noetherian / locally noeth. schemes

[affine + quasi-compact] $U \subseteq X$ open, affine $\rightsquigarrow U = \operatorname{Spec} A$: A = noeth. ring.

$f : X \to Y$ morphism of schemes

- "locally" finite type $\hat{=}$ $\varphi : A \to B$ algebra, B = f.g. A-algebra
- affine
- finite $\hat{=}$ $\varphi : A \to B$ algebra : B = finite over A (i.e. B = f.g. A-module)
- closed embedding : $\varphi : A \to A/I$

Def $f : X \to Y$ is of l.f.t. $\iff \exists\ Y = \bigcup_j \operatorname{Spec} B_j$ $(\rightsquigarrow B_j \to A_{ij})$, $f^{-1}(\operatorname{Spec} B_j) = \bigcup_i \operatorname{Spec} A_{ij}$ such that A_{ij} = fin. gen. B_j-algebra.

Prop: $f : X \to Y$ l.f.t.

If $\operatorname{Spec} B \subseteq Y$ (open), $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$

$\implies \operatorname{Spec} A \to \operatorname{Spec} B \ \hat{=}\ B \to A$, A = f.g. B-algebra.

Remark "f.t." $\iff$ "l.f.t." + quasi-compact

(i.e. f = "quasi-compact" $:\iff \forall \operatorname{Spec} B \subseteq Y : f^{-1}(\operatorname{Spec} B)$ = quasi-compact)

Proof Let $f : X \to Y$ l.f.t. (1) assume that $Y = \operatorname{Spec} B$.

Let us assume that: $X = \bigcup_i \operatorname{Spec} A_i$; $B \to A_i$ is f.g. algebras.

Let $\operatorname{Spec} A \subseteq X$ open, affine $\rightsquigarrow \operatorname{Spec} A \hookrightarrow X \to \operatorname{Spec} B \implies \varphi : B \to A$ **Claim** A = f.g. B-algebra.

($\operatorname{Spec} A = U$)

$U \cap \operatorname{Spec} A_i$ – covered by W_{iv} (open subsets) : [W_{iv} = nice w.r.t. A = –"– A_i] $\rightsquigarrow W_{iv} = \operatorname{Spec} A_{iv}$, $A_{iv} = (A_i)_{f_{iv}} = A_i\left[\frac{1}{f_{iv}}\right]$

$\implies A_{iv}$ = f.g. B-algebra.

W_{iv} give a (finite) cover of $\operatorname{Spec} A$.

$W_{iv} = \operatorname{Spec} A_{a_{iv}}$ i.e. $(a_{iv} \mid v) = (1)$ in $A \rightsquigarrow \exists\, b_{iv} \in A : \sum a_{iv} b_{iv} = 1$

We know $B \to A \to A_{a_i}$ (f.g. algebra) **claim**: A = f.g. B-algebra.

generators $\frac{f_i}{a_i}$ $\rightsquigarrow$ collect all these elements: $a_i, f_i, b_{iv} \in A \rightsquigarrow A' \subseteq A$ subalgebra generated by these el.

Claim $A' = A$ $\quad A' \hookrightarrow A$ / [apply functor localiz. in a_i] $= \otimes_{A'} A'_{a_i}$

$B \to A'_{a_i} \xrightarrow{\sim} A_{a_i}$ (both = B-algebras)

Recall B = ring, $M' \hookrightarrow M$ B-linear, φ = iso for $a_i \in B$, $(a_1, \dots, a_n) = (1) \implies \varphi$ = iso

Here instead of B: consider A', $b_{iv} \in A' \implies (a_{iv}) = (1)$ in A' ∎

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(2) $f : X \to Y$ l.f.t., $Y = \bigcup_v \operatorname{Spec} B_v$. $f^{-1}(\operatorname{Spec} B_v) \to \operatorname{Spec} B_v$ is l.f.t.

- Take $\operatorname{Spec} B \subseteq Y$ open : $\operatorname{Spec} B \cap \operatorname{Spec} B_v$ = covered by W_{iv} = nice w.r.t. B and B_v
- Take $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$ **goal** $B \to A$ f.g. alg.

$U = f^{-1}(\operatorname{Spec} B)$: $\operatorname{Spec} A \to \operatorname{Spec} B$; $f^{-1}(\operatorname{Spec} B_b) = U_b \to \operatorname{Spec} B_{v_i} = W_{vi}$; $f^{-1}(\operatorname{Spec} B_v) \to \operatorname{Spec} B_v$; $B_a = B_b$

$\left(U_b := U \times_{\operatorname{Spec} B} \operatorname{Spec} B_b\right)$

$(\operatorname{Spec} A)_b = \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} B_b = \operatorname{Spec}(A_b)$

Know: $U_b \to \operatorname{Spec} B_b$ is of l.f.t. for $b \in B$ forming an ideal $(b_1, \dots, b_r) = (1)$ in B.

