$(\underset{\text{top sp}}{X}, \underset{\text{sheaf of rings}}{\mathcal{O}_X})$ = **locally** ringed space: · $\forall x \in X$: $\mathcal{O}_{X,x}$ = local ring

· $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$: · $f: X \to Y$

· $\boxed{\mathcal{O}_Y \to f_* \mathcal{O}_X}$

local hom. $\downarrow$ $\Downarrow$

$\forall x: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \Longleftarrow f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

$\underset{\mathcal{O}_X}{} f: X \to Y \underset{\mathcal{O}_Y}{}$ ⇝ $\boxed{f_*: Sh(X) \to Sh(Y)}$

main object: $Sh(\mathcal{O}_X) :=$ (Sheaves of) $\mathcal{O}_X$-modules $\ni \mathcal{F}$

i.e. · $\mathcal{F}$ = sheaf of abelian groups

· $\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$, i.e. $\mathcal{O}_X$ acts on $\mathcal{F}$ such that all $\mathcal{F}(U)$ become $\mathcal{O}_X(U)$-modules.

example: · $X = \text{Proj } S$ ⇝ $\mathcal{O}(d) := \widetilde{S(d)} = \mathcal{O}_X$-module

· $M = A$-module $\Longrightarrow \tilde{M} = \mathcal{O}_X$-module, $X = \text{Spec } A$.

Upgrade $\mathcal{F} = \mathcal{O}_X$-module ⇝ $f_*\mathcal{F} = (f_*\mathcal{O}_X)$-module $\Longrightarrow$ apply $\mathcal{O}_Y \to f_*\mathcal{O}_X$

$(\forall V \subseteq Y: (f_*\mathcal{F})(V) = (f_*\mathcal{O}_X)(V)$-module $\Longleftrightarrow \mathcal{F}(\overset{U}{f^{-1}V}) = \mathcal{O}_X(\overset{U}{f^{-1}V})$-module

$f_*\mathcal{F} = f_*\mathcal{O}_X$-module $\Longrightarrow \mathcal{O}_Y \to f_*\mathcal{O}_X$ makes this into a $\mathcal{O}_Y$-module

(similar to: $R \to A$ ring hom., $M = A$-module $\Longrightarrow M = R$-module)

$f^{-1}: Sh(Y) \to Sh(X)$

$\mathcal{G} = \mathcal{O}_Y$-module $\Longrightarrow f^{-1}\mathcal{G} = f^{-1}\mathcal{O}_Y$-module

$f^*: Sh(\mathcal{O}_Y) \to Sh(\mathcal{O}_X)$ $\quad \downarrow \mathcal{O}_X \quad \Longrightarrow \overbrace{f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X}^{f^*\mathcal{G}} = \mathcal{O}_X$-module

Recall: f^{-1} was exact ⇝ f^* = right exact (recall: f_* = left exact)

Remark: $f^* \dashv f_*$ Sheaves of ab. grps

$f^{-1} \dashv f_*$ Sheaves of modules

Def: A **scheme** is: · a locally ringed space

· locally it is isomorphic to $(\overset{\text{"affine scheme"}}{\text{Spec } A, \mathcal{O}_A})$.

Recall: X = top. sp., $\mathcal{F}/X$ = sheaf, $U \subseteq X$ open ⇝ $\mathcal{F}|_U$ = sheaf on U ("restriction"): $U' \subseteq U$ ⇝ $\mathcal{F}|_U(U') := \mathcal{F}(U')$

alternatively: $j: U \hookrightarrow X$ open embedding, $\mathcal{F}|_U = j^*\mathcal{F} = j^{-1}\mathcal{F}$

· $j: U \hookrightarrow X$ open $(j^{-1}\mathcal{F})(\underset{U}{\overset{\cap}{U'}}) = \varinjlim_{V \supseteq j(U')} \mathcal{F}(V) = \mathcal{F}(U') \Longrightarrow j^{-1}\mathcal{F} = \mathcal{F}|_U$.

minimal elmt is U' !!

· $(U, [\mathcal{O}_U = \mathcal{O}_{X|U}]) \hookrightarrow (X, \mathcal{O}_X)$ $\Updownarrow$

· $j: U \hookrightarrow X$ ✓

· $\mathcal{O}_X \to j_*\mathcal{O}_U \Longleftrightarrow \underset{\mathcal{O}_U}{\underset{\parallel}{j^{-1}\mathcal{O}_X}} \xrightarrow{id} \mathcal{O}_U$ ⇝ $j^*\mathcal{F} = j^{-1}\mathcal{F} \otimes_{j^{-1}\mathcal{O}_X} \mathcal{O}_U = j^{-1}\mathcal{F}$.

