last class: Friday, 7/29, 10–14 }
last exercise session: –"– , 14–16 } Arnim 22, HS-A

reduced, integral, separated ....

Prop: S = base scheme ; let $f_1, f_2 : X \to Y$ be a morph. of S-schemes (e.g. $S = \operatorname{Spec} k$)

$$\begin{tikzcd} X \arrow[dr] \arrow[r, "f_1, f_2"] & Y \arrow[d] \\ & S \end{tikzcd}$$

Assume that $\cdot$ X = reduced
$\cdot$ Y = separated over S

If $U \subseteq X$ open, dense, if $f_1 = f_2$ on $U \Rightarrow f_1 = f_2$ (on X)

Consequence: $f : X \dashrightarrow Y$ rational, i.e. $\exists\, U \subseteq X : f : U \longrightarrow Y$, $U \subseteq X$

$\Rightarrow \exists$ maximal domain of definition of f: f is def. on U, –"– on U' $\} \Rightarrow$ on $U \cup U'$

Proof: $\cdot$ $f_1, f_2 : X \to Y \rightsquigarrow F = (f_1, f_2) : X \longrightarrow Y \times_S Y$

$\cdot$ on $U \subseteq X$: $f_1 = f_2$

$$\begin{tikzcd} X \arrow[r, "F"] & Y \times_S Y \\ U \arrow[u, hook] \arrow[r] & Y \arrow[u, "\Delta"'] \end{tikzcd}$$

$\boxed{U \subseteq F^{-1}(Y)} \subseteq X \Rightarrow F^{-1}(Y) = X$
open dense, closed

$$\begin{tikzcd} X_{red} = X \arrow[r, "F"] & Y \times_S Y \\ F^{-1}(Y) \arrow[u, hook, "\text{closed imm.}"] \arrow[r] & Y \arrow[u, "\Delta"'] \\ U \arrow[u, dashed] \arrow[ur] \arrow[uu, "?"] & \end{tikzcd}$$

scheme th. maps

$U \to X$ as a top. space.

X = reduced $\Rightarrow$ the given X factors via Δ.

$$\begin{tikzcd} X \arrow[dr, "F"] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "f_2"'] & & \\ & Y \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

Hilbert Polynomials

S = f.g. S_0-algebra

$S = \bigoplus_{d \geq 0} S_d$ graded ring ; S_0 = noetherian ring (e.g. $S_0 = k$)

λ = additive function on f.g. S_0-modules e.g.: $\lambda : \{\text{modules}\} \longrightarrow \mathbb{N}$, $M \longmapsto \lambda(M)$

$0 \to M_1 \to M_2 \to M_3 \to 0$
$\Rightarrow \lambda(M_1) + \lambda(M_3) = \lambda(M_2)$

M = f.g. S-module, graded : $M = \bigoplus_{d \in \mathbb{Z}} M_d$ ($\rightsquigarrow$ all M_d are f.g. S_0-modules)

Poincaré-function: $\Phi(d) := \lambda(M_d)$ $(\forall d)$

Poincaré-series: $\Phi(M) := \sum_{d \geq 0} \lambda(M_d) \cdot t^d \in \mathbb{N}[[t]]$ power series.

Theorem (Hilbert-Serre): Assume that S is generated (as an S_0-algebra) by $a_1, \dots, a_n \in S$ (homog.) ; $d_i := \deg a_i \in \mathbb{N}_{\geq 1}$

Then $\Phi(M) \cdot \prod_{i=1}^{n} (1 - t^{d_i}) \in \mathbb{Z}[t]$, i.e. $\exists$ pol. $f(t)$: $\Phi = \frac{f(t)}{\prod (1 - t^{d_i})}$

polynomials !!

Proof: Induction by n.

$n = 0$ $S = S_0 \Rightarrow M$ = f.g. S_0-module $\Rightarrow$ lives only in finitely many degrees
$\Rightarrow \Phi \in \mathbb{N}[t]$.

