(19) Sheeves

Pre-Sheaves: $\mathcal{C}$ = category (Ab, Rings, some modules) Let X = topological space.

Def $\mathcal{F}$ = „pre sheaf of $\mathcal{C}$" (on X) $:\Leftrightarrow$ $\mathcal{F}: \mathrm{Open}(X)^{op} \longrightarrow \mathcal{C}$ (functor)

$\mathrm{Open}(X)$: cat. of open subsets of X

- objects $= \{U \subseteq X \mid \text{open}\}$
- morphisms $\mathrm{Hom}(U,V) := \begin{cases} \emptyset & U \not\subseteq V \\ \mathrm{id}\colon U \hookrightarrow V & U \subseteq V \end{cases}$

  $\hookrightarrow$ $\exists\, \mathrm{id}_U \in \mathrm{Hom}(U,U)$

- $U \subseteq V \subseteq W$: $\mathrm{Hom}(U,V) \times \mathrm{Hom}(V,W) \to \mathrm{Hom}(U,W)$, $(i, j) \longmapsto j \circ i$

i.e.

- $U \subseteq X \rightsquigarrow \mathcal{F}(U) \in \mathcal{C}$
- $V \subseteq U \rightsquigarrow \mathcal{F}(U) \in \mathcal{C}$, $\;\mathcal{F}(V) \in \mathcal{C}$

  $\mathcal{F}(\iota)$, $\iota: V \hookrightarrow U$; $\;\mathcal{F}(\iota) = \mathrm{res}_{UV}$ („restriction")

  $f \in \mathcal{F}(U) \mapsto \boxed{f_{|V}} \in \mathcal{F}(V)$ just a notation

$W \subseteq V \subseteq U$: $\mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}(W)$

Example

- $\mathcal{F}(U) := \{\text{functions } f: U \longrightarrow \mathbb{R}\}$ e.g. continuous, differentiable

  $V \overset{\iota}{\hookrightarrow} U$, $f \mapsto f_{|V}$

  $\mathcal{F}(V) = \{g: V \to \mathbb{R}\}$

- $A \in \mathcal{C} \rightsquigarrow$ constant presheaf $\underline{A}^{pre}$: $\boxed{\underline{A}^{pre}(U) = A}$ $\forall$ open $U \subseteq X$

  $\underline{A}^{pre}(V) = A$, $\;\underline{A}^{pre}(\iota: V \hookrightarrow U) = \mathrm{id}_A$

X = alg. variety $\rightsquigarrow \mathcal{F}(U) = \{\text{regular fcts on } U\}$

$\mathcal{F}, \mathcal{G} \in \mathrm{PreSh}_X(\mathcal{C})$

$\eta: \mathcal{F} \to \mathcal{G}$ morphism of presheaves $:\Leftrightarrow$ η = natural transformation among the functors $\mathcal{F}$ and $\mathcal{G}$

i.e.

- $U \subseteq X$ open $\rightsquigarrow$ $\mathcal{F}(U) \xrightarrow{\eta(U)} \mathcal{G}(U)$ morphism within $\mathcal{C}$
- $V \subseteq X$ open $\rightsquigarrow$ $\mathcal{F}(V) \xrightarrow{\eta(V)} \mathcal{G}(V)$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta(U)"] \arrow[d, "\mathrm{res}^{\mathcal{F}}_{UV}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\eta(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

$\mathcal{F}, \mathcal{G}$ = presheafs $\rightsquigarrow$

- $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \{\text{morphisms } \eta: \mathcal{F} \to \mathcal{G}\}$
- $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ will become a presheaf (of sets)

$\mathrm{Hom}(\mathcal{F}, \mathcal{G})(U) := \{\eta(U): \mathcal{F}(U) \longrightarrow \boxed{\mathcal{G}(U)}\}$

V: $\cancel{= \{\eta(V): \mathcal{F}(V) \longrightarrow \boxed{\mathcal{G}(V)}\}}$? OK

$\mathrm{Hom}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$

$\mathcal{F}_{|U}$ = presheaf on U: $\mathcal{F}_{|U}(V \subseteq U) := \mathcal{F}(V)$ „restriction of $\mathcal{F}$ to U".

