$\mathcal{F} \rightsquigarrow \mathcal{F}^a$ $\quad \exists\ \mathcal{F} \to \mathcal{F}^a$, $\ \mathcal{F}_p \xrightarrow{\sim} \mathcal{F}^a_p$

$\mathcal{F} = \mathrm{PreSh}$, $\mathcal{G} = \mathrm{Sh}$

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ \mathcal{F}^a \arrow[ur, dashed, "\varphi^a"'] & \end{tikzcd}$$

$$(\mathcal{F} \otimes^{pre}_R \mathcal{G})(U) := \mathcal{F}(U) \otimes_{R(U)} \mathcal{G}(U) \rightsquigarrow (\mathcal{F} \otimes_R \mathcal{G}) = (\mathcal{F} \otimes^{pre}_R \mathcal{G})^a$$

$\mathcal{A}b \rightsquigarrow \underline{\mathrm{Pre}}\mathrm{Sh}(\mathcal{A}) \supseteq \mathrm{Sh}(\mathcal{A}b)$

$\|$ abelian category:

$\mathrm{Ker}^{pre}(\mathcal{F} \xrightarrow{f} \mathcal{G})(U) = \mathrm{Ker}(\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U))$

$\mathcal{F}/_{\mathcal{G}}(U) := \mathcal{F}(U)/\mathcal{G}(U)$ $\quad \mathrm{Im}^{pre} \quad \mathrm{Im}$

$\mathrm{Coker}^{pre} \quad \mathrm{Coker}$

$U \subseteq X$ open $\rightsquigarrow$ functor $\Gamma(U, .) : \mathrm{PreSh} \longrightarrow \mathcal{A}b$ $\quad$ exact by def.

$\mathcal{F} \longmapsto \Gamma(U, \mathcal{F}) := \mathcal{F}(U)$

$P \in X \quad \rightsquigarrow$ -- " -- (stalk in P) : $\mathcal{F} \longmapsto \mathcal{F}_p$ $\quad$ exact

Sheaves: $\mathcal{F} \xrightarrow{f} \mathcal{G} \rightsquigarrow \mathrm{Im}, \mathrm{coker}, \mathcal{F}_1/\mathcal{F}_2$ don't need to stay sheaves $\rightsquigarrow$ apply $(\mathcal{F} \mapsto \mathcal{F}^a)$ afterwards.

$\mathrm{Ker}(f) = \mathrm{Ker}^{pre}(f)$

$\mathrm{Im}(f) := (\mathrm{Im}^{pre}(f))^a$, same with coker, ...

$(\mathcal{O} \xrightarrow{\exp} \mathcal{O}^*)$

Def. $f : \mathcal{F} \to \mathcal{G}$ is inj. $\iff \mathrm{Ker}\, f = 0$

is surj $\iff \mathrm{Coker}\, f = 0$

Prop. $f\ \mathcal{F} \to \mathcal{G} \Rightarrow$

- $\mathrm{Ker}\, f \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G}$

$\exists!$

- $K \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G}$

$U \subseteq X$

- $\mathcal{F} \xrightarrow{f} \mathcal{G} \longrightarrow \mathrm{coker}\, f$

$\|\quad\|\quad \exists!$

$\mathcal{F} \to \mathcal{G} \to C$

0

Proof: $U \subseteq X$ open $\ \mathrm{Ker}[f(U)] \hookrightarrow \mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)$

$\Rightarrow$ is 0.

$\forall U.\ \ K(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \Rightarrow K \to \mathrm{Ker}\, f$

$\mathcal{F} \to \mathcal{G} \to \mathrm{coker}^{pre} f \xrightarrow{a} \mathrm{coker}\, f$

$\Rightarrow \quad 0 \quad 0 \quad 0$

$\mathcal{F} \to \mathcal{G} \to C$

$\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \longrightarrow \mathrm{coker}\, f(U) = \mathcal{G}(U)/\mathrm{im}\, f(U)$ $\quad \mathcal{F}(U) \to \mathcal{G}(U) \to C(U)$

0

Prop. $\mathcal{F} \xrightarrow{f} \mathcal{G}$ morph. of sheaves.

