(A, m) ⇝ 3 candidates for a "dimension": · $\dim(A) \le d(A) \le \delta(A)$

"Krull dimension" = $ht(m)$; deg Hilb pol of $Gr_m A$

· $\delta(A) := \min_{\substack{Q \subseteq A \\ m\text{-primary}}} \#\{\text{min. generators of } Q\}$

Theorem: $\dim(A) = d(A) = \delta(A)$

Proof: show: $\delta(A) \le \dim(A)$.

Method: ∀ $k \le \dim(A)$: produce $a_1, \dots, a_k \in A$: Each minimal prime over $(a_1, \dots, a_k)$ has height $\ge k$.

If proven, then: $k = \dim(A)$ ⇝ · $(a_1, \dots, a_{\dim}) \subseteq A$

· $P \supseteq$ -"- min. prime $\Rightarrow ht(P) \ge \dim$

$\Rightarrow m$ = min. prime above $(a_1, \dots, a_{\dim})$

on the other hand: · $ht(m) = \dim$; · $ht(P \neq m) < \dim$

$\Rightarrow (a_1, \dots, a_{\dim}) = m$-primary ⇝ this is a potential $\underline{Q}$

$\Rightarrow \delta(A) \le \dim$. □

Induction: $\underset{I}{\underbrace{(a_1, \dots, a_{k-1})}}$ ⇝ ∃ ~~Ass~~ $I \supseteq Min(I)$; $Ass(A/I)$; $\boxed{ht = k-1}$; $ht \ge k$

⇝ $\underbrace{P_1, \dots, P_\ell}_{\neq m} \supseteq I$: PI with $ht = k-1$ $\Rightarrow \bigcup_i P_i \subsetneq m \Rightarrow$ choose $a_k \in m \setminus \bigcup_i P_i$.

$I \subset A$ radical ideal $\Rightarrow I = \bigcap_{i=1}^{k} P_i$ (P_i = min. PI above I)

$V(I) \subseteq \operatorname{Spec} A \Rightarrow V(I) = \bigcup_{i=1}^{k} V(P_i)$ ← irred. components.

Example: $A = k[x,y,z]/(y^2 - xz)$ ⇝ localize: $A_{(x,y,z)}$ ≙ $X = V(xz - y^2) \subseteq \mathbb{A}^3$, $X \ni 0$

local ring: · $\underline{2\text{-dim.}}$, $\underline{\text{but}}$: $m = (x,y,z)$

$\Rightarrow m/m^2 = kx \oplus ky \oplus kz$ is 3-dim. $\Rightarrow 3$ = min # of gens of m.

$\Rightarrow \dim A_{(x,y,z)} \le 3$

· $Q := (x,z) \Rightarrow \sqrt{Q} \ni y \Rightarrow \sqrt{Q} = (x,y,z) \Rightarrow Q \in$ candidates, # gens of $Q = 2$. ($y^2 \in Q$)

Tangent cones: Let $Q \subseteq (A, m)$ an m-primary ideal s.t. # gens$(Q) = \dim A$. $Q = (\underbrace{a_1, \dots, a_d})$ "set of parameters of A".

$(A/Q)[z_1, \dots, z_d] \xrightarrow{\Phi} Gr_Q(A) := \bigoplus_{k=0}^{\infty} Q^k/Q^{k+1} \cdot t^k$

$z_i \longmapsto \bar{a}_i \cdot t$ ($\bar{a}_i \in Q/Q^2$)

Claim: If $f \in (A/Q)[z_1, \dots, z_d]$ homog. of deg. k

$f \overset{\Phi}{\longmapsto} 0 \Rightarrow \bar{f} \in (A/m)[z_1, \dots, z_d]$ is zero.

Proof: $f \mapsto 0$; assume $\bar{f} \neq 0$ in (-"-) ⇝ not a zero-divisor.

