Presheaves & sheaves

X = top. space

$\mathcal{F} \xrightarrow{f} \mathcal{G}$ hom. of presheaves of ab. groups $\rightsquigarrow$ $\ker f$, $\operatorname{im} f$, $\operatorname{coker} f$ …

$\mathcal{F} \subseteq \mathcal{G}$ — " — $\rightsquigarrow$ $\mathcal{G}/\mathcal{F}$

$\mathcal{F}, \mathcal{G}$ — " — $\rightsquigarrow$ $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \otimes \mathcal{G}$

<u>Sheaves</u> $\rightsquigarrow$ $\ker f$, $\mathcal{F} \oplus \mathcal{G}$ is still ok

<u>but</u>: $f : \mathcal{F} \to \mathcal{G}$: $\operatorname{im} f$ does not need to stay a sheaf!

<u>Example</u> ① $\widetilde{\mathbb{A}}^n \xrightarrow{\pi} \mathbb{A}^n$, $h : \widetilde{\mathbb{A}}^n \to \mathbb{P}^{n-1}$

$\rightsquigarrow$ sheaf of sections of h: $U \subseteq \mathbb{P}^{n-1}$ open $\rightsquigarrow$ look for $s : U \to \widehat{\mathbb{A}}^n$: $\pi \circ s = id_U$.

$\rightsquigarrow$ "$\mathcal{O}(-1)$"$(U) = \{ \text{ — " — } \}$ is a sheaf.

$V \subseteq U \Rightarrow$ need $\mathcal{O}(-1)(U) \to \mathcal{O}(-1)(V)$, $s \longmapsto s_{|V}$ ($U \to \widehat{\mathbb{A}}^n$, $V \to \widehat{\mathbb{A}}^n$)

$\rightsquigarrow$ general notation: $\mathcal{F}$ = (pre)sheaf $\rightsquigarrow$ $\mathcal{F}(U)$ = „<u>sections</u>" of $\mathcal{F}$.

② functions $f : X \to \mathbb{R}$ $\hat{=}$ sections of $X \times \mathbb{R} \xrightarrow{\pi} X$

$\mathcal{F}$ = (pre)sheaf $\rightsquigarrow$ $\forall P$ $\exists$ $\mathcal{F}_P := \varinjlim_{U \ni P} \mathcal{F}(U)$ $\rightsquigarrow$ $P \in U \subseteq X$: $\mathcal{F}(U) \longrightarrow \mathcal{F}_P$.

<u>Sheafification</u> $\quad \operatorname{PreSh}(X) \longrightarrow \operatorname{Sh}(X)$

X - top. space ; fix $\mathcal{U}$ = a basis of the topology, <u>i.e.</u> (i) $U \in \mathcal{U}$ are open subsets of X

(ii) $U \in X$ open $\Rightarrow \exists U_i \in \mathcal{U}$ $U = \bigcup_i U_i$

(ii)' $\forall P \in U \subseteq X$ (U = open) $\exists U' \in \mathcal{U}$: $P \in U' \subseteq U$

<u>Example</u>. $X = \operatorname{Spec} A$, $\mathcal{U} = \{ D(f) \mid f \in A \}$ ‖ $\operatorname{Proj} S$, $\mathcal{U} = \{ D_+(f) \mid \dots \}$

<u>Def</u>. $\cdot$ $\mathcal{F}$ = presheaf on $\mathcal{U}$ (values in $\mathcal{C}$) $:\Leftrightarrow$ $\mathcal{F} : \mathcal{U}^{op} \longrightarrow \mathcal{C}$

$\cdot$ $\mathcal{F}$ = sheaf on $\mathcal{U}$ $:\Leftrightarrow$ additionally : $\forall \underbrace{U, U_i}_{\in \mathcal{U}}$: $U = \bigcup U_i$:

- $s \in \mathcal{F}(U)$, $s_{|U_i} = 0$ $\forall i \Rightarrow s = 0$,
- $s_i \in \mathcal{F}(U_i)$, $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ $\Rightarrow \exists s \in \mathcal{F}(U) : s_{|U_i} = s_i$.