First "real" ex of a scheme: $\text{Proj } S := (\text{Proj } S, \mathcal{O}_S := \tilde{S}) = \bigcup_i \underset{\text{Spec } S_{(z_i)}}{\underset{\parallel}{D_+(z_i)}}$

Recall $\text{Spec } A := (\text{Spec } A, \mathcal{O}_A)$

$\text{Spec } B \xrightarrow{f} \text{Spec } A \Longleftrightarrow A \xrightarrow{\varphi} B$

$(\ , \mathcal{O}_B) \to (\ , \mathcal{O}_A)$ ⇝ $\mathcal{O}_A \to f_*\mathcal{O}_B$ ⤴ global sections

Q Does this hold true in general?

$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ ⇝ $\mathcal{O}_Y \to f_*\mathcal{O}_X$ ⇝ $\mathcal{O}_Y(Y) \to (f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$

Recall. $\mathbb{P}^n = \text{Proj } k[z_0, \dots, z_n]$ ⇝ $\Gamma(\mathbb{P}^n, \mathcal{O}) = k$

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & \text{Spec } k \end{tikzcd}$$

"over k" ⇝

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) & & \Gamma(Y, \mathcal{O}_Y) \arrow[ll, "F"'] \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\Gamma(X,\mathcal{O}_X) = k$, $\Gamma(Y,\mathcal{O}_Y) = k$ $\Longrightarrow F = id_k$

all sections $\mathcal{O}_X(U)$ become k-objects!

Theorem: X = scheme, $Y = \text{Spec } B$ = affine scheme. $[X \to \text{Spec } B] \longleftrightarrow [B \to \Gamma(X, \mathcal{O}_X)]$

Proof: $X = \bigcup_\alpha U_\alpha$, $\underset{(\text{Spec } A_\alpha, \mathcal{O}_{A_\alpha})}{\underset{\parallel}{U_\alpha}} \subseteq X$ open, affine (i.e. $\mathcal{O}_X|_{U_\alpha} \cong \mathcal{O}_{A_\alpha}$ on $U_\alpha = \text{Spec } A_\alpha$)

$X \to \text{Spec } B$ ⇝ $U_\alpha \hookrightarrow X \to \text{Spec } B$ $\Longrightarrow \text{Spec } A_\alpha \to \text{Spec } B \Longleftrightarrow B \to A_\alpha$

$U_\alpha \cap U_\beta = \bigcup_\gamma W_{\alpha\beta,\gamma}$, $W_{\alpha\beta,\gamma} = \text{Spec } C_{\alpha\beta,\gamma}$.

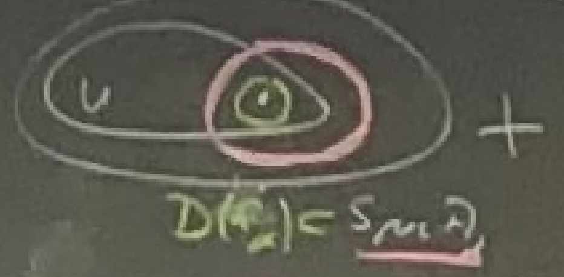
$$\begin{tikzcd} & U_\alpha \arrow[dr] & & \\ W_{\alpha\beta,\gamma} \arrow[ur, hook] \arrow[dr, hook] \arrow[rrr, bend right] & & X \arrow[r] & \text{Spec } B \\ & U_\beta \arrow[ur] & & \end{tikzcd} \Longrightarrow \begin{tikzcd} & A_\alpha \arrow[dl] & \\ C_{\alpha\beta\gamma} & & B \arrow[ul] \arrow[dl] \\ & A_\beta \arrow[ul] & \end{tikzcd}$$

Lemma $U \subseteq X$, X = scheme $\Rightarrow$ U = scheme $\quad ((X,\mathcal{O}_X) \rightsquigarrow (U, \mathcal{O}_U = \mathcal{O}_{X|U}))$

