open, closed embeddings
fiber product

Pre-images:

$f : X \longrightarrow Y$ (X, Y = schemes)
$\quad\cup \qquad\quad \cup$
$f^{-1}(Z) \longrightarrow Z$

generally defined as:

$f^{-1}(Z) = X \times_Y Z$

$$\begin{array}{ccc} A & \longleftarrow & B \\ \downarrow & & \downarrow \\ A/J\cdot A & \longleftarrow & B/J \end{array}$$

locally: $\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B$

$\operatorname{Spec}(A \otimes_B B/J) \longrightarrow \operatorname{Spec} B/J$

$f^{-1}(\operatorname{Spec} B/J) = \operatorname{Spec} A/J A$

via $\varphi : B \to A$, $\quad J \subseteq B$

Scheme-theoretic image: ① Local situation: $f : \operatorname{Spec} A \to \operatorname{Spec} B \rightsquigarrow f(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$

$\hookrightarrow$ is defined as $\overline{f(\operatorname{Spec} A)} \subseteq \operatorname{Spec} B$.

$\overline{f(\operatorname{Spec} A)} = V(J)$, $J \subseteq B$ $\rightsquigarrow$ scheme str.: $\operatorname{Spec} B/J$ – **problem**: what is J?

$\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B \supseteq \operatorname{Spec} B/J$ $\;\hat{=}\;$ $A \xleftarrow{\varphi} B \to B/J \to A$

best choice of J (maximal): $J = \operatorname{Ker} \varphi$.

$\Longrightarrow B/J \hookrightarrow A$ inj.

$\Longrightarrow \operatorname{Spec} A \longrightarrow V(J) \subseteq \operatorname{Spec} B$ dense.

$\Downarrow$

$\overline{f(\operatorname{Spec} A)} = V(J)$.

Thus: $J := \operatorname{Ker} \varphi \Longrightarrow \operatorname{Spec} B/J$ is the smallest scheme str. on $\overline{f(\operatorname{Spec} A)}$ such that f goes through it.

(i.e. if $J' \subseteq B$ s.t.: $\operatorname{Spec} A \longrightarrow \operatorname{Spec} B \supseteq \operatorname{Spec} B/J'$, then $\boxed{\operatorname{Spec} B/J} \subseteq \operatorname{Spec} B/J'$)

② $X \xrightarrow{f} \operatorname{Spec} B$ (⇝ general situation: $f: X \longrightarrow Y$, $\boxed{f : f^{-1}(\operatorname{Spec} B) \longrightarrow \operatorname{Spec} B}$ (open cover))

$\operatorname{Spec} A \xrightarrow{f_A}$ does not suffice ($\operatorname{im}(f_A)$ doesn't glue)

$X = \bigcup_i (\operatorname{Spec} A_i) \rightsquigarrow f_i : \operatorname{Spec} A_i \longrightarrow \operatorname{Spec} B \supseteq \operatorname{Spec} B/J$ $\;\hat{=}\;$ $A_i \xleftarrow{\varphi_i} B \to B/J \to A_i$ $\quad(*)$

$X = \bigcup_i \operatorname{Spec} A_i$

$A_i \to (A_i)_{red} = A_i/\sqrt{0}$

$X_{red} = \bigcup_i \operatorname{Spec} (A_i)_{red}$

J exists for $\varphi_i \Longleftrightarrow J \subseteq \operatorname{Ker} \varphi_i$

$\rightsquigarrow J_\varphi := \bigcap_i \operatorname{Ker} \varphi_i$ is the max. J satisfying $(*)$ $(\forall i)$

smallest sub scheme such that all f_i go through! = special subscheme $= \operatorname{Spec} B/J_\varphi \subseteq \operatorname{Spec} B$ $\Longleftarrow$

Remark: $\operatorname{Spec} B/J_\varphi = \overline{f(X)} \subseteq \operatorname{Spec} B$ (will do an Exerc.)

$\|$ sch. th. image of f.