($\mathcal{F}(X)$ might not even exist if $X \notin \mathcal{U}$.)

$\operatorname{PreSh}(\mathcal{U}) \underset{\iota}{\overset{a}{\rightleftarrows}} \operatorname{Sh}(X)$ $\quad$ ($U \subseteq X$ open)

$\mathcal{F} \longmapsto a(\mathcal{F}) = \mathcal{F}^a$ is defined as $\mathcal{F}^a(U) := \left\{ s \in \prod_{P \in U} \mathcal{F}_P \;\middle|\; \text{such that } s \text{ comes } \underline{\text{locally}} \text{ from sections from } \mathcal{F} \text{ over } \mathcal{U} \right\}$

$\mathcal{G}_{|\mathcal{U}} \longleftarrow \mathcal{G}$

$\cdot$ $\forall p \in X$ $\mathcal{F}_p := \varinjlim_{\substack{U \ni p \\ U \in \mathcal{U}}} \mathcal{F}(U)$

$\cdot$ $U \in \mathcal{U}$ $\Rightarrow$ $\mathcal{F}(U) \xrightarrow{\varphi_{U,p}} \mathcal{F}_p$, $s \longmapsto s_p$

$(A_\alpha \longrightarrow \varinjlim A)$

$id \xrightarrow{\Phi} \iota \circ a$

i.e. $\cdot$ $\exists U = \bigcup U_i$, $U_i \in \mathcal{U}$

$\cdot$ $\exists s_i \in \mathcal{F}(U_i)$: $s(P) \in \mathcal{F}_P$ equals $(s_i)_P$.

~~Variant: $\exists t \in \mathcal{F}(U)$: $s(P) = t_P$ might be not allowed~~ : $U \in \mathcal{U}$?

$s_i \in \mathcal{F}(U_i) \ni s_j$, $(s_i)_P$, $\mathcal{F}_P$, $\mathcal{F}_Q$, $(s_j)_Q$, $s(P)$, $s(Q)$

$- s(P), s(Q)$
$- \exists \boxed{s_i \in \mathcal{F}(U_i)}$, $\mathcal{F}_P \ni s(P)$, $\mathcal{F}_Q \ni s(Q)$

<u>Claim</u>: (i) $\cdot$ $\mathcal{F}^a$ = sheaf on X

(ii) $\cdot$ $\mathcal{F} \xrightarrow{\Phi} (\mathcal{F}^a)_{|\mathcal{U}}$ is a (canonical) map of presheaves on $\mathcal{U}$

(iii) $\cdot$ $\forall P \in X$ $\rightsquigarrow$ $\Phi_P : \mathcal{F}_P \longrightarrow (\mathcal{F}^a)_P$ are isomorphisms

(iv) $\cdot$ $\mathcal{G}/X$ sheaf $\rightsquigarrow$ $(\mathcal{G}_{|\mathcal{U}})^a = \mathcal{G}$

$a \circ \iota = id_{\operatorname{Sh}(X)}$

(i) $\mathcal{F}^a$ is a presheaf : $V \subseteq U \subseteq X$ open $\quad \mathcal{F}^a(U) \longrightarrow \mathcal{F}^a(V)$, $s \longmapsto s_{|V}$ ✓

((locally) a section on $U_i \in \mathcal{U}$ $\Rightarrow$ ((locally) a section on $V_i = U_i \cap V$ $\quad$ <u>problem</u>: $U_i \cap V \in \mathcal{U}$? no - but: $\exists U_i \cap V = \bigcup_j W_j$, $W_j \in \mathcal{U}$.)