1) $K \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$ $\qquad K = \mathrm{Ker}\, f \iff \forall P.\ K_p = \mathrm{Ker}(\mathcal{F}_p \to \mathcal{G}_p)$, i.e. $0 \to K_p \to \mathcal{F}_p \to \mathcal{G}_p$ is exact.

stalk $\Downarrow \quad \downarrow$

$\underline{\forall P}$: $K_p \to \mathcal{F}_p \to \mathcal{G}_p$

2) $\mathcal{F} \xrightarrow{f} \mathcal{G} \to C$, $\ C = \mathrm{Coker}\, f \iff \forall P : \mathcal{F}_p \to \mathcal{G}_p \to C_p \to 0$ is exact.

$\mathrm{coker}\, f \qquad (\mathrm{coker}^{pre} f)^a$

Remark. this is wrong for presheaves!

Proof (2): $(\Rightarrow)$ $\cdot\ \mathcal{F} \to \mathcal{G} \to C$ is $0 \Rightarrow \mathcal{F}_p \to \mathcal{G}_p \to C_p$ is 0.

$\cdot\ C = (\mathrm{coker}^{pre} f)^a \Rightarrow \mathcal{F}_p \to \mathcal{G}_p \to (\mathrm{coker}^{pre} f)_p \to 0$ exact

$\| \ C_p$

$(\Leftarrow)$ $\mathcal{F} \to \mathcal{G} \to C$, $\underline{\text{Know}}$. $\forall P : \mathcal{F}_p \to \mathcal{G}_p \to C_p \to 0$ exact

$= 0$ on all stalks ... ? $\qquad 0$

$\mathcal{F} \longmapsto \Gamma(U, \mathcal{F})$

no longer exact!

$\underline{\text{but}}$: it is $\boxed{\text{left exact!}}$

$\mathrm{Ker}^{pre} = \mathrm{Ker}$

Assume $\mathcal{F} \xrightarrow{f} \mathcal{G} \longrightarrow C'$ is zero

look for

$(\mathrm{coker}\, f) \overset{\sim}{\dashrightarrow} C$ $\quad$ (2)

(we know only $\ \mathcal{F}_p \to \mathcal{G}_p \to C_p \to 0$ exact $\forall P$)

$\| \quad \| \quad \uparrow ? \quad$ (5-Lemma)

$\mathcal{F}_p \to \mathcal{G}_p \to (\mathrm{coker}\, f)_p \to 0$ exact

$(\mathcal{F} \xrightarrow{f} \mathcal{G}$ hom. of sheaves, $\mathcal{F}_p \xrightarrow{\sim} \mathcal{G}_p\ \forall P \Rightarrow f = \mathrm{isom})$

$\square$

Corollary: $\cdot\ \mathrm{Sh}_X(\mathcal{A}b)$ turn into an abelian category

$\cdot\ \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$ complex $\iff \forall P \in X : \mathcal{F}_p \to \mathcal{G}_p \to \mathcal{H}_p$ complex

exact $\iff$ exact

$$\left(\mathrm{Ker}(\mathcal{G} \to \mathcal{H}) / \mathrm{Im}(\mathcal{F} \to \mathcal{G})\right)_p = \mathrm{Ker}(\mathcal{G}_p \to \mathcal{H}_p) / \mathrm{Im}(\mathcal{F}_p \to \mathcal{G}_p)$$

$(\mathcal{F} \otimes \mathcal{G})_p = \mathcal{F}_p \otimes \mathcal{G}_p \rightsquigarrow \otimes_R \mathcal{G}$ is right exact:

$(R =$ ring sheaf, $\mathcal{G} = R$-module$)$

$0 \to (\mathcal{F}_1)_p \to (\mathcal{F}_2)_p \to (\mathcal{F}_3)_p \to 0 \ |\ \otimes \mathcal{G}_p$

$(\mathcal{F}_1 \otimes \mathcal{G})_p \to (\mathcal{F}_2 \otimes \mathcal{G})_p \to (\mathcal{F}_3 \otimes \mathcal{G})_p \to 0$

$\mathcal{F}_{1p} \otimes \mathcal{G}_p \to \mathcal{F}_{2p} \otimes \mathcal{G}_p$

Prop: $U \subseteq X$ open $\rightsquigarrow \Gamma(U,\cdot): Sh_X(AS) \longrightarrow AS$ is left exact.

Proof. Let $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$ be an exact sequence of sheaves (i.e. $\forall P$: $0 \to \mathcal{F}_P \xrightarrow{\varphi} \mathcal{G}_P \xrightarrow{\psi} \mathcal{H}_P \to 0$ exact)

· $\operatorname{Ker}\varphi = 0$ $\quad (\operatorname{Ker}\varphi)_P = \operatorname{Ker}\varphi_P = 0 \ \forall P$ $\quad$ 0!