Remark: $f : X \to \operatorname{Spec} B \;\hat{=}\; B \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X) \longrightarrow A_i$

Q. $J_\varphi \overset{(?)}{=} \operatorname{Ker} \varphi$

Special cases: (i) $D(f) \subseteq \operatorname{Spec} A$ ($f \in A$), what is $\overline{D(f)} \subseteq \operatorname{Spec} A$?

$\operatorname{Spec} A_f \overset{①}{\hookrightarrow} \operatorname{Spec} A$

$\Longrightarrow \overline{D(f)} = \operatorname{Spec} A/(0 : f^\infty)$

recipe of ①: $A \xrightarrow{\varphi} A_f$

$\operatorname{Ker} \varphi = \{ a \in A \mid \frac{a}{1} = 0 \text{ in } A_f \} = \{ a \mid \exists n\; f^n a = 0 \}$
$= (0 : f^\infty) \subseteq A$.

(ii) generalize: $\operatorname{Spec} A \setminus V(J) \hookrightarrow \operatorname{Spec} A$ ($J \subseteq A$ ideal) **ex:** $\mathbb{A}^2 \setminus \{(0,0)\} = \operatorname{Spec} k[x,y] \setminus V(x,y)$

Goal: find a canonical scheme str. on $\overline{U}$, $\quad U$

(we met this before: $\overline{V(I) \setminus V(J)}$, e.g. strict transform of the blow up)

Situation of ②: $J = (f_1, \dots, f_n) \Longrightarrow V(J) = \bigcap_i V(f_i)$, $\quad U = \bigcup_i D(f_i)$ $\Longrightarrow A \longrightarrow A_{f_i}$ $(\forall i)$

before: $\overline{\operatorname{Spec} A \setminus V(J)} = V((0 : J^\infty))$ $\qquad I = \bigcap_i (0 : f_i^\infty) = (0 : J^\infty)$

now: $\operatorname{Spec} A/(0 : J^\infty) =$ s.t.… ($\operatorname{Spec} A/I \subseteq \operatorname{Spec} A$ will be the s.t.…)

does not need to be reduced.

$X_{red} \hookrightarrow X \xrightarrow{f} Y \supseteq Z_{red} \hookrightarrow Z = \text{„}\overline{f(X)}\text{“}$

Elimination

$\mathbb{A}^n \xrightarrow{\pi} \mathbb{A}^{n-1}$

$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$

$V(J) \longrightarrow \overline{\pi(V(J))}$

$k[x_1, \dots, x_n] \hookleftarrow k[x_1, \dots, x_{n-1}]$, $\underline{x} = (x_1,\dots,x_n)$, $\underline{x}' = (x_1,\dots,x_{n-1})$

$\leadsto \operatorname{Spec} k[\underline{x}]/J \hookrightarrow \operatorname{Spec} k[\underline{x}] \xrightarrow{\pi} \operatorname{Spec} k[\underline{x}']$

$k[\underline{x}]/J \twoheadleftarrow k[\underline{x}] \hookleftarrow k[\underline{x}']$, φ

$\ker \varphi = \{ f \in k[\underline{x}'] \mid f(x_1, \dots, x_{n-1}) \in J \}$
$= J \cap k[\underline{x}']$, $J \subseteq k[\underline{x}]$

ex: $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$

$V(J)$: $\longrightarrow V(J \cap k[x,y]) = V(x-y)$

$J = (x-z, y-z) \ni x-y$

Ex (K-rat. pts)

$X := \operatorname{Spec} \mathbb{R}[x] = \mathbb{A}^1_{\mathbb{R}} \overset{?}{\longleftrightarrow} \mathbb{R}^1 \ni r$, $(x-r) \longleftarrow r$

Problem: (x^2+1)

$Y = \operatorname{Spec} \mathbb{C}[x] = \mathbb{A}^1_{\mathbb{C}} \overset{!!}{\longleftrightarrow} \mathbb{C}^1 \ni r$, $(x-r) \longleftarrow r$

"K-rational points of a scheme" K = field

Yoneda-Lemma: replace objects by functors: X = scheme $\leadsto X(S) := \operatorname{Mor}_{Sch}(S, X)$, $X \in Sch^{op} \longrightarrow \operatorname{Fun}(\cdot, X)$

Special case: $S = \operatorname{Spec} K$

$X(K) := X(\operatorname{Spec} K) =$ "set of K-rational points".