<u>Sheaf</u>: $U \subseteq X$, $U = \bigcup U_i$ ($U, U_i \notin \mathcal{U}$) [don't need]

$\cdot$ Let $s \in \mathcal{F}^a(U)$, $s_{|U_i} = 0$ $\forall i$ $\Rightarrow s = 0$

$\cdot$ $s_i \in \mathcal{F}^a(U_i)$; $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ $\Rightarrow$ glue to $s \in \prod_{P \in U} \mathcal{F}_P$

$\Downarrow$ comes from sections over $W_{ij} \in \mathcal{U}$, $\bigcup_j W_{ij} = U_i$

$\Rightarrow s$ comes from sect. over W_{ij} ; $\bigcup_{i,j} W_{ij} = \bigcup_i U_i = U$.

$(a, b, c) \in \mathbb{R}^3$ $\;\|\;$ $(c, d, e) \in \mathbb{R}^3$ $\;\Downarrow\;$ (a, b, c, d, e)

(ii) $\Phi : \mathcal{F} \longrightarrow \mathcal{F}^a$

$\mathcal{F}(U) \longrightarrow \mathcal{F}^a(U)$

$s \longmapsto (s_P)_{P \in U}$, $s_P \in \mathcal{F}_P$

$\Phi_P : \mathcal{F}_P \longrightarrow (\mathcal{F}^a)_P$ <u>surjective</u>: let $x \in (\mathcal{F}^a)_P \Rightarrow \exists \overset{P}{\overset{\cap}{U}} \subseteq X$: x comes from some $s \in \underline{\mathcal{F}^a(U)}$ ($U \in \mathcal{U}$ if it helps)

even better: we may assume that $s \in \mathcal{F}^a(U)$ comes from some $s \in \mathcal{F}(U)$, $U \in \mathcal{U}$.

(ii) $\Phi: \mathcal{F} \to (\mathcal{F}^a)|_{\mathcal{U}}$; claim Φ_P is surjective:

let $x \in \mathcal{F}^a_P \rightsquigarrow \exists\, U \in \mathcal{U}$ ($U \ni P$) : x comes via $s \in \mathcal{F}(U) \to \mathcal{F}^a(U)$

$$s_P \in \mathcal{F}_P \xrightarrow{\Phi_P} \mathcal{F}^a_P \ni x$$

claim: Φ_P is injective.

Assume $\mathcal{F}_P \to \mathcal{F}^a_P$, $s \mapsto 0$

$\Rightarrow \exists\, U \in \mathcal{U}: \mathcal{F}(U) \to \mathcal{F}^a(U)$, $s \mapsto$ is zero in the stalk over P

shrink U further $\Rightarrow$ w.l.o.g.: $s \mapsto 0 \in \mathcal{F}^a(U) = \{ \prod_{Q\in U} \mathcal{F}_Q \mid \text{locally from } \mathcal{F} \}$, $s \in \mathcal{F}(U)$

$\Rightarrow s_Q = 0 \ \forall Q \in U$ $\quad \mathcal{F}(U) \to \mathcal{F}_Q$, $s \mapsto s_Q$

and P is one of these points $Q \Rightarrow s_P = 0$

Recall: $\mathcal{F} \xrightarrow{f} \mathcal{G}$ hom. of sheaves

$f = \text{ISO} \iff \forall P \in X: f_P: \mathcal{F}_P \to \mathcal{G}_P$ is an isom!

(iii) ✓ (iv) $\mathcal{G}|X$ is a sheaf.

$(\mathcal{G}|_{\mathcal{U}})^a$: $(\mathcal{G}|_{\mathcal{U}})_P = \mathcal{G}_P$ $(\forall P \in X)$

$(\mathcal{G}|_{\mathcal{U}})^a(U) = \{ s \in \prod_{P\in U} \mathcal{G}_P \mid \text{locally it comes from } \mathcal{G}\text{-sections} \}$

$= \mathcal{G}(U)$

i.e. $\exists\, U_i \in \mathcal{U}: \bigcup U_i = U$; s comes from $s_i \in \mathcal{G}(U_i)$ $\quad s_i, s_j$ have the same stalks in $P \in U_i \cap U_j$;

$\Rightarrow s_i|_{U_i\cap U_j} = s_j|_{U_i \cap U_j}$ (because $\mathcal{G}$ = sheaf) $\Rightarrow$ glue $\Rightarrow \exists\, s \in \mathcal{G}(U)$ $s_P = s(P)$.