$(n-1) \rightsquigarrow n$: $M \xrightarrow{\cdot a_n} M$ S-linear map, increases the degree by d_n

$$\begin{tikzcd} & & & & 0 \\ 0 \arrow[r] & K \arrow[r] & M \arrow[r, "\cdot a_n"] & M \arrow[r] & L \arrow[u] \end{tikzcd}$$

$\Rightarrow K, L$ = graded S-modules. $\left(\Phi(M(e)) = \Phi(M) \cdot t^e + \text{polynomial in } t \right)$

$\Rightarrow 0 \to K \to M \xrightarrow{\cdot a_n} M(d_n) \to L(d_n) \to 0 \rightsquigarrow \forall d \in \mathbb{Z}$: $0 \to K_d \to M_d \longrightarrow M(d_n)_d \to L(d_n)_d \to 0$

$\Rightarrow \lambda(K_d) - \lambda(M_d) + \lambda(M(d_n))_d - \lambda(L(d_n))_d = 0 \quad | \cdot t^{d+d_n}$

$t^{d_n} \cdot \lambda(K_d) \cdot t^d \quad t^{d_n} \cdot \lambda(M_d) \cdot t^d + t^{d+d_n} \lambda(M_{d+d_n}) - t^{d+d_n} \cdot \lambda(L_{d+d_n}) = 0$

$\sum_{d \geq 0}$: $-t^{d_n} \cdot \Phi(M) + t^{d_n} \cdot \Phi(K) + \Phi(M) - \Phi(L) = g(t) := \sum_{e=0}^{d_n - 1} (\lambda(M_e) - \lambda(L_e))\, t^e$, $g(t) \in \mathbb{Z}[t]$

$(-t^{d_n} + 1) \cdot \Phi(M) = \Phi(L) - t^{d_n} \Phi(K) + g(t)$ $\quad$ L, K: $a_n \cdot K = 0$, $a_n \cdot L = 0 \Rightarrow S_0[a_1, \dots, \cancel{a_n}] = S$

suffices to prove that: $\prod_{i=1}^{n-1} (1 - t^{d_i}) \cdot$ RHS = polynomial in t. $\quad$ $S' := S/(a_n)$

$L, K = S'$-modules $\xrightarrow{\text{induction}}$ (–"–) !! $\quad \Rightarrow S'$ is gen. by $a_1, \dots, a_{n-1}$.

Example: $S = k[x_1, \dots, x_n]$ with $\deg x_i := 1$

$\lambda = \dim_k$, $\dim_k k[\underline{x}]_d = \binom{n+d-1}{n-1}$

$$\Phi(S) = \sum_{d=0}^{\infty} \binom{n+d-1}{n-1} \cdot t^d = \frac{1}{(1-t)^n} \cdot 1$$

ex: $n=2 \Rightarrow \dim k[x,y]_d = d+1$

$\binom{2+d-1}{1} = d+1$ $\quad \boxed{x^i y^{d-i}},\ i=0,\dots,d$

$(1-t^{d_i})$

trick: $\mathbb{Z}$-grading is refined by the $\mathbb{Z}^n$-grading: $\Phi(t_1,\dots,t_n) := \sum_{d_1,\dots,d_n \geq 0} \dim_k S_{(d_1,\dots,d_n)} \cdot t_1^{d_1} \cdots t_n^{d_n}$

$S_{(d_1,\dots,d_n)} = k \cdot x_1^{d_1} \cdots x_n^{d_n}$, $\dim = 1$

then: $\Phi(S) = \Phi(t,\dots,t)$ $\quad (t_i \mapsto t\ \forall i)$

$\Phi(t_1,\dots,t_n) = \sum_{\underline{d} \in \mathbb{N}^n} \underline{t}^{\underline{d}}$ $\quad (\underline{t} = (t_1,\dots,t_n))$ $\qquad \sum_{d=0}^{\infty} t^d = \frac{1}{1-t}$

$$= \left(\sum_{d_1=0}^{\infty} t_1^{d_1}\right) \cdot \left(\sum_{d_2=0}^{\infty} t_2^{d_2}\right) \cdot \ldots \cdot \left(\sum_{d_n=0}^{\infty} t_n^{d_n}\right)$$

$$= \frac{1}{1-t_1} \cdot \frac{1}{1-t_2} \cdots \frac{1}{1-t_n} \Rightarrow \Phi(S) = \frac{1}{(1-t)^n}$$