Proof $X = \bigcup_\alpha U_\alpha$, $U_\alpha = \operatorname{Spec} A_\alpha$ $\rightsquigarrow$ $\forall p \in U \rightsquigarrow \exists \alpha = \alpha(p)$: $p \in U \cap \operatorname{Spec} A_\alpha \underset{\text{open}}{\subseteq} \operatorname{Spec} A_\alpha$

$\Rightarrow \exists f_\alpha \in A_\alpha$: $p \in D(f_\alpha) \subseteq U \cap \operatorname{Spec} A_\alpha$, $\quad D(f_\alpha) = \operatorname{Spec}(A_\alpha)_{f_\alpha}$

$\Rightarrow U = \bigcup_{p \in U} \operatorname{Spec}(A_{\alpha(p)})_{f_{\alpha(p)}}$ □



$\underset{\text{affine}}{U,V} \subseteq X$ open, assume (w.l.o.g.). $W = U \cap V$ = ~~affine~~, too. $\quad$ (assume: locally ringed spaces)

$U \cup V = \boxed{X}$

Claim.
$$\begin{tikzcd} & U \arrow[r, "f"] & \boxed{Y} \\ W \arrow[ur, hook] \arrow[r, hook] & V \arrow[ur, "g"] & \end{tikzcd}$$
i.e. $f_{|W} = g_{|W} \iff X \xrightarrow{h} Y$

Proof. · topological part: clear.

· Know $\left.\begin{matrix} U \xrightarrow{f} Y \\ W \hookrightarrow V \xrightarrow{g} Y \end{matrix}\right\} \Rightarrow \begin{matrix} \mathcal{O}_Y \to f_*\mathcal{O}_U \\ \mathcal{O}_Y \to g_*\mathcal{O}_V \end{matrix}$

Want. $\mathcal{O}_Y \to h_*\mathcal{O}_X$

$L \subseteq Y$ open subset

$\mathcal{O}_Y(L) \longrightarrow \mathcal{O}_X(h^{-1}L)$ $\quad (h^{-1}L \cap U) \cup (h^{-1}L \cap V) = h^{-1}L$

$\mathcal{O}_X(h^{-1}L \cap U) = \mathcal{O}_X(\boxed{f^{-1}L}) = f_*\mathcal{O}_U(L)$

$\mathcal{O}_X(h^{-1}L \cap V) = \mathcal{O}_X(\boxed{g^{-1}L}) = g_*\mathcal{O}_V(L)$

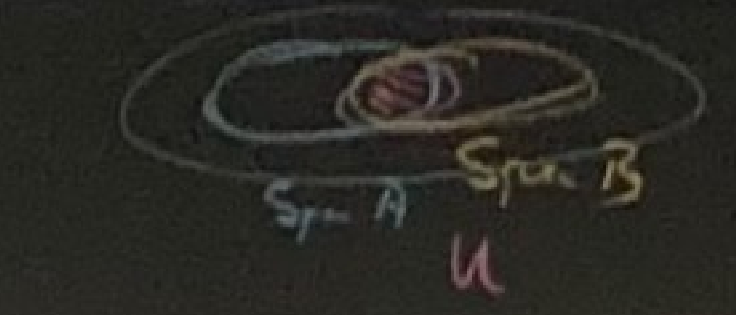
$\mathcal{O}_X(h^{-1}L \cap W) = \mathcal{O}_X(f^{-1}L \cap g^{-1}L)$

□

Lemma. $\operatorname{Spec} A$, $\operatorname{Spec} B \underset{\text{open}}{\hookrightarrow} X$ = scheme.

$U = \operatorname{Spec} A \cap \operatorname{Spec} B$

$\Rightarrow \exists\, U = \bigcup_\alpha U_\alpha$: $U_\alpha = \operatorname{Spec} A_{a_\alpha} = \operatorname{Spec} B_{b_\alpha}$.