Need: $U \to \operatorname{Spec} B$ is of l.f.t. ($\operatorname{Spec} A \subseteq U$). Goal $B \to A$; tool : $B_b \to A_b$

A' = B-algebra, gen. by gen's of A_{b_i} (as B_b-alg)

$\implies A' \hookrightarrow A$ B-modules

$\forall i : A'_{b_i} \hookrightarrow A_{b_i}$ is an isom. $(b_1, \dots, b_r) = (1)$ in B. ∎

Def $f : X \to Y$ is affine $\iff \forall/\exists\ \operatorname{Spec} B \subseteq Y$: $f^{-1}(\operatorname{Spec} B)$ = affine $(= \operatorname{Spec} A)$

Prop. $\exists \implies \forall$ $\quad$ ($X \xrightarrow{f} Y$, $U = f^{-1}(\operatorname{Spec} B) \to \operatorname{Spec} B$)

$\operatorname{Spec} B \subseteq Y$ open $\rightsquigarrow \exists\ (b_1, \dots, b_r) = (1)$ in B : $f^{-1}(\operatorname{Spec} B_{b_i}) = \operatorname{Spec} A_i \subseteq X$,

Claim $f^{-1}(\operatorname{Spec} B) = U$ is affine.

Simplify notation: $f : X \to \operatorname{Spec} B$, $(b_1, \dots, b_r) = (1)$ in B, $f^{-1}(\operatorname{Spec} B_{b_i}) = \operatorname{Spec} A_i \subseteq X$

Claim: X = affine.

$B \to \Gamma(X, \mathcal{O}_X)$ ($\hat{=}$ system of rings A_i and their localization)

$\Gamma(\mathcal{O}_X) \to A_1, A_2 \to A_{12} = A_{1,2}$

$B_b \to [\text{system of } A_i] \otimes_B B_b = [\text{system of } (A_i)_b]$

$\operatorname{Spec} B_b \to \boxed{\Gamma(X, \mathcal{O}_X)_b} = \Gamma(f^{-1}(\operatorname{Spec} B_b), \mathcal{O}_X)$,

$\implies f^{-1}(\operatorname{Spec} B_b) = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)_b$

- $X \xrightarrow{f} \operatorname{Spec} B \;\hat{=}\; B \to \Gamma(\mathcal{O}_X) \rightsquigarrow \boxed{X \xrightarrow{\pi} \operatorname{Spec} \Gamma(\mathcal{O}_X)} \xrightarrow{f} \operatorname{Spec} B$ | $X \to \operatorname{Spec} B \;\hat{=}\; B \to \Gamma(X, \mathcal{O}_X) \;\hat{=}\; \operatorname{Spec}\Gamma(\mathcal{O}_X) \to \operatorname{Spec} B$

$\Gamma(\mathcal{O}_X) \xleftarrow{id} \Gamma(\mathcal{O}_X)$

- $(b_1, \dots, b_r) = (1)$ in B

$(f^{-1})(D(s)) = \operatorname{Spec} \Gamma(\mathcal{O}_X)_s$

$\Rightarrow \boxed{f^{-1}(D(s)) = \operatorname{Spec}\Gamma(\mathcal{O}_X)_s} \to D(s)$ Claim: π = isomorphism!

Summarize (avoid the usage of B)

$\Gamma(\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$

$\pi: X \xrightarrow{\pi} \operatorname{Spec}\Gamma(\mathcal{O}_X)$, $U' \quad U$

$\Rightarrow \pi^{-1}(\operatorname{Spec}\Gamma(\mathcal{O}_X)_s) \xrightarrow{\sim} \operatorname{Spec}\Gamma(\mathcal{O}_X)_s$ is an isomorphism

$= \operatorname{Spec}\Gamma(\mathcal{O}_X)_s$ $\Rightarrow \pi$ is locally an isom. $\Rightarrow$ is an isom.! $\square$

$X \to \operatorname{Spec}\Gamma(\mathcal{O}_X)$ ‖ $f: X \to Y$, $g: Y \to Z$

- f, g = affine $\Rightarrow g \circ f$ = affine.
- $g \circ f$ = affine, g = affine $\overset{?}{\Rightarrow} f$ = affine?