$V \subseteq X$

implies $\mathcal{F}(V) \to \mathcal{G}(V)$ for all $V \subseteq U$ (in particular: $\mathcal{F}(U) \to \mathcal{G}(U)$)

$\mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{G}_{|U}) \longrightarrow \mathrm{Hom}(\mathcal{F}_{|V}, \mathcal{G}_{|V})$ is obvious if $V \subseteq U$.

$\mathcal{F}(U)$ = „sections of $\mathcal{F}$ over U"

$\pi: Y \to X$ (es: $Y = X \times \mathbb{R}$), $s: X \to Y$; s = sect $\Leftrightarrow \pi \circ s = \mathrm{id}_X$

$\mathcal{F}(U) = \{\text{those } s\text{'s}\}$

$U \subseteq X$; $\mathcal{F} \rightsquigarrow \mathcal{F}(U)$ = „set" of sections ($\in \mathcal{C}$)

$\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$

$\mathcal{F}$:

- $U \subseteq X \Longrightarrow \mathcal{F}(U)$
- $P \in X \Longrightarrow \mathcal{F}_P := \varinjlim_{U \ni P} \mathcal{F}(U)$ „direct limit"

(recall:

- I = index set (here: all $U \subseteq X$ open s.t. $P \in U$)
- I = poset „$i \leq j$" („$U \leq V$" $\Leftrightarrow V \subseteq U$)
- I = filtered: $\forall i, j \in I\ \exists k \in I: i, j \leq k$ ($U, V \subseteq X$ open $\Rightarrow U \cap V$ = open in X)

$\varinjlim_I A_\bullet$: 1st: Universal property: Given „inductive system on $\mathcal{C}$", i.e.:

- $i \in I \rightsquigarrow A_i \in \mathcal{C}$
- $i \leq j \rightsquigarrow A_i \xrightarrow{\varphi_{ij}} A_j$ (functorial)

$\Downarrow$

ask for a map (within $\mathcal{C}$) $A_i \longrightarrow \varinjlim_I A_\bullet$ s.t. this is an initial object, i.e.

$\forall i \in I$, $i \leq j$: $A_i \to A_j \to B$, $\varinjlim_I A_\bullet \to B$ ⟷ form a category

$I = \{0, 1\}$

$A_0 \to B$, $A_1 \to B$

$A_0 \hookrightarrow A_0 \oplus A_1$, $A_1 \hookrightarrow A_0 \sqcup A_1$

$\varinjlim_I A_i$, A_i = abelian groups $\qquad \varinjlim A_i := \coprod_{i\in I} A_i \Big/ \left.\begin{array}{l} i\le j \\ a\in A_i \rightsquigarrow \varphi_{ij}(a)\in A_j \end{array}\right\} \Rightarrow a \sim \varphi_{ij}(a)$

- $A_i \longrightarrow \varinjlim_I A_i$ ✓
- how to add $a\in A_i$; $b\in A_j$ $\overset{\text{filtered}}{\rightsquigarrow} \exists\, k\ge i,j$

$\quad a' \in A_k \quad b'\in A_k \Longrightarrow a'+b'\in A_k$ is well defined.

addl. property: $a\in\varinjlim_I A_i \Rightarrow$ repr. by $a_i\in A_i$
$\qquad b\in\varinjlim_I A_i \Rightarrow$ - " - $b_j\in A_j$

Q: $a=b \iff \exists\, k\ge i,j : \varphi_{ik}(a_i)=\varphi_{jk}(b_j)$ inside A_k

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$\mathcal{F}$ = presheaf on X, $P\in X$

$\mathcal{F}_P := \varinjlim_{U\ni P} \mathcal{F}(U)$:

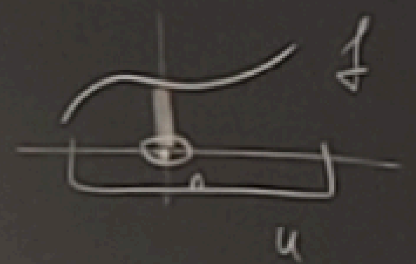
- $\mathcal{F}(\overset{P}{\hat{U}}) \longrightarrow \mathcal{F}_P$
  $s \longmapsto s_P$ germ of s in P
- $U\supseteq V\ni P$
  $\mathcal{F}(U)\longrightarrow\mathcal{F}(V)\longrightarrow\mathcal{F}_P$, "stalk at P"
  $s\longmapsto s_{|V}\longmapsto s_P$