· $\mathcal{F} \to \mathcal{G}$ is injective on all $U \subseteq X$

$\operatorname{Ker}(\mathcal{F}(U) \to \mathcal{G}(U)) \underset{\text{Def.}}{=} \operatorname{Ker}^{pre}(\mathcal{F} \to \mathcal{G})(U) = \operatorname{Ker}(\mathcal{F} \to \mathcal{G})(U)$

$0 \to \boxed{\mathcal{F} \overset{\varphi}{\hookrightarrow} \mathcal{G} \to \boxed{\operatorname{coKer}^{pre}\varphi}} \to \operatorname{coker}\varphi \to 0$ exact $\quad$ 0 is ok $\quad s \mapsto 0$

$\Downarrow$ $\qquad \Rightarrow \mathcal{H}$ #

$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to (\operatorname{coKer}^{pre}\varphi)(U) \to 0$ exact

Goal: $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ exact.

$\forall P \ \exists \ f_P \longmapsto s \longmapsto t \quad t_P = 0 \ \forall P$

$\mathcal{F} \to \mathcal{F}^{\#}$ and ...

Now: $\mathcal{F} \subseteq \mathcal{G}$ sheaves; $s \in \mathcal{G}(U)$ s.t. $\forall P$: $s_P \in \mathcal{F}_P \subseteq \mathcal{G}_P$ $\qquad$ **Goal:** $s \in \mathcal{F}(U)$.

$s_P \in \mathcal{F}_P \rightsquigarrow \exists\, U(p),\ \exists\, t \in \mathcal{F}(U(p)) \subseteq \mathcal{G}(U(p))$: $t_p = s_p$ [open neighborhood of p] $\Rightarrow \exists\, V(p) \subseteq U(p)$: $t = s$ on $V(p)$

$\longrightarrow s \in \mathcal{G}(U)$ has the property: $\boxed{s|_{V(p)} \in \mathcal{F}(V(p))} \subseteq \mathcal{G}(V(p))$

$\boxed{\mathcal{G} \longmapsto \mathcal{G}^a}$

↳ glue to some $t \in \mathcal{F}(U) \Rightarrow t|_{V(p)} = s|_{V(p)}$ "$\mathcal{F} \Rightarrow \mathcal{G}$"

$\Rightarrow t = s$ in $\mathcal{G}(U)$, $t \in \mathcal{F}(U)$

$PreSh_X \xrightarrow{a} Sh_X$

$PreSh_X \xleftarrow{\iota} Sh_X$

$\mathcal{F} \longleftarrow \mathcal{F}$

· $\mathcal{G} \longmapsto \mathcal{G}^a$ is exact. ($PreSh \to Sh_a$)

· $\mathcal{F} \longmapsto \iota(\mathcal{F}) = \mathcal{F}$ ($Sh \to PreSh$)

$[\mathcal{O} \xrightarrow{\exp} \mathcal{O}^*] \longmapsto [\mathcal{O} \xrightarrow{\exp} \mathcal{O}^*]$ $\qquad \iota$ = left exact!

$\operatorname{Hom}_{Sh}(\mathcal{F}^a, \mathcal{G}) = \operatorname{Hom}_{Pre}(\mathcal{F}, \iota\mathcal{G})$

$a \dashv \iota$

$f: X \longrightarrow Y$ (U, V; $\mathcal{F}$, $\mathcal{G}$) continuous map between top. spaces.

① $f_*\mathcal{F}$ = sheaf on Y: $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ is a sheaf!!

$\mathcal{G}|P \longleftrightarrow \mathcal{G}(P)$ = abel. grp.

Example Y = point P. $\quad f: X \to P \quad (\mathcal{G}(\emptyset) = 0$ anyway$)$

$\mathcal{F} \rightsquigarrow f_*\mathcal{F}$ = sheaf on P

$\hat{=}$ abelian grp $(f_*\mathcal{F})(P) = \mathcal{F}(X) = \Gamma(X,\mathcal{F})$

$\Rightarrow \boxed{f_* = \Gamma(X,\cdot)}$

In general ($\forall f: X \to Y$) · f_* is left exact.

· f_* is covariant: $\mathcal{F} \to \mathcal{F}' \Rightarrow f_*\mathcal{F} \to f_*\mathcal{F}'$

② $\mathcal{G} \rightsquigarrow f^{-1}\mathcal{G}$ · $(f^{-1}\mathcal{G})(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$ for $V \supseteq f(U)$ open (pre)

$(f^{-1}\mathcal{G})$ · sheafify afterwards.

