)} = k[x_1, x/y_1]$

$\cap$

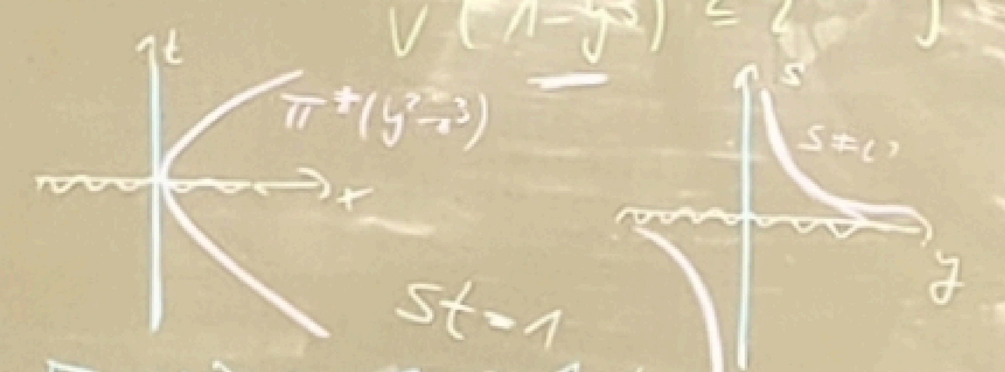
$k[x, y](y, y_1)$ — say y_1: $k[\underline{x}, \frac{y}{y_1}, y_1]$

$(x, y \neq 0)$: $k[x, \frac{y}{x}, y, \frac{1}{y}] = k[x, y, s, t]/(st - 1, \dots)$

$(y \neq 0)$

$V(x^2 y^2) \subseteq \mathbb{C}^2$

$\pi^*(V(y^2 - x^3)) = V(t^2 - x) \subseteq 1^{st}$, $V(1 - ys^3) \subseteq 2^{nd}$ chart



$E \cong \mathbb{P}^1$, $s = 0 \Leftrightarrow t = \infty$

2^{nd} chart: $s \neq 0$, $(1 - ys = 0)$; 1^{st} chart $t \neq \infty$ $\Longrightarrow$ $t = \frac{1}{s}$: $\pi^*(V(y^2 - x^3)) \subseteq 1^{st}$ chart $\Rightarrow$ is affine.

(4) Proj-description of blow ups.

$A = k[\underline{x}] = k[x_1, \dots, x_n]$, $I = (x_1, \dots, x_n)$

$\hat{\mathbb{A}}^n$ claim $= \mathrm{Proj} \bigoplus_{\nu=0}^{\infty} I^\nu$

$k[\underline{x}][\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}] = k[x_i, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}]$

$A = V(\mathfrak{a})$, $I \subseteq A$ ideal

$X = \mathrm{Spec}\, A \xleftarrow{\pi} \tilde{X} = \mathrm{Proj} \bigoplus I^\nu t^\nu$

$I = (f_1, \dots, f_n)$, $I^d = (f_1, \dots, f_n)^d$

$D_+(f_i t) = \mathrm{Spec}\left(\bigoplus I^\nu t^\nu\right)_{(f_i t)}$ $\Rightarrow \frac{F}{f_i^d}$ of deg $d \in I^d$