Def: M = f.g. S-module (S = f.g. S_0-algebra, λ) $\quad$ (recall $\Phi(M) = \frac{\text{pol.}}{\prod_i (1-t^{d_i})} = \frac{g(t)}{(t-1)^{d(M)} \cdot q(t)}$, $g, q \in \mathbb{Q}[t]$; $g(1), q(1) \neq 0$)

$d(M) :=$ pole order of $\Phi(M)$ (in $t=1$) $\leq n$ $\quad$ (n = # gens of S)

Remark: $a \in S$ homog.; a = non-zero-divisor for M, i.e. $M \xrightarrow{\cdot a} M$ is injective

$\Rightarrow d(M/aM) = d(M) - 1$

Proof: $0 \to M \xrightarrow{\cdot a} M \to M/aM \to 0 \Rightarrow \Phi(M/aM) = \Phi(M) - t^{\deg a} \cdot \Phi(M) + \underbrace{g(t)}_{\text{pol.}}$

$= (1-t^{\deg a}) \cdot \Phi(M) + g(t)$

$$\Rightarrow \frac{\Phi(M/aM)}{(1-t^{\deg a})} = \Phi(M) + \frac{g(t)}{(1-t^{\deg a})} \quad \square$$

Numerical polynomials

Def: $f(t) \in \mathbb{R}[t]$ is called numerical $:\Leftrightarrow \forall d \in \mathbb{Z}, d \gg 0: f(d) \in \mathbb{Z}$.

Ex: $f \in \mathbb{Z}[t]$; $f(t) = \frac{t(t-1)}{2} = \frac{1}{2}t^2 - \frac{1}{2}t = \binom{t}{2}$ $\quad \| \quad \binom{t}{d} = \frac{t(t-1)\dots(t-d+1)}{d!}$, $d \in \mathbb{N}$

Prop: $f \in \mathbb{R}[t]$ is numerical $\Rightarrow \exists c_i \in \mathbb{Z}$: $f(t) = \sum_{i=0}^{N} c_i \cdot \binom{t}{i}$ $\quad$ (c_i = unique!) $\quad \deg = i$

(highest coeff: $c_N = N! \cdot$ [true highest coeff. of f in t])

Proof: · induction by degree: $\deg f = N = 0 \Rightarrow f(t) - \text{const} = c_0 \in \mathbb{Z}$.

· $g(t) := f(t+1) - f(t)$ ⇝ num. pol. of degree $\deg f - 1$.

· $f(t) = \sum_{i=0}^{N} c_i \cdot \binom{t}{i}$ with $c_i \in \mathbb{R}$ $\Rightarrow g(t) = \sum_{i=0}^{N} c_i \left(\binom{t+1}{i} - \binom{t}{i}\right)$, $= \binom{t}{i-1}$

$\deg = N$, $\deg \binom{t}{N} = N$

$(?) \cdot t^N = \boxed{c_N} \cdot \frac{1}{N!} t^N$

$$= \sum_{i=1}^{N} c_i \cdot \binom{t}{i-1} = \sum_{i=0}^{N-1} c_{i+1} \binom{t}{i}$$

· $\deg g = N-1$

· g = numerical

↳ $c_1, \dots, c_N \in \mathbb{Z}$

$f(t) = \sum_{i=1}^{N} c_i \cdot \binom{t}{i} + c_0 \Rightarrow c_0 \in \mathbb{Z}$.

Theorem: Assume that $\deg a_1 = \dots = \deg a_n = 1$ $\quad$ (a_i = gens of S over S_0) (and S, S_0, M as usual)

$\Rightarrow d \longmapsto \lambda(M_d)$ is a function

Claim: this becomes a polynomial for $d \gg 0$. I.e. $\exists H_M \in \mathbb{Q}[t]$ (num. pol.): $H_M(d) = \lambda(M_d)\ \forall d \gg 0$.