$P\in U,V$
$s\in\mathcal{F}(U)$, $t\in\mathcal{F}(V)$
$s_P = t_P$ in $\mathcal{F}_P \iff$
$\exists\, W\subseteq U\cap V : s_{|W}=t_{|W}$
$\quad \overset{\cup\!\!\!\!\shortmid}{P}$

Example: $\mathcal{O}^{an}$:= presheaf of analytic functions $\mathbb{R}\to\mathbb{R}$

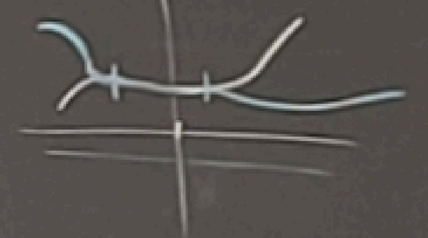
- $0\in\mathbb{R} \rightsquigarrow U\subseteq\mathbb{R}$ contains 0
  $\mathcal{O}^{an}(U) = \{ f: U\longrightarrow\mathbb{R} \text{ analytic}\} \ni f$
  $\downarrow$
  $\mathcal{O}^{an}_0 \overset{?}{=} \mathbb{R}\{z\}$ = ring of convergent power series (in 0)

$f \mapsto f(0)\in\mathbb{R}$, f_0: $a_0 + a_1 z + a_2 z^2 + \ldots$



$f(z) = \frac{1}{z-3}$

- $\mathcal{C}^\infty$-fcts on $\mathbb{R}$



Remark: $\mathcal{F}(U)\longrightarrow\mathcal{F}_P$ $(P\in U)$ is an exact functor.
$\quad \mathcal{G}(U) \qquad \mathcal{G}_P$

Def: $s\in\mathcal{F}(U) \rightsquigarrow \operatorname{supp} s := \{P\in U \mid s_P\neq 0 \text{ in } \mathcal{F}_P\}$ ($\mathcal{F}$ = presheaf of abelian groups)
$\rightsquigarrow U$ closed subset

Proof: Let $P\in U\setminus\operatorname{supp}(s) \rightsquigarrow s_P=0$ in $\mathcal{F}_P$ $(\leftarrow\mathcal{F}(U))$ $\quad (\mathcal{F}_P = \varinjlim_{U\ni P}\mathcal{F}(U))$
$\Longrightarrow \exists\, V\ni P : s_{|V}=0$ in $\mathcal{F}(V) \Longrightarrow s_Q = 0 \ \forall Q\in V$
$\quad V\subseteq U \qquad \Longrightarrow V\cap\operatorname{supp} s = \emptyset$ □

$\mathcal{F},\mathcal{G}$ = presheaves of ab. groups $\rightsquigarrow$
- $\mathcal{F}\oplus\mathcal{G}$: $(\mathcal{F}\oplus\mathcal{G})(U) := \mathcal{F}(U)\oplus\mathcal{G}(U)$
- $\mathcal{H}om(\mathcal{F},\mathcal{G})$
- $\mathcal{F}\otimes_{\mathbb{Z}}\mathcal{G}$ $(\mathcal{F}\otimes\mathcal{G})(U) := \mathcal{F}(U)\otimes_{\mathbb{Z}}\mathcal{G}(U)$

$\mathcal{R}$ = ring presheaf

Def. $\mathcal{F}$ (= presh. of ab. gr.) is an $\mathcal{R}$-module $:\iff \forall\, U\subseteq X$: $\cdot\ \mathcal{F}(U) = \mathcal{R}(U)$-module
$V\subseteq U$ $\cdot\ \mathcal{F}(V) = \mathcal{R}(V)$-module
$\Downarrow$
$\mathcal{F}(V) = \mathcal{R}(U)$-module