Def $f: X \to Y$ is finite $\Leftrightarrow$
- f = affine
- $\forall/\exists \operatorname{Spec} B \subseteq Y \rightsquigarrow f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A \rightsquigarrow B \xrightarrow{\varphi} A$; φ = finite, i.e. A = f.g. B-module

(Recall $f: \operatorname{Spec} B \to \operatorname{Spec} A$ finite $\Rightarrow f$ = quasi-finite, i.e. fibers are finite)

Ex $\operatorname{Spec} A_a \hookrightarrow \operatorname{Spec} A$ ~~local~~ open embedd., $A \to A_a = A[\frac{1}{a}]$ is not finite.

Def $Z \hookrightarrow X$ closed emb. $\Leftrightarrow$
- affine ($\rightsquigarrow$ locally $\operatorname{Spec} A \to \operatorname{Spec} B$)
- $B \to A$ is always surj.

Ex $X \to \operatorname{Spec} k$, X f.t. $\Rightarrow X$ noetherian

Def k = field. X = "k-variety" $\Leftrightarrow$
- $X \to \operatorname{Spec} k$
- f.t.
- reduced ($X = X_{red}$)
- irreducible
- separated

"integral scheme"

Def X = "integral" $\Leftrightarrow$ X = reduced + irreducible.

- $X = \bigcup \operatorname{Spec} A_v$: reduced $\Leftrightarrow \forall v: A_v$ has no nilpotent elements.
- X = irred. $\Leftrightarrow$ every non-empty open subset $U \subseteq X$ is dense in X ($\overline{U} = X$)

Assume
- X = reduced
- X = irred.

$\Big\}$ $\operatorname{Spec} A \subseteq X$ open $\Rightarrow \operatorname{Spec} A$ = red. + irred.

($Z \subseteq \operatorname{Spec} A$, Z = irr $\Leftrightarrow I(Z) = PI$)

$\boxed{\operatorname{Spec} A = \text{red} \Leftrightarrow \sqrt{0} = 0 \\ \operatorname{Spec} A = \text{irred} \Leftrightarrow \sqrt{0} = PI}$ $\Rightarrow A$ = integral domain

Remark: $X \supseteq \operatorname{Spec} A$ open dense, A = domain $\Downarrow$ (X = reduced) X = integral

Ex $\operatorname{Spec} k[x,y]/(x^2+y) \to \mathbb{A}^2$

$X \setminus \{(0,0)\} = D(y) \hat{=} A_y$ = integral domain

Def $f: X \dashrightarrow Y$ is "birational" $\Leftrightarrow \exists g: Y \dashrightarrow X$: $f \circ g$, $g \circ f = id$

$\exists U \subseteq X, V \subseteq Y$: $U \xrightarrow[f]{\sim} V$

$K(X) = K(Y)$

(X = integral) $\supseteq \operatorname{Spec} A$ open, dense ($\rightsquigarrow A$ = integral domain)

Claim: $\exists!\, \eta = \eta_X \in X$: $\overline{\eta} = X$ "generic point"

($\forall U \subseteq X$ open, $U \neq \emptyset \Rightarrow \eta \in U \Rightarrow \eta \in \operatorname{Spec} A$: $\eta = (0) \subseteq A$)

$(X, \mathcal{O}_X) \rightsquigarrow \mathcal{O}_{X,\eta} = \operatorname{Quot} A$ = field! "function field" $= K(X)$

$f: X \dashrightarrow Y$ dominant $\rightsquigarrow \eta_X \mapsto \eta_Y \rightsquigarrow K(Y) \hookrightarrow K(X)$

Also: $\boxed{X, Y = \text{integral} \\ \text{f.t. over } k}$, $K(Y) \hookrightarrow K(X) \Rightarrow f: U \subseteq X \dashrightarrow Y$

$Y \supseteq \operatorname{Spec} B$, $X \supseteq \operatorname{Spec} A$

$Q(B) \hookrightarrow Q(A)$

$k \to B = k[s_1, \dots, s_r]$, $\varphi(s_i) = \frac{a_i}{n_i} \rightsquigarrow$ common denominator $N \rightsquigarrow \varphi(s_i) \in A_N$

$\rightsquigarrow B \hookrightarrow A_N \rightsquigarrow D(N) \subseteq X$, $\operatorname{Spec} B \subseteq Y$

next class: 7/20 (last class one week later)

Def $f: X \to Y$ is separated iff $\Delta: X \to X \times_Y X$ (diagonal; id) is a closed embedding
$\iff \Delta(X) \subseteq X \times_Y X$ is a closed subset.