Example: $M = S = k[x_1, \dots, x_n]$ ⇝ $\dim k[\underline{x}]_d = \binom{n+d-1}{n-1} = H_{k[\underline{x}]}(d)$

Proof: $\Phi_M(t) = \frac{f(t)}{(1-t)^{d(M)}} = f(t) \cdot (1 + t + t^2 + \dots)^{d(M)}$

$= f(t) \cdot \sum_{d=0}^{\infty} \binom{n+d-1}{n-1} \cdot t^d$ $\quad | \quad f(t) = a_0 + a_1 t + \dots + a_n t^n$ $(a_i \in \mathbb{Z})$

$\sum_{d=0}^{\infty} \lambda(M_d) \cdot t^d$

↳ t^d-coeff. ⇝ t^d-coeff on RHS: $a_\nu \cdot t^\nu \cdot \binom{n+d-\nu-1}{n-1} \cdot t^{d-\nu}$ $\quad \forall \nu$

$\Rightarrow \lambda(M_d) = \sum_{\nu=0}^{K} a_\nu \cdot \binom{n+d-\nu-1}{n-1}$ if $d > K$. $\Rightarrow$ for $d \gg 0$. $\square$

Remark: $\lambda(M_\nu) = \sum_{k=0}^{N} a_k \cdot \underbrace{\binom{d+\nu-k-1}{d-1}}_{\text{pol in } \nu,\ \deg = d-1}$ $\Longrightarrow$ $\boxed{\deg H_M \leq d(M)-1}$ ("=" because of $f(1) = \sum_{k=0}^{N} a_k \neq 0$)

$\| \quad H_M(\nu)$ "Hilbert polynomial" $\quad d = d(M)$ $\quad \left(\Phi = \frac{f(t)}{(1-t)^d},\ f(t) = \sum a_k t^k\right)$

Example: $M = S/I$ $\quad (\hat{=}\ V_+(I) \subseteq \mathbb{P}^N$, $\operatorname{Proj}(S/I)$ for $S = k[x_1, \dots, x_n])$

- $\dim \operatorname{Proj} S/I := \deg H_M = d(M) - 1$
- $\deg \operatorname{Proj} S/I = [\text{highest coeff. of } H_M] \cdot (\deg H_M)!$

① $\mathbb{P}^{n-1} = \operatorname{Proj} k[x_1, \dots, x_n] \rightsquigarrow H_{\mathbb{P}^{n-1}}(\nu) = \binom{n-1+\nu}{n-1} \Longrightarrow \deg H = n-1 \longrightarrow \dim \mathbb{P}^{n-1} = n-1$

$\deg \mathbb{P}^{n-1} = \frac{1}{(n-1)!} \cdot (n-1)! = 1$

② $X_d = \operatorname{Proj} k[x_1, \dots, x_n]/F(x)$ with $\deg F = d$

$= V_+(F) \subseteq \mathbb{P}^{n-1}$ hypersurface of degree d

$0 \to k[x] \xrightarrow{\cdot F} k[x] \to S \to 0$

$H_S(\nu) = H_{k[x]}(\nu) - H_{k[x](-d)}(\nu)$, $\quad H_{k[x](-d)}(\nu) = H_{k[x]}(\nu - d)$

$H_{X_d}(\nu) = H_{\mathbb{P}^{n-1}}(\nu) - H_{\mathbb{P}^{n-1}}(\nu - d)$

$= \binom{\nu+n-1}{n-1} - \binom{\nu+n-1-d}{n-1} \rightsquigarrow \deg \leq n-2$ (ν^{n-1}-terms kill each other)

$= \boxed{\frac{(\nu+n-1)\cdot \ldots \cdot (\nu+1)}{(n-1)!} - \frac{(\nu+n-1-d)\cdot \ldots \cdot (\nu-d+1)}{(n-1)!}}$

$\underline{\underline{\nu^{n-2}}}: \left((n-1)+(n-2)+\ldots+1\right) - \left((n-1-d)+\ldots+(1-d)\right)$

$= (+d+d+\ldots+d) = (n-1)\cdot d$

$= \frac{(n-1)d \cdot \nu^{n-2}}{(n-1)!} + \ldots = \boxed{\frac{d}{(n-2)!}}\nu^{n-2} + \ldots$ $\quad \dim = n-2$; $\deg X = d$ //

Local rings

$(A, \mathfrak{m})$ = local noetherian ring. Let Q = $\mathfrak{m}$-primary ideal, i.e. $\exists r: \boxed{\mathfrak{m}^r \subseteq Q \subseteq \mathfrak{m}}$.