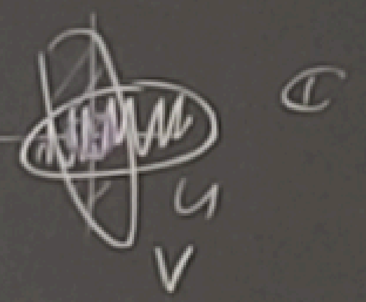
$\Downarrow$

$(\mathcal{F}\otimes_{\mathcal{R}}\mathcal{G})(U) = \mathcal{F}(U)\otimes_{\mathcal{R}(U)}\mathcal{G}(U)$

Ex: $(\mathcal{F}\oplus\mathcal{G})_P = \mathcal{F}_P\oplus\mathcal{G}_P$, $(\mathcal{F}\otimes_{\mathcal{R}}\mathcal{G})_P \to \mathcal{F}_P\otimes_{\mathcal{R}_P}\mathcal{G}_P$

$\mathcal{O}^{an}_0 \ni 3,\ x+1,\ 1+x+x^2\ldots$

$\mathcal{O}^{an}_0$



top. space $\mathbb{R}$
$(U\subseteq\mathbb{R} \text{ open}) \longmapsto \mathcal{F}(U) := \mathbb{C}$.

Def $\mathcal{F},\mathcal{G}$ = presheaves (in Ab)
- $f:\mathcal{F}\to\mathcal{G}$ is injective $:\iff \forall\, U\subseteq X$: $f(U):\mathcal{F}(U)\to\mathcal{G}(U)$ inj.
- f is surj. $\iff$ $f(U)$ surj.
- $\operatorname{Ker} f$ = presheaf $(\operatorname{Ker} f)(U) = \operatorname{Ker} f(U)$
  $\operatorname{Im} f$
  $\operatorname{Coker} f$

$\mathcal{F}\subseteq\mathcal{G}$
$\Downarrow$
$\mathcal{G}/\mathcal{F}(U) = \mathcal{G}(U)/\mathcal{F}(U)$

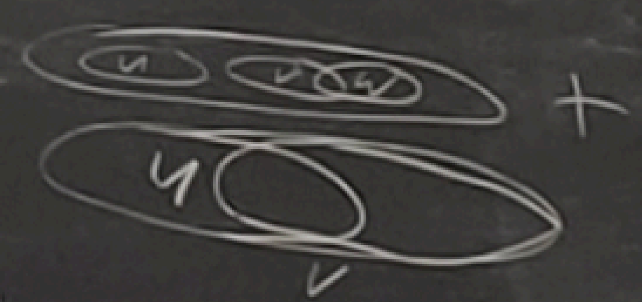
Example: presheaf $\mathcal{F}$: $\mathcal{F}\subseteq\underline{A}^{pre}$; $\mathcal{F}(X) = 0 \subsetneq A$

Sheaf Def. $\mathcal{F}$ = presheaf is called sheaf $:\iff$ (Ab, Rings)

(0) $\mathcal{F}(\emptyset) = 0$

(1) $U\subseteq X$ open, $U = \bigcup U_i$, $s\in\mathcal{F}(U)$ s.t. $s_{|U_i}=0\ \forall i \Longrightarrow s=0$

(2) $U=\bigcup U_i$, $s_i\in\mathcal{F}(U_i)$ s.t. $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ $\Longrightarrow \exists\, s\in\mathcal{F}(U) : s_{|U_i} = s_i$ $\quad (\rightsquigarrow s_{|U_i\cap U_j} = s_{|U_i}{}_{|U_i\cap U_j} = s_{i|U_i\cap U_j})$



$\mathcal{F} \longmapsto \mathcal{F}_P$ ($P \in X$ fixed) is an exact functor $\mathrm{PreSh}_{\mathcal{C}}(X) \xrightarrow{\mathcal{F} \mapsto \mathcal{F}_P} \mathcal{C}$

($\mathcal{F} \hookrightarrow \mathcal{G}$ „ $\Leftrightarrow \forall U$ … „ …)

$$\begin{tikzcd} \mathrm{PreSh}_{\mathcal{C}}(X) \arrow[d, "\mathcal{F} \mapsto \mathcal{F}(U)"] \\ \mathcal{C} \end{tikzcd}$$

Main point about sheaves: $s \in \mathcal{F}(U) \rightsquigarrow [s = 0 \iff \forall P \in U : s_P = 0]$