Proof: 1st part: w.l.o.g. $\boxed{\operatorname{Spec} A \subseteq \operatorname{Spec} B}$

Assume that in this case the claim is proven. $U = \bigcup_\alpha \operatorname{Spec} C_\alpha \Rightarrow \boxed{\operatorname{Spec} C_\alpha \subseteq \operatorname{Spec} A}$

· $\Rightarrow \boxed{\operatorname{Spec} C_\alpha} = \bigcup_\beta \operatorname{Spec} C_{\alpha\beta}$: $C_{\alpha\beta}$ = loc. of C_α and of A

· -"- $= \bigcup_\gamma \operatorname{Spec} C'_{\alpha\gamma}$: $C'_{\alpha\gamma}$ = loc. C_α and of B

· $\operatorname{Spec} C_{\alpha\beta} \cap \operatorname{Spec} C'_{\alpha\gamma} = \operatorname{Spec}(C_\alpha)_{bc}$

$C_{\alpha\beta} = (C_\alpha)_b$ = Loc of A; $\quad C'_{\alpha\gamma} = (C_\alpha)_c$ = Loc of B; $\quad (C_\alpha)_{bc}$ = both. Loc of A and of B, and these spectra cover U. //

Assume $\operatorname{Spec} A \subseteq \operatorname{Spec} B$ (open embedding) $\quad$ claim: $\exists$ open cov. nice for both.

$\exists\, b \in B$: $\operatorname{Spec} A = \bigcup_{(\text{those } b \in B)} \operatorname{Spec} B_b$ $\quad (\operatorname{Spec} B_b = D(b))$: $\quad \operatorname{Spec} B_b \hookrightarrow \operatorname{Spec} A \underset{\text{open}}{\hookrightarrow} \operatorname{Spec} B$

$$\begin{tikzcd} B_b & A \arrow[l] & B \arrow[l, "\varphi"'] \arrow[ll, bend left, "\text{localization}"] \\ & \varphi(b) \arrow[u, phantom, "\in" rotate=90] & b \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

⇙

claim $B_b \xrightarrow{\sim} A_{\varphi(b)}$

$$\begin{tikzcd} B_b \arrow[d, equal] & A_{\varphi(b)} \arrow[l] \arrow[r, hook] & B_b \arrow[d, equal] \arrow[ll, bend right, "\text{isom}" description] \\ R & S \arrow[l] \arrow[r, hook] & R \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, hook, "\text{open}"] \arrow[rr, bend right, "\text{id}"'] & \operatorname{Spec} S \arrow[r, hook, "\text{open}"] & \operatorname{Spec} R \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r, "\varphi"] & S \\ r \in R & \Downarrow \\ R_r \arrow[r] & S_{\varphi(r)} \end{tikzcd}$$

□

· $(X, \mathcal{O}_X)$, $U \subseteq X \rightsquigarrow (U, \mathcal{O}_U)$, $\mathcal{O}_U = \mathcal{O}_{X|U}$

· $U \overset{\iota}{\hookrightarrow} X$ open emb. of sch., $U \hookrightarrow X$ iso of top. spa. $\Rightarrow \iota$ = isom.

(ex: $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \xleftarrow[\text{embedding}]{\text{closed}} \operatorname{Spec} k[\varepsilon]/(\varepsilon)$)

$\operatorname{Spec} A$, $F \subseteq \operatorname{Spec} A$ closed

$\Rightarrow$ there are several schemes $Z_\alpha \hookrightarrow X$

Construction with schemes $\quad \operatorname{Spec} A \supseteq V(\mathfrak{J}) = \operatorname{Spec} A/\mathfrak{J}$

closed embeddings $\quad$ ex: $\operatorname{Spec} A \xleftarrow[\text{closed emb.}]{} \operatorname{Spec} A/\mathfrak{J}$

↓

Def: $f: X \to Y$ (both are schemes) is called a closed embedding $:\iff$ locally on the target it is of the form $\operatorname{Spec} A/\mathfrak{J} \hookrightarrow \operatorname{Spec} A$.

i.e. · $\boxed{\exists}$ $Y = \bigcup_\alpha V_\alpha$, $V_\alpha = \operatorname{Spec} B_\alpha$:
· $\forall$ Y = ... :
$\forall/\exists$ Y = . . . :

$$\begin{tikzcd} f^{-1}(V_\alpha) \arrow[r, "f"] \arrow[d, equal] & V_\alpha \arrow[d, equal] \\ \operatorname{Spec} B_\alpha/\mathfrak{J}_\alpha \arrow[r, hook] & \operatorname{Spec} B_\alpha \end{tikzcd}$$
for some ideal $\mathfrak{J}_\alpha \subseteq B_\alpha$.