Prop: $(\nu \in \mathbb{N}) \overset{\varphi}{\longmapsto} \operatorname{length}\left(\cancel{A/Q^\nu}\ M/Q^\nu \cdot M\right)$ this is a polynomial ($\nu \gg 0$) of degree$_{d(A)}$ $\leq \#(\text{gens of } Q)$.

M = f.g. A-module

Proof: $\cdot\ S = \operatorname{Gr}_Q A := \bigoplus_{\nu=0}^{\infty} Q^\nu/Q^{\nu+1} \rightsquigarrow S_0 = A/Q$ = Artinian ring $\rightsquigarrow \lambda := \operatorname{length}$

gen as S_0-algebra by the generators of Q as an ideal in A.

$\cdot\ \operatorname{Gr}_Q M := \bigoplus_{\nu=0}^{\infty} Q^\nu M/Q^{\nu+1}M$

$\lambda\left((\operatorname{Gr}_Q M)_\nu\right) = \lambda\left(M/Q^{\nu+1}M\right) - \lambda\left(M/Q^\nu M\right) = \varphi(\nu+1) - \varphi(\nu)$

$\|$

polynomial in ν, $\deg(\text{polynomial}) = d(\operatorname{Gr}_Q M) - 1 \Longrightarrow \varphi$ = polynomial in ν

$\deg \varphi = d(\operatorname{Gr}_Q M) \leq \#$ gens of Q in A. //

① notion of dim: $d(A) := d(\operatorname{Gr}_Q A) \leq \#$ gens of Q.

Remark: $d(A)$ does not depend on Q!

$\varphi_Q(\nu) = \operatorname{length}(M/Q^\nu M)$

$\boxed{\varphi_{\mathfrak{m}}(\nu)} \leq \varphi_Q(\nu) \leq \varphi_{\mathfrak{m}^r}(\nu)$

$\varphi_{\mathfrak{m}}(\nu) = \ell(M/\mathfrak{m}^\nu M)$, $\quad \varphi_{\mathfrak{m}^r}(\nu) = \ell(M/\mathfrak{m}^{r\cdot\nu}M) = \boxed{\varphi_{\mathfrak{m}}(r\cdot\nu)}$

$\longrightarrow \deg \varphi_{\mathfrak{m}} = \deg \varphi_{\mathfrak{m}^r}$

$\longrightarrow = \deg \varphi_Q$!

$c_N \nu^N + \ldots$ $\quad (r^N c_N) \cdot \nu^N = c_N \cdot (r \cdot \nu)^N + \ldots$

② $\delta(A) = \min_{Q = \mathfrak{m}\text{-primary}} \{\#\text{gen's of } Q\} \Longrightarrow \boxed{d(A) \leq \delta(A)}$, $\quad \dim A \leq d(A)$

③ "Krull dimension" $\dim A := \operatorname{ht}(\mathfrak{m}) \leq \infty$ $\quad \dim \mathfrak{m}/\mathfrak{m}^2 = \#$ gens of $\mathfrak{m}$

Prop: $\dim A \leq d$

Proof: $\cdot\ d(A) = 0 \Longrightarrow \varphi_{\mathfrak{m}}(\nu) = \operatorname{length}(A/\mathfrak{m}^\nu) = \text{const in } \nu\ (\nu \gg 0) \Longrightarrow \mathfrak{m}^\nu = \mathfrak{m}^{\nu+1} \Longrightarrow \exists N\ \mathfrak{m}^N = 0 \Longrightarrow$ Artinian $\Longrightarrow \dim A = 0$

$\cdot$ Let $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_\ell$ tower of PI; choose $a \in P_1 \setminus P_0$ $\quad A \twoheadrightarrow A/P_0 \longrightarrow A/P_0 + (a) = \bar{A}$

$d(\bar{A}) \leq d(A/P_0) \leq d(A)$ $\quad$ | in $\bar{A}$: $\bar{P}_1 \subsetneq \bar{P}_2 \subsetneq \ldots \subsetneq \bar{P}_\ell \Longrightarrow d(\bar{A}) \geq \dim \bar{A} \geq \ell - 1$

$\Longrightarrow d(A) \geq \ell$ □