$\boxed{p \in X(K)} \iff \operatorname{Spec} K \xrightarrow{p} X$:

- cont. map: $pt \longrightarrow X \mathrel{\hat=} \boxed{\text{choosing } x \in X}$, $x \in \operatorname{Spec} A \subseteq X$ open neighborhood of x
- (replace X by $\operatorname{Spec} A$)
- $\operatorname{Spec} K \to \operatorname{Spec} A \iff A \xrightarrow{\varphi} K$ ($x \in \operatorname{Spec} A$, i.e. $x \mathrel{\hat=} [P \subseteq A$ PI$]$)

$P = \ker \varphi = \varphi^{-1}(0) \hookleftarrow (0)$

$\iff A/P \overset{\varphi}{\hookrightarrow} K \iff \boxed{K(P) \hookrightarrow K}$, $K(P) = K(x)$

$\boxed{A} = \{\text{reg. fct. on } \boxed{\operatorname{Spec} A}\}$, $a \in A$, $P \in \operatorname{Spec} A$

$A \twoheadrightarrow A/P \hookrightarrow \operatorname{Quot} A/P =: \underline{K(P)}$ residue field, $= (A/P)_{(0)}$

$a \longmapsto$ "$a(P)$" $:= \bar{a}$

Remark: $\mathcal{O}_{X,P} \supseteq m_{X,P} \leadsto K(P) = \mathcal{O}_{X,P}/m_{X,P} = A_P/P_P = (A/P)_{(0)}$, $K(P)$; $\mathcal{O}_{X,P} = A_P$, $m_{X,P} = P_P$

Y: Y = scheme over $\operatorname{Spec} \mathbb{C}$ ($Y \longrightarrow \operatorname{Spec} \mathbb{C}$, $\operatorname{Spec} K$, $\mathbb{C} \hookrightarrow K$)

$Y(\mathbb{C}) = \operatorname{Mor}(\operatorname{Spec} \mathbb{C}, Y) = \{ p \in Y;\ K(p) \hookrightarrow \mathbb{C} \}$ ($\mathbb{C}$-schemes, $\mathbb{C}$, id) $\Rightarrow K(p) = \mathbb{C}$

$\Downarrow$

$\mathbb{C}$-rat. pts of a $\mathbb{C}$-scheme = closed pts of this scheme.

$Y = \operatorname{Spec} \mathbb{C}[x,y]$

P: $(x-c, y-d)$ (max ideal)

$(0) \Rightarrow K((0)) = \operatorname{Quot} \mathbb{C}[x,y]/(0) = \mathbb{C}(x,y)$

$K(P) = \mathbb{C}[x,y]/(x-c, y-d) = \underline{\mathbb{C}}$

$X = \operatorname{Spec} \mathbb{R}[x] \leadsto$ scheme over $\operatorname{Spec} \mathbb{R}$ $\leadsto$ candidates for K (as $\mathbb{R}$-algebras): $\mathbb{R} \to \underline{\mathbb{R}}$, $\mathbb{R} \hookrightarrow \underline{\mathbb{C}}$

$X(\mathbb{R}) = \{ p \in X$ and $K(p) \hookrightarrow \mathbb{R} \}$ ($\mathbb{R}$, id)

$\Downarrow$

$\boxed{\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{R}^1}$

i.e. $K(p) = \mathbb{R}$, $p = (x-c), (x^2+1), (0)$

$K(x-c) = \mathbb{R}[x]/(x-c) = \mathbb{R}$

$K((0)) = \mathbb{R}(x)$

$K(x^2+1) = \mathbb{R}[x]/(x^2+1) \cong \underline{\mathbb{C}}$, $x \mapsto i, -i$