$\Rightarrow f \in (A/Q)[z_1, \dots, z_d]$ is ≠ zero divisor

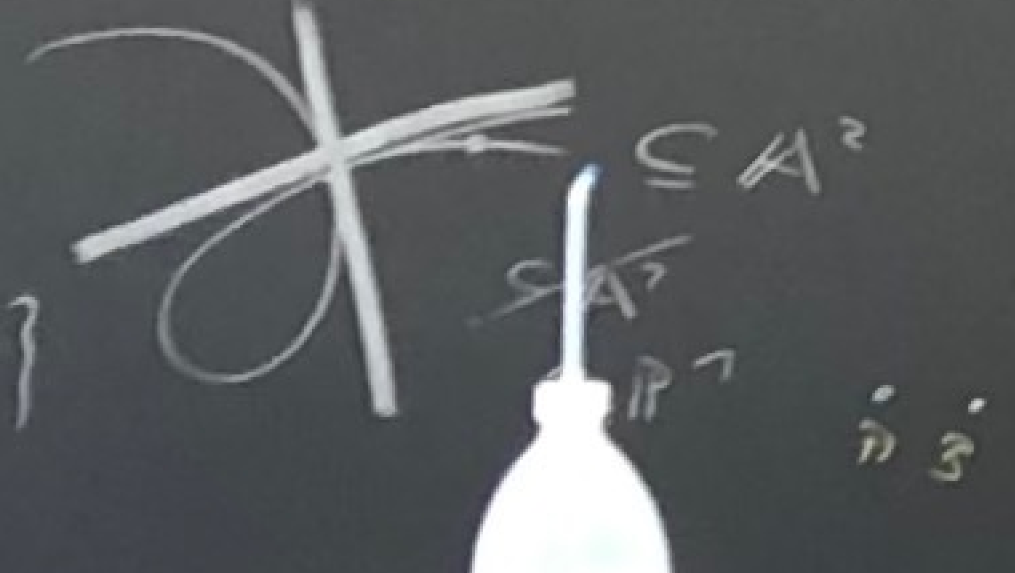
$$d(Gr_Q(A)) \le d\left(\frac{A/Q[z]}{f(z)}\right) < d(A/Q[z_1, \dots, z_d]) = d \quad ↯$$

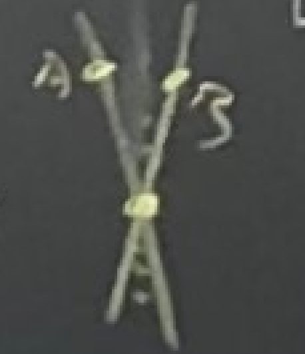
$d(Gr_Q(A)) = d$

⇝ $\Phi : A/Q[z] \to \frac{A/Q[z]}{f(z)} \to Gr_Q(A)$

IF $Q = m$ (i.e. # gens of $m = d = \dim A$) $\Rightarrow (A/m)[z_1, \dots, z_d] \xrightarrow{\sim} \boxed{Gr_m A = \bigoplus_{k=0}^{\infty} m^k/m^{k+1}}$; $S_1 = m/m^2$

Def: (A, m) = "regular" (local) ring, $\Leftrightarrow \dim A = \dim_k m/m^2 = \#\{\text{gen. of } m\}$ ($k = A/m$)





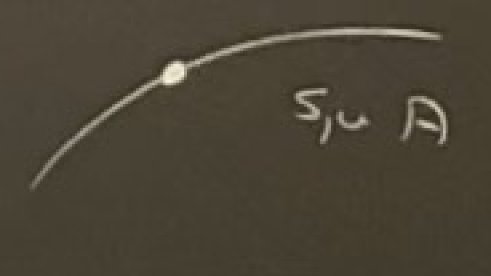
[Proj of Gr = tangent cone]

($Gr A = S = k[y_1, \dots, y_\ell]/\text{ideal}$; gen in S_1 = ℓ-dim. $\Rightarrow \operatorname{Proj} S \subseteq \mathbb{P}^{\ell-1}$

Remark: · (A, m) = regular $\Rightarrow A$ = domain

Proof: $A/m[z] = k[z]$ = domain $\Rightarrow Gr_m A$ = domain $\Rightarrow A$ = domain.

· (A, m) regular, $P \in \operatorname{Spec} A \Rightarrow A_P$ = regular.



$C = V(P) \subseteq \operatorname{Spec} A$

⇑

· (A, m) = regular $\Leftrightarrow \exists N \in \mathbb{N} : \operatorname{Tor}^{A}_{\ge N}(A/m, A/m) = 0$

$\Leftrightarrow \forall M$ = f.g. A-module : M has a finite free resolution.

($\operatorname{Tor}(M, A/m)$)

A = noeth. (not nec. local) ring.

Prop: $a_1, \dots, a_r \in A$; $P \supseteq (a_1, \dots, a_r)$ $\underline{\text{minimal}}$ prime ideal $\Rightarrow ht\, P \le r$.

Proof: Consider A_P · $ht(P) = \dim A_P$

· $\underset{Q}{(a_1, \dots, a_r)} = P \cdot A_P$-primary in $A_P \Rightarrow r \ge \dim A_P$ //

Tool for calculating dim

$A \subseteq B$ rings. if B = integral over A ⇝ $\operatorname{Spec} B \xrightarrow{\Phi} \operatorname{Spec} A$ ⇝ Going up/down

$\boxed{Q_{r+1}}$ $Q_r \mapsto P_r = \Phi(Q_r)$ P_{r+1}

Q_1 $\vdots$ P_r

Q_0 $Q_0 \mapsto \boxed{P_0 = \Phi(Q_0)}$ P_0

$\Rightarrow \dim A = \dim B$!