$\operatorname{Spec} A \supseteq F$ = closed subset $\rightsquigarrow$ closed subscheme?

i.e. $\exists\, \mathfrak{J} \subseteq A$: $F = V(\mathfrak{J})$ $\rightsquigarrow$ yes: take $\operatorname{Spec} A/\mathfrak{J}$

Problem: there is no unique $\mathfrak{J}$! But: [illegible] $V(\mathfrak{J}) = V(\mathfrak{J}') \Rightarrow$

$\bigcap P$

$\operatorname{Spec} A \hookrightarrow \operatorname{Spec} B$ open embedding $\overset{Q}{\rightsquigarrow}$ Is $B \to A$ injective ?

in general NO!

Lemma $B \hookrightarrow A$ injective $\Rightarrow$ $\operatorname{Spec} A \to \operatorname{Spec} B$ is dominant (i.e. image is dense)

Proof Assume $\exists F \subsetneq \operatorname{Spec} B$, closed s.t. $\operatorname{Spec} A \xrightarrow{f} F \subsetneq \operatorname{Spec} B$ ($X \xrightarrow{f} Y$ $f(X) \subseteq U \overset{open}{\subseteq} Y$ $\Rightarrow f: X \to U$)

$$\left(\begin{tikzcd} \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[r, hook, "f"] \arrow[dr, hook, "/"'] & \operatorname{Spec} k[\varepsilon]/\varepsilon^3 \arrow[d, phantom, "\cup"] \\ & \operatorname{Spec} k[\varepsilon]/\varepsilon \end{tikzcd}\right)$$

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "f"] \arrow[dr] & F \arrow[d, equal, "①"] \arrow[r, phantom, "\subsetneq"] & \operatorname{Spec} B \\ & \operatorname{Spec} B/\mathfrak{a} \arrow[ur] & \end{tikzcd}$$

$(V(\mathfrak{a}) = F)$

$\Updownarrow$

$$\begin{tikzcd} A & & B \arrow[ll, hook', "inj"'] \arrow[dl] \\ & B/\mathfrak{a} \arrow[ul] & \end{tikzcd}$$

$\Rightarrow \mathfrak{a} = 0 \Rightarrow F = \operatorname{Spec} B.$

Fiber product

$$\begin{tikzcd} \circledast \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

Schemes:

$X \times_S Y :=$ „fiber product" $\Leftrightarrow$ it maps to X, Y (over S) as (∗) and is universal (terminal) with this property.

$$\begin{tikzcd} Z \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & S \end{tikzcd}$$

(0) Fiber prod. in (Aff Sch)

① Fiber prod. $\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B$ in Sch

⓪
$$\begin{tikzcd} \operatorname{Sp} C \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \operatorname{Sp} X \arrow[r] \arrow[d] & \operatorname{Sp} B \arrow[d] \\ & \operatorname{Sp} A \arrow[r] & \operatorname{Sp} R \end{tikzcd}$$

$$\begin{tikzcd} & C & & \\ A \times B_{(a,0)} & & X \arrow[ul, "\exists!"'] & B \arrow[l] \arrow[ull, bend right] \\ A \arrow[u] & a \arrow[u, mapsto] & A \arrow[u] \arrow[uul] & R \arrow[l] \arrow[u] \end{tikzcd}$$

$X := A \otimes_R B$

$A = B = R = \mathbb{Z}$

$$\begin{tikzcd} & \mathbb{Z} & \\ \mathbb{Z} & \mathbb{Z} \times \mathbb{Z} \arrow[ul] \arrow[u, "(a,1)"'] & \mathbb{Z} \arrow[l, "(a,1)\mapsto(a,1)"'] \\ a & \mathbb{Z} \arrow[u] \arrow[uul] & \mathbb{Z} \arrow[l] \arrow[u] \end{tikzcd}$$

$A \xrightarrow{\varphi} C$, $B \xrightarrow{\psi} C$ $\Rightarrow$ $A \otimes_R B \longrightarrow C$

$(a, b) \longmapsto \varphi(a) \cdot \psi(b)$

ex. $\boxed{\text{Sets}}$ $X \times_S Y = \{(x,y) \mid f(x) = g(y) \text{ in } S\}$

. $\exists$ good special cases (i) $S = pt$ $X \times_{pt} Y = X \times Y$

($\mathbb{P}^m, \mathbb{P}^n \rightsquigarrow \mathbb{P}^m_k \times_{\operatorname{Spec} k} \mathbb{P}^n_k$ needed Segre)

(ii)
$$\begin{tikzcd} f^{-1}(Y) \arrow[r] \arrow[d, hook] & Y \arrow[d, phantom, "\cap"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

$Y =$ subset of S, $Y \overset{g}{\hookrightarrow} S$

$\Rightarrow X \times_S Y = f^{-1}(Y)$

(iii) $Z_1, Z_2 \hookrightarrow X$ closed emb.