Corollary. $a \dashv \iota$ $\quad$ (iii) $\iota \circ a \leftarrow \mathrm{id}_{\mathrm{Pre}(\mathcal{U})}$ ISO on the stalks; (iv) $a \circ \iota = \mathrm{id}_{\mathrm{Sh}(X)}$

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}^a, \mathcal{G}) = \mathrm{Hom}_{\mathrm{Pre}(\mathcal{U})}(\mathcal{F}, \underbrace{\mathcal{G}|_{\mathcal{U}}}_{=\iota(\mathcal{G})})$$

$\mathcal{F} \in \mathrm{PreSh}_{\mathcal{U}}$, $\mathcal{G} \in \mathrm{Sh}(X)$

$\longleftarrow$ i.e. $\longrightarrow$

$\mathcal{F} \xrightarrow{f} \mathcal{G}$ (= sheaf on X), $\Phi: \mathcal{F} \to \mathcal{F}^a$, $\mathcal{F}^a \overset{?}{\dashrightarrow} \mathcal{G}$

$f_P: \mathcal{F}_P \to \mathcal{G}_P$; $\mathcal{F}_P = (\mathcal{F}^a)_P \to \mathcal{G}_P$

$(I,<)$ poset $\rightsquigarrow \varinjlim_I A.$

$(I',<)$ restriction $\rightsquigarrow \varinjlim_{I'} A.$

cofinal: $\forall a \in I\ \exists\, b \in I': a \leq b. \Rightarrow \psi = \text{ISO}$.

A_a ($a \in I'$) $\to \varinjlim_I A. = B$; ψ canonical; $\varinjlim_{I'} A. \leftarrow A_c$ ($a \in I'$)

2 main special cases ① $\mathcal{U} = \{\text{open subsets of } X\} \Rightarrow \mathrm{PreSh}(X) \underset{\iota}{\overset{a}{\rightleftarrows}} \mathrm{Sh}(X)$

② $\mathcal{U}$ = true subset of $\mathrm{Open}(X)$, s.t. $\mathcal{F}$ = sheaf on $\mathcal{U}$ (Subbasis of topology)

claim: $(\mathcal{F}^a)|_{\mathcal{U}} = \mathcal{F}$ $\quad$ Proof: $\mathcal{F} \xrightarrow{\Phi} (\mathcal{F}^a)|_{\mathcal{U}}$, $\Phi_P = \text{ISO}$ (ii + iii)

$\mathcal{F} \xrightarrow{\Phi} \mathcal{G}$ map of sheaves on $\mathcal{U}$; $\Phi_P = \text{iso}\ \forall P \Rightarrow \Phi = \text{isom.}$ □

Application. $X = \operatorname{Spec} A$; consider $M = A$-module (e.g. $M = A$)

$M \rightsquigarrow \boxed{\text{sheaf } \tilde{M} \text{ on } X}$ $\quad$ (e.g. $\tilde{A}$ on X, $= \mathcal{O}_A = \mathcal{O}_X = \mathcal{O}_{\operatorname{Spec} A}$)

$\mathcal{U} = \{ D(f) \mid f \in A \}$ $\quad \boxed{D(fg) \subseteq D(f)}$ $f,g \in A$

$\tilde{M}(D(f)) := M_f$.

Presheaf: $\boxed{D(g) \subseteq D(f)} \longrightarrow D(g) = D(g) \cap D(f) = D(fg)$; $g \notin P$; $\{P \in \operatorname{Spec} A \mid f \notin P\}$

$\tilde{M}(D(f)) \to \tilde{M}(D(fg))$; $M_f \to M_{fg}$

Remark: $\tilde{M}_P = \varinjlim_{f \notin P} \tilde{M}(D(f)) = \varinjlim M_f = M_P$

Claim. $\tilde{M}$ is a sheaf on $\mathcal{U}$.