$\mathbb{A}^1_{\mathbb{R}}(\mathbb{C}) = \{ p \in X$ and $K(p) \hookrightarrow \mathbb{C} \}$ ($\mathbb{R}$)

$(\mathbb{A}^1_{\mathbb{R}} \to \mathbb{R})$

$\Rightarrow (x^2+1) \in \mathbb{A}^1_{\mathbb{R}}$ carries two $\mathbb{C}$-rat. pts!

$(0) \Rightarrow K((0)) = \mathbb{R}(x)$

$(x-c) \Rightarrow K((x-c)) = \mathbb{R}$

$\hookrightarrow$ induce exactly one $\mathbb{C}$-rat. pt.

$(x^2+1) \leadsto \mathbb{R}[x]/(x^2+1)$

$\mathbb{R}[x]/(x^2+1) \hookrightarrow \mathbb{C}$, $x \longmapsto i, -i$, $\mathbb{R}$

Remark: $\boxed{Y \longrightarrow \operatorname{Spec} k,\ k = \bar{k}}$ $\leadsto \underline{Y(k)} = \{$ closed pts of $Y \}$

$\mathbb{Q}(\sqrt{2}) \longrightarrow \mathbb{Q}(\sqrt{2})$, $\sqrt{2} \longmapsto -\sqrt{2}$

- $S := \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \supseteq S_{red} = \operatorname{Spec} k[\varepsilon]/(\varepsilon) = \underline{\operatorname{Spec} k}$
- $X = k$-Scheme $\leadsto X(S) = \operatorname{Der}_k(A, k)$ (remind: $X(\operatorname{Spec} k) = \{ \operatorname{Spec} k \to X \}$)

Let f ($\operatorname{Spec} k[\varepsilon]/\varepsilon^2 \longrightarrow X$, $\operatorname{Spec} k$) $\mathrel{\hat=} x \in X$, closed point $\leadsto$ choose $x \in \operatorname{Spec} A \subseteq X$ (open) ($x \mathrel{\hat=} P \subseteq A$ maximal ideal)

$a \in A \xrightarrow{\varphi} k[\varepsilon]/\varepsilon^2$, ring hom: $\varphi(a) = \bar{a} + \varepsilon \cdot d(a)$, $d(a) \in k$

$\Rightarrow k$ becomes an A-module via $A/P \xrightarrow{\sim} k$, $a \mapsto \bar{a}$

$d: A \to k$, additive, k-linear

$\overline{ab} + \varepsilon \cdot d(ab) = (\bar{a} + \varepsilon\, d(a))(\bar{b} + \varepsilon\, d(b))$
$= \bar{a}\bar{b} + \cancel{\varepsilon^2} + \varepsilon(d(a)\bar{b} + \bar{a} \cdot d(b))$

$\boxed{d(ab) = d(a) \cdot \bar{b} + \bar{a} \cdot d(b)}$ Leibniz rule

A = ring, M = A-module

$k \to A$ (k alg. closed)

Def $d: A \to M$ "k-derivation" $\Longleftrightarrow$
- d = k-linear
- $d(ab) = a \cdot d(b) + b \cdot d(a)$

We set X = k-scheme, $k = \bar{k}$

$X(k[\varepsilon]/\varepsilon^2) = \{ x \in X(k)$ ($\cong$ closed pts) and a derivation $d: \mathcal{O}_{X,x} \to k \}$, $k = \mathcal{O}_{X,x}/m_{X,x}$

(replaces A from before: $\mathcal{O}_{X,x} = A_p$)

(A, m) = local ring

$\mathrm{Der}_k(A, k) = \mathrm{Hom}_A(m, k)$, $k = A/m$

$d \longmapsto d|_m$: $d(a \cdot m) = a \cdot d(m) + m \cdot d(a) = a \cdot d(m)$, $a \in A$, $m \in m$, $m \cdot d(a) = 0$ in $k = A/m$