$Z_1 = \operatorname{Spec} A/\mathfrak{a}_1$, $Z_2 = \operatorname{Spec} A/\mathfrak{a}_2$, $X = \operatorname{Spec} A$

$\rightsquigarrow Z_1 \cap Z_2 \subseteq \operatorname{Spec} A$

$Z_1 \cap Z_2 = \operatorname{Sp} A/\mathfrak{a}_1 + \mathfrak{a}_2$

$Z_1 \times_X Z_2$ $\overline{A/\mathfrak{a}_1 \otimes_A A/\mathfrak{a}_2} = A/\mathfrak{a}_1 + \mathfrak{a}_2.$

$$\begin{tikzcd} U_\alpha \arrow[r, hook, "\psi"] \arrow[dr, dashed, "\Phi_\alpha"'] \arrow[ddr, bend right, "\Phi_\alpha"'] & Z \arrow[d, dashed, "\Phi"] \arrow[dr, bend left] \arrow[dd, bend right] & \\ & \operatorname{Spec} A \otimes_R B \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

①

$Z = \bigcup_\alpha U_\alpha$ $U_\alpha = \operatorname{Spec} C_\alpha$

- $\forall \alpha$: $\exists!\ \Phi_\alpha$
- if Φ exists $\Rightarrow$ $\Phi \circ \iota_\alpha = \Phi_\alpha$ $\Rightarrow$ Φ is unique (if it exists)
- „define" Φ by ↗, i.e. we glue it from the Φ_α.
- $W =$ affine $\hookrightarrow U_\alpha \cap U_\beta$

$\Rightarrow \Phi_{\alpha|W}, \Phi_{\beta|W}$ satisfy the existence prop for W

uniq-ness $\Rightarrow \Phi_{\alpha|W} = \Phi_{\beta|W}$

②
$$\begin{tikzcd} & Y \arrow[d] \arrow[dr] & \\ X \arrow[r] \arrow[drr, bend right] & S \arrow[r, "\varphi"] & S' \end{tikzcd}$$

$X \times_S Y \supset X \times_{S'} Y$

③
$$\begin{tikzcd} p^{-1}U \arrow[r, phantom, "\subseteq"] \arrow[d] & X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d] \\ U \arrow[r, phantom, "\subseteq"] & X \arrow[r] & S \end{tikzcd}$$

$U \times_S Y$

If $X \times_S Y$ exist $\Rightarrow$ $p^{-1}(U) \subseteq X \times_S Y$ equals $U \times_S Y$

④
$$\begin{tikzcd} X \times_S Y = \bigcup U_\alpha \times_S Y \arrow[rr] & & Y \arrow[d] \\ p_\alpha^{-1}(U_\alpha \cap U_\beta) \arrow[d] & \arrow[d, "p_\alpha"] \quad \arrow[d, "p"] & \\ X = \bigcup_\alpha U_\alpha \quad U_\alpha \cap U_\beta & U_\alpha \hookrightarrow X \arrow[r] & S \end{tikzcd}$$

$\exists U_\alpha \times_S Y$ $(\forall \alpha)$ is assumed

$\Rightarrow X \times_S Y$ is glued from these $U_\alpha \times_S Y$.

⑤ we may assume: $X, Y =$ affine.

$U_\alpha \times Y$ $U_\beta \times Y$

$\Sigma =$ fan $\rightsquigarrow \sigma \in \Sigma$. $TV(\sigma) =$ affine

$TV(\Sigma) = \varinjlim TV(\sigma)$.

$(U_\alpha \cap U_\beta) \times Y \hookrightarrow U_\alpha \times Y$

$\| \quad p_\beta^{-1}(U_\alpha \cap U_\beta)$ $\| \quad p_\alpha^{-1}(U_\alpha \cap U_\beta)$

Before: $S = \bigcup_\gamma S_\gamma$, S_γ = affine.

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

$g^{-1}(S_\gamma)$ $f^{-1}(S_\gamma)$

Before ④: may assume:

$f(X) \subseteq S_\gamma$

$g(Y) \subseteq S_\gamma$

$X \times_{S_\gamma} Y = X \times_S Y$