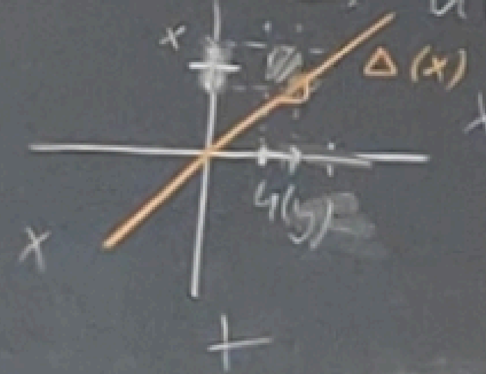
$X \overset{\Delta}{\hookrightarrow} X \times_Y X$, $V_i \to f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$

X = scheme $\not\Rightarrow$ Hausdorff

Prop: X = top. space. X = Hausdorff $\iff X \overset{\Delta}{\hookrightarrow} X \times X$, $x \mapsto (x,x)$ is a closed subset

Proof $(\Rightarrow)$ $(X \times X) \setminus \Delta(X) = \{(x,y) \in X \times X \mid x \neq y\}$

$\Rightarrow U(x)$ = open neighborhood, $U(y)$: $U(x) \cap U(y) = \emptyset$



$U(x) \times U(y) \subseteq X \times X$

disj $\Rightarrow U(x) \times U(y) \subseteq (X \times X) \setminus \Delta(X) \Rightarrow (X \times X) \setminus \Delta(X) = \bigcup_{x \neq y} U(x) \times U(y)$ (open; $\Uparrow$ open; $\underset{(x,y)}{\cup}$)

$(\Leftarrow)$ $x \neq y \Rightarrow (x,y) \in (X \times X) \setminus \Delta(X)$ (open $\Rightarrow$ = $\bigcup$ products $U \times V$, $U \subseteq X$, $V \subseteq X$ open)

$\Rightarrow \exists U, V: (x,y) \in U \times V \subseteq (X \times X) \setminus \Delta(X)$

$\Updownarrow$ $x \in U, y \in V$. $\Updownarrow$ $U \cap V = \emptyset$

Def X = „~~fake Hausdorff~~" separated $:\iff X \overset{\Delta}{\hookrightarrow} X \times_{\mathrm{Spec}\,\mathbb{Z}} X$ is a closed embedding (or $\Delta(X)$ = closed subset)
$\iff X \to \mathrm{Spec}\,\mathbb{Z}$ is separated.

Remark $X \to \mathrm{Spec}\,R$ $\rightsquigarrow$ ask for $X \to \mathrm{Spec}\,R$ being separated ($R = \mathbb{R}$?)

Claim $\Delta(X) \subseteq X \times_Y X$ closed $\Rightarrow \Delta: X \to X \times_Y X$ is a closed embedding.

- $X \overset{f}{\to} Z$ closed emb $\iff$ locally: all $f^{-1}(U_i) \to U_i$ ($U_i \subseteq Z$) is a closed embedding.
- $f: X \to Y$ is sep. $\iff \forall V_i \subseteq Y$ open: $f^{-1}(V_i) \to V_i$ is separated („separated" is local on the target)

$\rightsquigarrow$ w.l.o.g. Y = affine = $\mathrm{Spec}\,B$

- $X = \bigcup \mathrm{Spec}\,A_i = \bigcup U_i \Rightarrow X \times_Y X = \bigcup_{i,j} U_i \times_Y U_j = \left(\bigcup_i U_i \times_Y U_i\right) \cup \left((X \times X) \setminus \Delta(X)\right) \overset{\Delta}{\longleftarrow} X$ (open)

$U \subseteq X \rightsquigarrow U \times_S Y = p_X^{-1}(U)$ (p_X: $X \times_S Y \to X$; $\downarrow$; $Y \to S$)

$X \to \bigcup_i U_i \times_Y U_i$ Show: closed embedding.

locally $U_i \to U_i \times_Y U_i$ $\hat{=}$ $\mathrm{Spec}\,A_i \to \mathrm{Spec}\,A_i \times_{\mathrm{Spec}\,B} \mathrm{Spec}\,A_i = \mathrm{Spec}(A_i \otimes_B A_i)$
$\hat{=}$ $A_i \otimes_B A_i \xrightarrow{\text{surj!!}} A_i$ (for $B \to A_i$)
$(a \otimes a') \mapsto a \cdot a'$ $\square$