$d(c \in k) = 0$ (?) $\longleftarrow \varphi$

(d = derivation:
- k-linear: $d(c \cdot 1) = c \cdot d(1)$
- derivation: $d(c \cdot 1) = c \cdot d(1) + 1 \cdot d(c)$

$\Longrightarrow \underline{d(c) = 0}$)

$$\boxed{k \hookrightarrow A \to A/m = k, \quad c \in k, \quad d}$$

$\varphi: m \to k$ A-linear is given

$d: A \to k$ is wanted: $d(a)$, $a \in A$

$a \in A \to A/m = k$, $a \longmapsto \bar{a}$, $a - \bar{a} \longmapsto 0$

$a - \bar{a} \in m \subset A$; $\varphi(a - \bar{a})$ is defined; $d(a) = d(a) - \underbrace{d(\bar{a})}_{0} = d(a - \bar{a}) = \varphi(a - \bar{a})$

$\mathrm{Der}_k(A, k) = \mathrm{Hom}_A(m, k) = \mathrm{Hom}_A(m/m^2, k) = \mathrm{Hom}_k(m/m^2, k) = (m/m^2)^*$

$\varphi \rightsquigarrow \varphi|_{m^2} = 0$ ($\varphi(ab) - a \cdot \varphi(b) = 0$, $a, b \in m$)

m/m^2, k = A-modules $\supset$ m – kills both $\Downarrow$ m/m^2, k = A/m-modules i.e. k-vs.

Thm is: $\mathrm{Mor}(\mathrm{Spec}\, k[\varepsilon]/\varepsilon^2, X) = (x \in X,\ d \in \mathrm{Der}_k(\mathcal{O}_{X,x}, k))$ and $d \in T_x X$

$= (m_{X,x}/m_{X,x}^2)^*$

Remark: $\boxed{m_{X,x}/m_{X,x}^2 = T^*_{X,x}}$

2 reasons to believe this: X = mf, $x \in X$

e.g.: $X = \mathbb{R}^n$, $x = 0$; tangent vector $t \in T_{\mathbb{R}^n, 0} = \mathbb{R}^n$

$t \in T_{t,0}$, $f: X \to \mathbb{R} \rightsquigarrow \frac{\partial f}{\partial t}(0)$ = dir. der. of f in dir. t $= \langle t, f'(0) \rangle$

$\frac{\partial}{\partial t}: (\text{fcts}) \to \mathbb{R}$, $f \longmapsto \frac{\partial f}{\partial t}(0)$ is a derivative!

{directional derivatives}.

f = fct. $\rightsquigarrow$ t operates on $f \in m/m^2$.

$m/m^2 = T^*_{X,0}$.

$X = V(y^2 - x^3) \subseteq \mathbb{C}^2$; $T_0 X = T_0 \mathbb{C}^2 = \mathbb{C}^2$

Example: $X = \mathbb{A}^n_{\mathbb{C}} \rightsquigarrow \mathrm{Spec}\, \mathbb{C}[x_1, \dots, x_n]$

$k[\underline{x}]_{(\underline{x})}/(\underline{x}) = k[\underline{x}]/(\underline{x}) = k$

$(\underline{x})/(\underline{x})^2 = \mathrm{span}_{\mathbb{C}}(x_1, \dots, x_n) \cong \mathbb{C}^n$

Finiteness assumptions
- X = scheme (locally $X \supset \mathrm{Spec}\, A$)
- "X = scheme over S" $\triangleq$ $X \to S$, e.g. $S = \mathrm{Spec}\, k$ "k-scheme"

"Y = –"–"; $Y \to X$, $Y \to S$

Def X = "locally noetherian" $\Longleftrightarrow$ $\exists$ open covering $X = \bigcup_v \mathrm{Spec}\, A_v$ s.t. A_v = noeth. rings.