Proof. • Let $U \in \mathcal{U}$, i.e. $U = D(f) = \operatorname{Spec} A_f$, w.l.o.g.: $U = \operatorname{Spec} A$

$s \in \tilde{M}(\operatorname{Spec} A = D(1)) = M$, let $s_P = 0\ \forall P \in \operatorname{Spec} A \overset{(1)}{\Longrightarrow} s = 0$.

Reason: $m \in M$, $\frac{m}{1} = 0$ in all $M_P \Longrightarrow m = 0$. $\quad$ ($\mathrm{Ann}(m) \subseteq A$ ideal; $m \neq 0 \Rightarrow \mathrm{Ann}(m) \neq (1)$ $\Rightarrow \exists P: \mathrm{Ann} \subseteq P$ max ideal in A_P)

• $\bigcup D(f_i) = \operatorname{Spec} A$, i.e. $(f_i \mid i \in I) = (1)$

$\frac{m_i}{f_i} \in M_{f_i}$ — assume that they glue. $\frac{m_i}{f_i} \mapsto\ \in M_{f_i f_j}$, $\frac{m_j}{f_j}$ ✓ $\overset{(2)}{\Longrightarrow} \exists\, m \in M: \frac{m}{1} = \frac{m_i}{f_i}$ in M_{f_i}

avoid powers: $D(f) = D(f^n)$

$\frac{m_i}{f_i} = \frac{m_j}{f_j}$ in $M_{f_i f_j}$ $\Leftrightarrow$ $(f_i f_j)^N (m_i f_j - m_j f_i) = 0$ in M for some N

$\Updownarrow$

$\boxed{m_i f_i^N f_j^{N+1} = m_j f_i^{N+1} f_j^N}$

$(f_i) = (1)$

$D(f_i) = D(f_i^N)$

$\Rightarrow (f_i^{N+1}, i \in I) = (1) \Rightarrow \exists\, l_j \in A : 1 = \sum_j l_j \cdot f_j^{N+1}$

<u>Ansatz</u>: $m = \sum_j l_j m_j f_j^N \in M$

<u>Check</u>: What is $\frac{m}{1} \in M_{f_i}$?

$$\frac{m}{1} = \sum_j l_j f_j^N \cdot \frac{m_j}{1} = \frac{\sum_j l_j \boxed{f_j^N \cdot m_j \cdot f_i^{N+1}}}{f_i^{N+1}}$$

$$= \sum_j l_j \frac{m_i \cdot f_i^N \cdot f_j^{N+1}}{f_i^{N+1}} = \frac{m_i}{f_i} \underbrace{\sum_j l_j f_j^{N+1}}_{=1} = \frac{m_i}{f_i} \text{ in } M_{f_i}$$

<u>Remark</u>:
- $\tilde{M}$ = sheaf on Spec A
- $\tilde{M}(D(f)) = M_f$
- $\widetilde{M \oplus N} = \tilde{M} \oplus \tilde{N}$ (because $(M \oplus N)_f = M_f \oplus N_f$)

$\widetilde{M \otimes_A N} = \tilde{M} \otimes_{\tilde{A}} \tilde{N}$

$\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G} := (\mathcal{F} \otimes^{pre}_{\mathcal{R}} \mathcal{G})^a$

$\mathcal{R}$ = ring sheaf

$\mathcal{F}, \mathcal{G}$ = $\mathcal{R}$-modules, i.e. sheaves of <u>AS</u>, $\mathcal{R}$-str.

$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U)$

$(M \otimes_A N)_f = M_f \otimes_{A_f} N_f$

$\qquad = \tilde{M}(D(f)) \otimes N(D(f))$

$\Downarrow$

$\widetilde{(M \otimes_A N)}(D(f)) = (\tilde{M} \otimes^{pre} \tilde{N})(D(f))$

$\Longleftarrow$

$\uparrow$ is already a sheaf on $\{D(f)\} = \mathcal{U}$

It might happen that $(\tilde{M} \otimes^{pre} \tilde{N})(U) \neq (\tilde{M} \otimes \tilde{N})(U)$ for $U \notin \mathcal{U}$.