Corollary. $\mathrm{Spec}\,B \overset{f}{\to} \mathrm{Spec}\,A \Rightarrow$ separated. <u>more</u> [f = affine $\Rightarrow f$ = separated]

Prop: $X \to \mathrm{Spec}\,B$ is separated $\iff$
- $\mathrm{Spec}\,A, \mathrm{Spec}\,A' \subseteq X$ open (U, V) $\Rightarrow \mathrm{Spec}\,A \cap \mathrm{Spec}\,A'$ = affine = $\mathrm{Spec}\,C$
- $\mathrm{Spec}\,C \hookrightarrow \mathrm{Spec}\,A$, $\searrow \mathrm{Spec}\,A'$; $C \leftarrow A$, $C \leftarrow A'$; $A \otimes_B A' \xrightarrow{\text{surj}} C$

Proof. $U \cap V \to (U \times_Y V)$; $\downarrow$ $\downarrow$; $X \xrightarrow{\Delta} X \times_Y X$

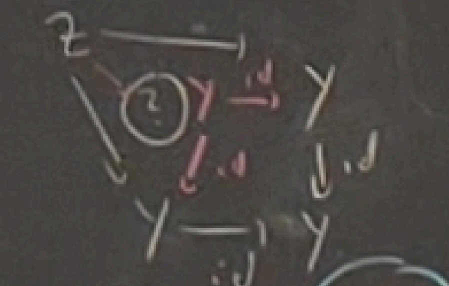
Σ = fan in $N_{\mathbb{R}} = \mathbb{R}^n \rightsquigarrow TV(\Sigma) = \bigcup_{\sigma \in \Sigma} TV(\sigma) = \varinjlim_{\sigma \in \Sigma} TV(\sigma)$ $\quad TV(\sigma) \cap TV(\sigma') = TV(\sigma \cap \sigma')$
$k[\sigma^\vee \cap M] \otimes k[\sigma'^\vee \cap M] \to k[(\sigma^\vee + \sigma'^\vee) \cap M]$

$(\Rightarrow)$ Δ = closed emb. $\Rightarrow U \cap V \hookrightarrow U \times_Y V$ closed emb
$X \to \mathrm{Spec}\,R \to \mathrm{Spec}\,\mathbb{Z}$ $\Rightarrow$ $\boxed{\mathrm{Spec}\,C}$ $\quad \mathrm{Spec}(A \otimes_B A')$

$(\Leftarrow)$ $\cdot\}$ cond $\Rightarrow U \cap V \hookrightarrow U \times_Y V$ closed embedding, $\bigcup_{U,V \text{ aff}} U \times_Y V = X \times_Y X$ //

① ~~$X \hookrightarrow Y$~~ open embedding ✓
$V \hookrightarrow Y$ $\rightsquigarrow V \xrightarrow{\Delta} V \times_Y V = \left(V \overset{j}{\hookrightarrow} Y, j^{-1}(V)\right) = V$ (id)

$Y \times_Y Y = Y$

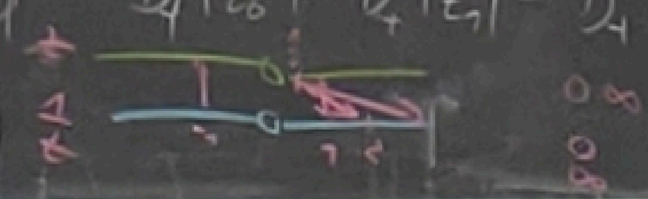


② $Z \hookrightarrow Y$ closed emb. $\Rightarrow$ affine $\Rightarrow$ separated.

③ $\mathbb{P}^1_k = \mathrm{Proj}\,k[z_0, z_1] = D_+(z_0) \cup D_+(z_1)$; $\downarrow$ $\mathrm{Spec}\,k$; $D_+(z_0) \cap D_+(z_1) = D_+(z_0 z_1)$

$k[\frac{z_1}{z_0}] \hookrightarrow k[\frac{z_1}{z_0}, \frac{z_0}{z_1}]$, $k[\frac{z_0}{z_1}] \nearrow$

$k[t] \hookrightarrow k[t, \frac{1}{t}]$, $k[\frac{1}{t}] \nearrow$



④ $\mathbb{A}^1 = \mathrm{Spec}\,k[t]$, $\mathbb{A}^1 = \mathrm{Spec}\,k[s]$; $t = u$, $s = \frac{1}{u}$, $s = u$
$\mathrm{Spec}\,k[u, u^{-1}] = \mathbb{A}^1 \setminus 0$