Proof. $(\Rightarrow)$ ✓

$(\Leftarrow)$ $\forall P : s_P = 0 \Rightarrow \exists \underset{\substack{\cup\\ P}}{U(P)} : s_{|U(P)} = 0 \Rightarrow \bigcup_{P \in U} U(P) = U \Rightarrow s = 0$

$\mathcal{F}, \mathcal{G}$ = sheaves … $\mathcal{F} \to \mathcal{G}$ (?) $\qquad \mathrm{Sh}_{\mathcal{C}}(X) \subseteq \mathrm{PreSh}_{\mathcal{C}}(X)$ full subcategory

Def. $f : \mathcal{F} \to \mathcal{G}$ map between sheaves $\rightsquigarrow \mathrm{Ker}\, f := (\mathrm{Ker}\, f)^{pre}$, i.e.: $U \mapsto \mathrm{Ker}(\mathcal{F}(U) \to \mathcal{G}(U))$

Claim: $\mathrm{Ker}\, f$ is again a sheaf! $\qquad \mathrm{Ker}\, f \hookrightarrow \mathcal{F}$

$$\begin{tikzcd} \mathrm{Ker}\, f(U) \arrow[r, hook] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathrm{Ker}\, f(V) \arrow[r, hook] & \mathcal{F}(V) \end{tikzcd} \quad V \subseteq U$$

(1) $U = \bigcup_i V_i \qquad s \in (\mathrm{Ker}\, f)(U),\ s_{|V_i} = 0 \overset{?}{\Longrightarrow} s = 0$

$\hookrightarrow s \in \mathcal{F}(U) \Rightarrow s_{|V_i} = 0$ in $\mathcal{F}(V_i) \Rightarrow s = 0$ in $\mathcal{F}(U)$ //

(2) $U = \bigcup_i V_i, \quad s_i \in (\mathrm{Ker}\, f)(V_i) \hookrightarrow \mathcal{F}(V_i)$

$s_{i|V_i \cap V_j} = s_{j|V_i \cap V_j}$ also for $s_i \in \mathcal{F}(V_i) \Rightarrow \exists s \in \mathcal{F}(U) : s_{|V_i} = s_i$

Claim: $s \in \mathrm{Ker}\, f(U)$

$$\begin{tikzcd} s \in \mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d] & \mathcal{G}(U) \arrow[d] \ni f(U)(s) \\ s_i \in \mathcal{F}(V_i) \arrow[r, "f(V_i)"] & \mathcal{G}(V_i) \end{tikzcd}$$

Q: $f(U)(s) = 0$? $\qquad f(V_i)(s_i) = 0$ by def. of s_i $\square$

$f : \mathcal{F} \to \mathcal{G} \rightsquigarrow (\mathrm{Im}\, f)^{pre}(U) := \mathrm{Im}\, f(U)$ $\qquad \mathrm{Im}\, f$ is in general not a sheaf

attempted proof for „sheaf":

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, two heads, "f"] \arrow[d] & \mathrm{Im}\, f(U) \subseteq \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V_i) \arrow[r, "f"] & \mathrm{Im}\, f(V_i) \subseteq \mathcal{G}(V_i) \end{tikzcd}$$

$t \in \mathrm{Im}\, f(U)$

$\exists\ s_i \longmapsto t_i$, $\quad s_i \in \mathcal{F}(V_i)$

$\boxed{s_{i|V_i \cap V_j}} \overset{\text{need}}{=} \boxed{s_{j|V_i \cap V_j}}$

$t_{i|V_i \cap V_j} = t_{j|V_i \cap V_j}$

Let $t_i \in (\mathrm{Im}\, f)(V_i) \subseteq \mathcal{G}(V_i)$

$t_{i|V_i \cap V_j} = t_{j|V_i \cap V_j}$

Exponential sequence $\quad X = \mathbb{C}$, $\mathcal{O}^{an}$ = sheaf of analytic fcts = holomorphic fcts.