Example $f: P \hookrightarrow Y$, $\mathcal{G}$ = sheaf: $(f^{-1}\mathcal{G}) = \mathcal{G}_P$.

---

pre sheaf $\quad U \subseteq U' \subseteq X$

$\boxed{\varinjlim_{V \supseteq f(U')} \mathcal{G}(V)} \to \boxed{\varinjlim_{V \supseteq f(U)} \mathcal{G}(V)}$ $\quad \mathcal{G}(V)$

We should know how the stalks behave under f_*, f^{-1}.

· $(f_*\mathcal{F})_{f(x)} \ne \mathcal{F}_x$?? $\quad \nexists$ no guess $\Rightarrow$ no result.

· $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)} \Longrightarrow f^{-1}$-functor is exact

$(\mathcal{G}^1 \to \mathcal{G}^2 \to \mathcal{G}^3$ exact on Y

$(f^{-1}\mathcal{G}^1)_x \to (f^{-1}\mathcal{G}^2)_x \to (f^{-1}\mathcal{G}^3)_x$ exact on X ?

Proof: $x \overset{g}{\hookrightarrow} X \xrightarrow{f} Y$ $\qquad (f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$\Rightarrow (f^{-1}\mathcal{G})_x = g^{-1}(f^{-1}\mathcal{G}) = (f \circ g)^{-1}(\mathcal{G}) = \mathcal{G}_{f(x)}$.

$X \xrightarrow{f} Y \xrightarrow{g} Z$

$U \quad V \supseteq f(U) \quad W \supseteq g(V) \supseteq (gf)(U)$ $\qquad \varinjlim \quad \varinjlim$

$\mathcal{F}_P = \varinjlim_{U \ni P} \mathcal{F}(U) \ \Big\| \ \varinjlim_{D(s) \ni P} \mathcal{F}(D(s))$

$f^{-1} \dashv f_*$ i.e. $\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$

$f: X \to Y$, $\mathcal{F}$ on X, $\mathcal{G}$ on Y

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

$\forall U \quad f^{-1}\mathcal{G}(U) \to \mathcal{F}(U)$, where $f^{-1}\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, so $\varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$

for $\boxed{V \supseteq f(U): \ \mathcal{G}(V) \longrightarrow \mathcal{F}(U) \text{ compatible}}$

$\forall V: \ \mathcal{G}(V) \to f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$

$\forall V: \ \boxed{\forall U \subseteq f^{-1}V: \ \mathcal{G}(V) \to \mathcal{F}(U)}$

Def. • Ringed space $(X, \mathcal{O}_X)$: • X = top space
• $\mathcal{O}_X$ = sheaf of rings on X.

($U \subseteq X \rightsquigarrow \mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X)$ = ring)

example. • $X = \mathrm{Spec}\, A$

$\mathcal{O}_X = \mathcal{O}_{\mathrm{Spec}\, A} = \mathcal{O}_A = \widetilde{A}$ (on $D(f)$: A_f)

• Locally ringed space $(X, \mathcal{O}_X)$ such that $\forall x \in X$: $\mathcal{O}_{X,x}$ = local ring ($\mathfrak{m}_{X,x}$ = max. ideal)

Def. • $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is • $f: X \to Y$ continuous map.
• $\mathcal{O}_Y \xrightarrow{f^\#} f_*\mathcal{O}_X$ or $f^{-1}\mathcal{O}_Y \xrightarrow{f^\#} \mathcal{O}_X$ (both give the same information!)

• locally ringed space: $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$

$x \in X \Rightarrow (f^{-1}\mathcal{O}_Y)_x \longrightarrow \mathcal{O}_{X,x}$, $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)} \supseteq \mathfrak{m}_{Y,f(x)} \to \mathfrak{m}_{X,x}$

$\hookrightarrow$ hom should impose local homomorphism among the stalks.