Prop: If $\mathrm{Spec}\, B \subseteq X$ is an open subset, X = l. noeth $\Longrightarrow$ B = noeth.

Proof: $\mathrm{Spec}\, B \subseteq X \supseteq \mathrm{Spec}\, A_v$ $\Longrightarrow$ $\exists$ covering of $\mathrm{Spec}\, B \cap \mathrm{Spec}\, A_v$ by $U_{v\mu}$ (A_v = noeth)

$U_{v\mu}$ = "nice" w.r. to both

$U_{v\mu} = \mathrm{Spec}\, C_{v\mu}$; $C_{v\mu} = B_\bullet = [(A_v)_\bullet = \text{noeth.}]$

$\Longrightarrow$ $B_\bullet$ = noeth.

$\Longrightarrow$ we get an open $\boxed{\text{covering}}$ of $\mathrm{Spec}\, B$ by $\mathrm{Spec}\, B_\bullet$ s.t. $B_\bullet$ = noeth.

$(B_{f_i},\ f_i \in B,\ \boxed{i \in I})$

$(f_i \mid i \in I) = (1)$ $\Longrightarrow$ $\exists$ finite $I' \subseteq I$: $(f_i \mid i \in I') = (1)$

$\Longrightarrow \mathrm{Spec}\, B = \bigcup_{i \in I'} \mathrm{Spec}\, B_{f_i}$

Claim: B_{f_i} = noeth. $\overset{(?)}{\Longrightarrow}$ B = noeth., $(f_i) = (1)$

Proof: $J \subseteq B$ ideal $\rightsquigarrow$ $J_{f_i} \subseteq B_{f_i}$ $\Longrightarrow$ $\exists$ $J_{f_i} = \left(\frac{b_{i1}}{f_i^\bullet}, \dots, \frac{b_{in}}{f_i^\bullet} \right)$ s.t. $b_{ij} \in J$

$J' := (b_{ij}) \subseteq B$ $\rightsquigarrow$
- $J' \subseteq J$
- $J'_{f_i} = J_{f_i}$

$(M_1 \overset{\eta}{\hookrightarrow} M_2) \otimes_B B_{f_i} = \text{iso}\ \forall i \Longrightarrow \eta = \text{iso}$, $(f_1, \dots, f_n) = (1)$

$\widetilde{M}_1 \hookrightarrow \widetilde{M}_2$

$(\widehat{M}_1 \to \widehat{M}_2) = \text{iso} \Longrightarrow (M_1 \to M_2) = \text{iso}$ □

$\forall P \in \mathrm{Spec}\, B\ \exists i: P \in \mathrm{Spec}\, B_{f_i} \rightsquigarrow B \to B_{f_i} \to B_P$

$B \to B_f$ f.g. algebra: $B_f = B[\frac{1}{f}]$; B_P is loc.

$I \subseteq A$ $\boxed{P}$ $I_p = (f_1, \dots, f_n) \rightsquigarrow \exists g$: $(f_1, \dots, f_n) \subseteq A_g$, $\supseteq I$ $\Longleftarrow$

Def: X = noeth. scheme $\Longleftrightarrow$
- l. noeth.
- $\exists$ finite open covering by affine noetherian $\mathrm{Spec}\, A_i$

$\Longleftrightarrow$ X = quasi compact. ($\forall$ open coverings you may select a finite sub covering) □

Proof: $X = \bigcup_{i=1}^n \mathrm{Spec}\, A_i$, goal: qu. comp.

$X = \bigcup_{v \in I} U_v \Longrightarrow \mathrm{Spec}\, A_i = \bigcup (\mathrm{Spec}\, A_i \cap U_v)$: here, $\forall i$: finitely many v suffice!

Remark: X = noeth. scheme $\Longrightarrow$ X = noeth. top. space (i.e. ascending chains of open subsets terminate)

$U \subseteq X$, $Z \subseteq X$ subschemes $\Longrightarrow$ U, Z = noeth.