$\mathcal{O}^{an} \supseteq \mathcal{O}^*$ = sheaf of invertible fcts h inside $\mathcal{O}$ ($\frac{1}{h}$ is defined)

$\underline{\mathbb{Z}}$ = const. sheaf

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}^{an} \xrightarrow{\exp} \mathcal{O}^* \to 1$$

$\underline{\mathbb{Z}}$ = const. fct from $(U \subseteq \mathbb{C}) \to \mathbb{Z}$

$g \longmapsto \exp(g)\ (= e^g)$

$f \xmapsto{\cdot 2\pi i} 2\pi i \cdot f$

$f \longmapsto$ -„- $\longmapsto e^{2\pi i \cdot f} = 1$

$\mathbb{C}^* \subseteq \mathbb{C}$

$[z \overset{id}{\mapsto} z] \in \mathcal{O}^*(\mathbb{C}^*)$

$\hookrightarrow \nexists g \in \mathcal{O}^{an}(\mathbb{C}^*) \mapsto id$

$g \longmapsto 1$ on some open disk $U \subseteq \mathbb{C}$

$\Rightarrow g = $ integer $\cdot 2\pi i$.

$\Rightarrow$ this sequence is locally exact,

in particular, exp is locally surjective!

$\mathrm{Im}(\exp)$? $\qquad \forall$ open, bounded disks U: $\mathcal{O}^{an}(U) \twoheadrightarrow \mathcal{O}^*(U)$

assume: is a sheaf $\qquad \boxed{\mathrm{Im}(\exp) = \mathcal{O}^*}$

$\Rightarrow \cdot\ \mathrm{Im}(\exp) \subseteq \mathcal{O}^*$

$\Leftarrow \cdot\ \mathrm{Im}(\exp)(U) = \mathcal{O}^*(U) \quad \forall$ disks U

A = abelian grp $\rightsquigarrow \underset{\substack{\|\\ \mathcal{F}}}{\underline{A}^{pre}}(U) := A \quad \forall U$

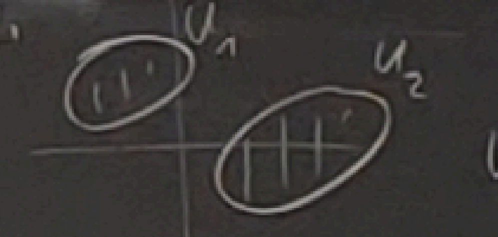
$\Downarrow$

$\underline{A}^{pre}$ is replaced by the „constant sheaf"

$\Downarrow$

$\underline{A}(U) := A^{\pi_0(U)}$ $\qquad \pi_0(U)$ = set of connected comp.s of U.

eg: $X = \mathbb{C}$, ordinary top.



$U = U_1 \cup U_2$

$\mathcal{F}(U_1) = A$
$\mathcal{F}(U_2) = A$
$\mathcal{F}(U) = A$
$\mathcal{F}(\emptyset) = 0$

$$\begin{tikzcd} & A = \mathcal{F}(U) \arrow[dl, "id"'] \arrow[dr, "id"] & \\ A = \mathcal{F}(U_1) & & \mathcal{F}(U_2) = A \end{tikzcd}$$

$a \in A = \mathcal{F}(U_1)$, $\quad a' \in \mathcal{F}(U_2)$

$a_{|U_1 \cap U_2} = a'_{|U_1 \cap U_2}$ $(= 0)$

$0 \to \mathbb{Z} \to \mathcal{O}^{an} \to \mathcal{O}^* \to 0$

$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact $\iff \forall P$ $\quad 0 \to \mathcal{F}_{1,P} \to \mathcal{F}_{2,P} \to \mathcal{F}_{3,P} \to 0$ is exact

$\Gamma(U, \cdot) : \mathcal{F} \longmapsto \Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ $\quad$ section functor

not exact, but left exact. $\rightsquigarrow \forall U \subseteq X$: $0 \to \underline{\mathbb{Z}}(U) \to \mathcal{O}^{an}(U) \to \mathcal{O}^*(U)$ exact

$0 \to H^0(U, \underline{\mathbb{Z}}) \to H^0(U, \mathcal{O}^{an}) \to H^0(U, \mathcal{O}^*) \to \boxed{H^1(U, \underline{\mathbb{Z}})} \to H^1(U, \mathcal{O}^{an})$

$H^1(U, \mathbb{Z})$

$U = \mathbb{C}^* \sim S^1$