Def. $(A, \mathfrak{m}) \xrightarrow{\varphi} (B, \mathfrak{n})$ (both are local rings)

$\varphi(\mathfrak{m}) \subseteq \varphi(\mathfrak{n})$ here (do not pose)

$\varphi(A \setminus \mathfrak{m}) \subseteq \varphi(B \setminus \mathfrak{n})$ always

„φ = local homomorphism" (Ex $k[x]_{(x)} \hookleftarrow k(x)$, (x), (0))

$g \in \mathcal{O}_{X,x} \rightsquigarrow$ value of g in x? $g(x) \in$ field, $g(x) = \bar{g} \in k$, $k = \mathcal{O}_{X,x}/\mathfrak{m}_x$

$g(x) = 0 \iff g \in \mathfrak{m}_{X,x}$. $X \xrightarrow{f} Y$

$x \longmapsto f(x)$

$f^*(g) = g \circ f \longleftarrow g$, $g(f(x)) = 0$

$(g \circ f)(x) = 0$

(1)

Example • A = ring $\rightsquigarrow (\mathrm{Spec}\, A, \mathcal{O}_A)$ = locally ringed space: $P \in \mathrm{Spec}\, A$

$(\mathrm{Spec}\, A, A)$ $(\mathcal{O}_A(\mathrm{Spec}\, A) = A)$

$\Downarrow$

$\mathcal{O}_{A,P} = (\mathcal{O}_A)_P = A_P$

$PA_P = \mathfrak{m}_P \subseteq A_P$

• $A \xrightarrow{\varphi} B \to \mathrm{Spec}\, B \xrightarrow{f} \mathrm{Spec}\, A$, $f = \mathrm{Spec}\, \varphi$

and $\boxed{\widehat{A} \to f_*\widehat{B}}$, $\widehat{A} = \mathcal{O}_A$, $\widehat{B} = \mathcal{O}_B$

$f^{-1}\widehat{A} \to \widehat{B}$

on $\mathrm{Spec}\, A$: base of top'gy $D(a)$, $a \in A$

$\widehat{A}(D(a)) \longrightarrow f_*\widehat{B}(D(a))$

$\varphi: A \to B \rightsquigarrow \Big[\ \widehat{A}(D(a)) = A_a \qquad \widehat{B}(f^{-1}D(a)) = B_{\varphi(a)}\ \Big]$, $f^{-1}D(a) = D(\varphi(a))$

$\frac{a'}{a^n} \longmapsto \frac{\varphi(a')}{\varphi(a)^n}$

$\Rightarrow$ functor: image = „affine schemes"

$\mathrm{Rings}^{op} \xrightarrow{\Phi}$ Locally ringed spaces

$A \longmapsto (\mathrm{Spec}\, A, \mathcal{O}_A)$

(A, B), $\varphi: A \to B$, $A \longmapsto$ Aff Schemes $(\mathrm{Spec}\, A, A)$

Claim. Φ = fully faithful (ISO on Hom-spaces)

$M = A$-module $\rightsquigarrow \widetilde{M} / \mathrm{Spec}\, A$

$\mathcal{F} / \mathrm{Spec}\, A$ with $\mathcal{F}(\mathrm{Spec}\, A) = M \overset{?}{\Longrightarrow} \mathcal{F} = \widetilde{M}$ NO

Proof. RHS: $(X, \mathcal{O}_X) = (\mathrm{Spec}\, A, \mathcal{O}_A)$, $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \rightsquigarrow$ Q: why does this come from $B \xrightarrow{\varphi} A$?

$(Y, \mathcal{O}_Y) = (\mathrm{Spec}\, B, \mathcal{O}_B)$

Rings $\rightsquigarrow$ [l.r.sp.] $\longrightarrow$ Rings

$A \longrightarrow \boxed{(\mathrm{Spec}\, A, \mathcal{O}_A)} \longmapsto A$

$(X, \mathcal{O}_X) \longmapsto \Gamma(X, \mathcal{O}_X)$

Assume: $f: (\mathrm{Spec}\, B, \mathcal{O}_B) \longrightarrow (\mathrm{Spec}\, A, \mathcal{O}_A)$ map of loc. r. sp.

• $\boxed{f: \mathrm{Spec}\, B \to \mathrm{Spec}\, A}$

• $\mathcal{O}_A \to f_*\mathcal{O}_B \xRightarrow{\Gamma(\mathrm{Spec}\, A, -)} \boxed{A \xrightarrow{\varphi} B}$

$\mathcal{O}_{A,f(Q)} = A_{f(Q)} \qquad \mathcal{O}_{B,Q} = B_Q$

Need to show: $f(Q) = \varphi^{-1}Q$!!, $Q \in \mathrm{Spec}\, B$

• $\varphi(A \setminus f(Q)) \subseteq B \setminus Q$
• $\varphi(f(Q)) \subseteq Q$

$\iff \varphi^{-1}(Q) = f(Q)$, $Q = PI$

$\square$