$D(fg) \hookrightarrow D(f)$

$\mathrm{Hom}_A(M, N) \xrightarrow{\text{Prop Alg I}}$
- $S \subset A$ mult. closed
- M = finitely presented

$\Big\}\ S^{-1}\mathrm{Hom}_A(M,N) = \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$

$\Downarrow$

$\mathrm{Hom}_A(M,N)_f = \mathrm{Hom}_{A_f}(M_f, N_f)$

$\Rightarrow \widetilde{\mathrm{Hom}_A(M,N)} = \mathrm{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N})$

$\mathcal{F}_1, \dots, \mathcal{F}_n$ = sheafs $\rightsquigarrow$ operations

<u>instead</u>:
- do only the presheaf-operations
- at the end: sheafify once!

<u>Predef.</u> Sheaves of $\mathcal{O}_A$-modules which are of the form $\tilde{M}$ (for some A-module M) are called quasi coherent sheaves on Spec A.

$X = \mathrm{Proj}\ S$ $\rightsquigarrow$ <u>goal</u>: construct $\tilde{S}$, <u>or</u>: M = graded S-module $\rightsquigarrow \tilde{M}$

S = $\mathbb{N}$-graded ring

$\mathcal{U} := \{D_+(f) \mid f \in S \text{ homog.}, \deg \geq 1\} \rightsquigarrow X \notin \mathcal{U}$

$D_+(f) = \mathrm{Spec}\ S_{(f)}$

$\hookrightarrow$ otherwise: $\exists f \in S_d$: all $P \subseteq S$ homog. PI with $P \not\supseteq S_1$ do not contain f!

<u>BUT</u>: (f) = homog. ideal, $(f) \neq (1)$

$\Downarrow$

$\exists$ homog. max ideal : $(f) \subset m$.

<u>Ex</u>: $S = k[z_0] \doteq \mathbb{P}^0$

$\longrightarrow \mathrm{Proj}\ S = \mathbb{P}^0 = D_+(z_0)$

- global sections of $\tilde{S}$ = constants
- sections over $D_+(f)$ = $S_{(f)}$

- $\tilde{M}(D_+(f)) = M_{(f)}$ $\rightsquigarrow$ sheaf (as before)
- $\tilde{M}_{|D_+(f)}$ = sheaf on $D_+(f) = \mathrm{Spec}\ S_{(f)}$ $\Rightarrow \tilde{M}_{|D_+(f)} = \boxed{\tilde{M}_{(f)}}$ ($S_{(f)}$-module) on Spec A

$S_{(f)} = A$, $M_{(f)}$ = A-module.

f, g = homog. (of deg 1)

Proj S

$M_{(f)}$

$M_{(fg)} = (M_{(f)})_{\frac{g}{f}}$

<u>Ex</u>: $S = k[x, y]$

$S_{(x)} = k[\frac{y}{x}]$ $\qquad x \cdot \frac{y^k}{x^k} = \frac{y^k}{x^{k-1}}$

$S(1)_{(x)} = [x, y, \frac{y^k}{x^{k-1}}]$

$\qquad = k[\frac{y}{x}]$-module generated by x, y

$S(l) = S$-module

$\Rightarrow S(l)_{(f)} = S_{(f)}$-module

S = graded ring $\rightsquigarrow S = S$-module.

$\rightsquigarrow \forall l \in \mathbb{Z}$: $S(l) = S$-module (graded)

$S(l)_d = S_{l+d}$

$S(l)_{(f)} = \{\frac{a}{f^k} \mid a \in S \text{ homog.}, k \in \mathbb{N}, \deg \frac{a}{f^k} = l\}$

$(\deg f = d)$

$\rightsquigarrow \widetilde{S(l)} = \mathcal{O}(l) \longrightarrow \mathrm{Proj}\ S$

$\tilde{S} = \mathcal{O}_{\mathrm{Proj}\ S}$

$\hat{A}^2 \longrightarrow \mathbb{P}^1$

$\rightsquigarrow \mathcal{O}(-1)$ = sh of sections