$A \oplus It \oplus I^2 t^2 \oplus \dots$

$A[\frac{f_1}{f_i}, \dots, \frac{f_n}{f_i}] \subseteq A_{f_i}$

$E = \pi^{-1}(V(I))$ $\hat{=}$ i-th chart $(f_1, \dots, f_n) = (f_i)$

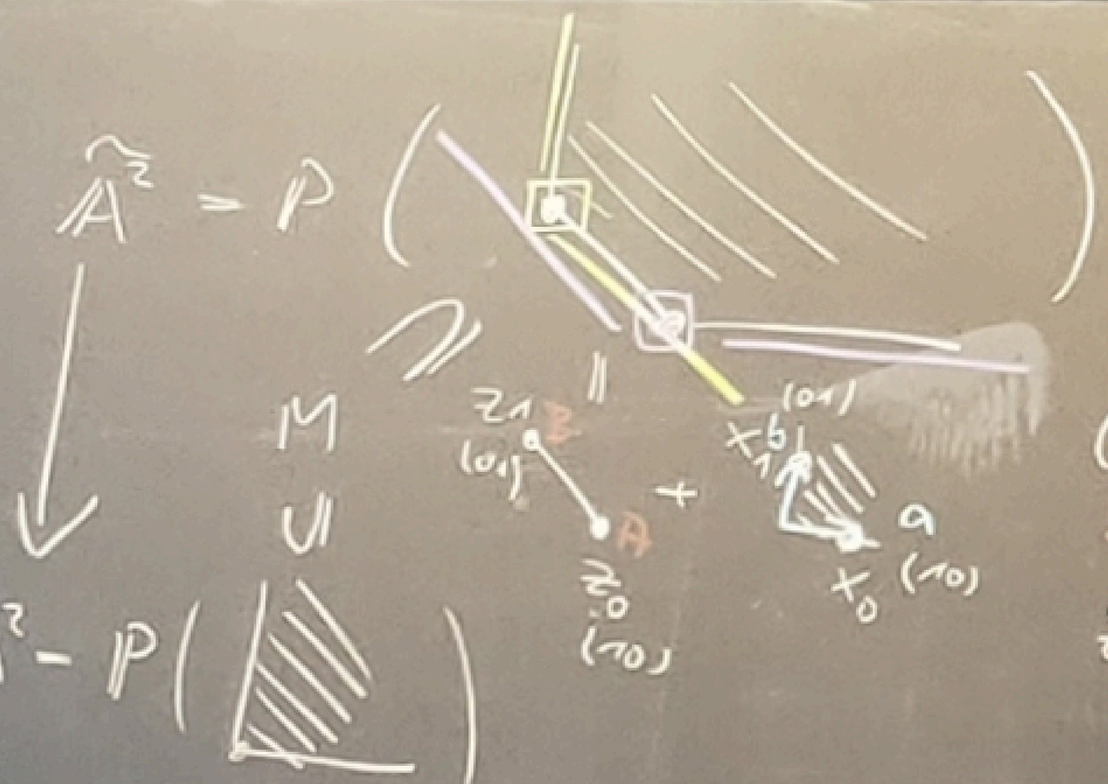
(5) Toric blow up of $\mathbb{A}^2$ in 0

$k[x,y] \hookrightarrow k[x, \frac{y}{x}]$, $k[\frac{x}{y}, y]$ $\} \hat{\mathbb{A}}^2$

claim: $\hat{\mathbb{A}}^2 - P$

$\mathbb{A}^2 - P$

$k[y, \frac{x}{y}] \qquad k[x, \frac{y}{x}]$

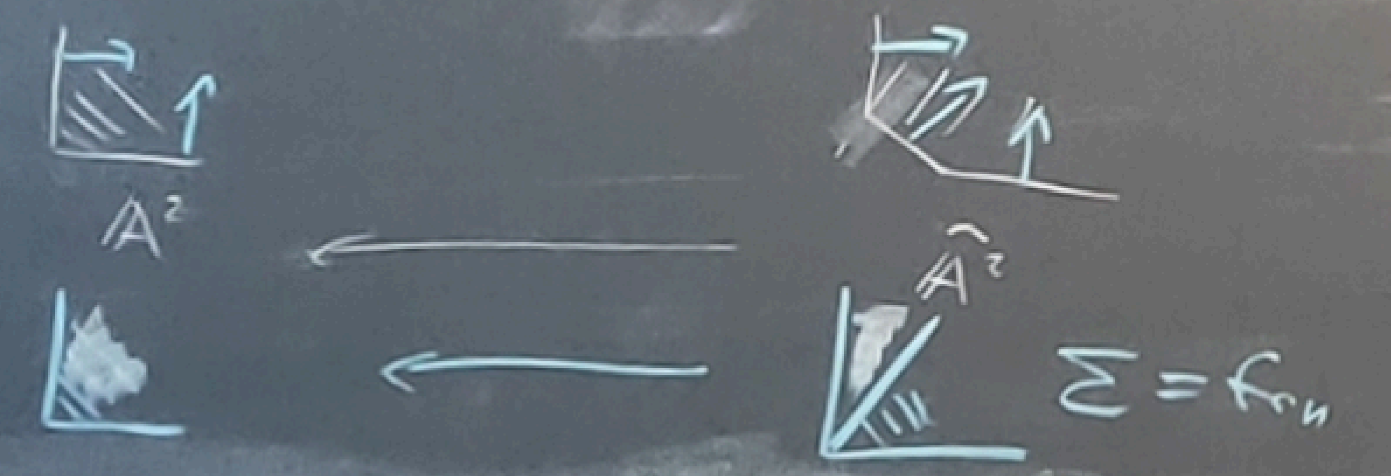


$(01) + (10) = (10) + (01)$

$B + a = A + b$

$z_1 x_0 = z_0 x_1$

translate into the N-language:



$\hat{\mathbb{A}}^2 \to \mathbb{A}^2$

$N \cong \mathbb{Z}^2$, $\Sigma = \mathrm{fan}$