Application: ∃ Noether normalization. A = f.g. k-algebra, gens $= a_1, \dots, a_r$

$\Rightarrow y_1, \dots, y_d \subseteq \operatorname{Span}_k(a_1, \dots, a_r)$: $k[y_1, \dots, y_d] \xhookrightarrow{\text{integral}} A$

Ex: $\dim k[x_1, \dots, x_n]$ ② Know that $\dim k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} = n$

$P_0 \subset P_1 \subset \dots \subset P_\ell = m \overset{!}{=} (x_1 - c_1, \dots, x_n - c_n)$ $(c \in k)$

$k[x_1, \dots, x_n] \underset{\text{integral}}{\hookrightarrow} \bar{k}[x_1, \dots, x_n]$

Theorem. A = f.g. k-algebra, A = domain. $\dim A = \text{tr-deg}_k\, Q(A)$

Quot $k[x_1 \dots x_d]$

Proof. $k[y_1, \dots, y_d] \xhookrightarrow{\text{finite}} A \Rightarrow \dim A = d$

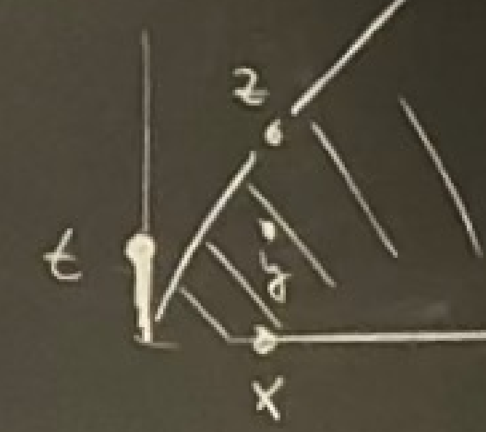
$\Downarrow$

$k(y_1, \dots, y_d) \xhookrightarrow{\text{finite}} Q(A)$ //

$Q(A)$ = finite extension of $k(x_1, \dots, x_d)$

$\Rightarrow d =$ „$\text{tr-deg}_k\, Q(A)$".

Ex: $A = k[x,y,z]/(y^2 - xz)$ = domain $= k[H]$: $H =$



$y/x = t$

$\Rightarrow k[H] \subseteq k[x^{\pm 1}, y^{\pm 1}]$ $= k[Z^2]$

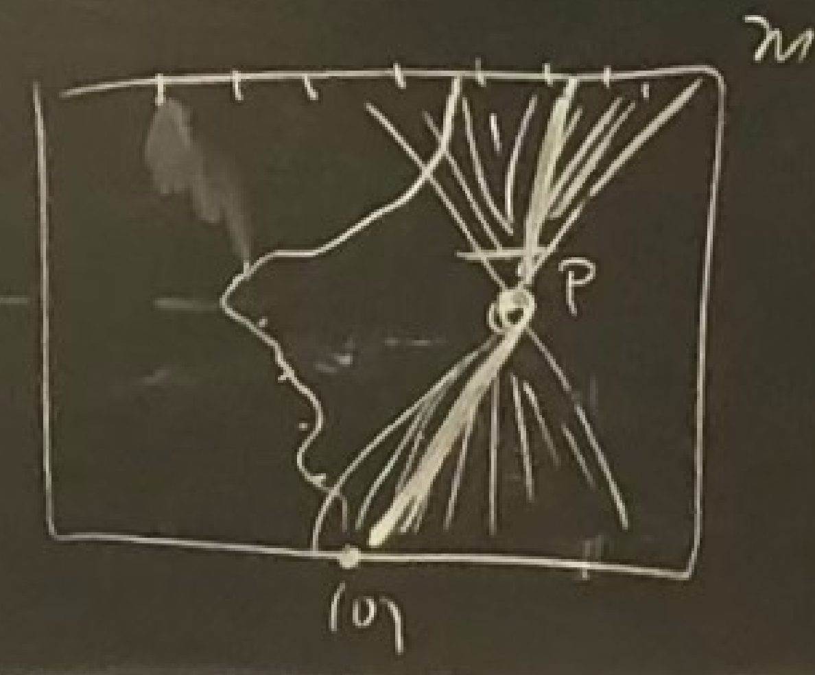
$\Rightarrow$ Quot $k[H] = k(x,y)$

$\Rightarrow \dim A = 2$.

$\dim A = \max\limits_{m \in \operatorname{MaxSpec}} \dim A_m$.

∃ noeth A: $\dim A = \infty$.

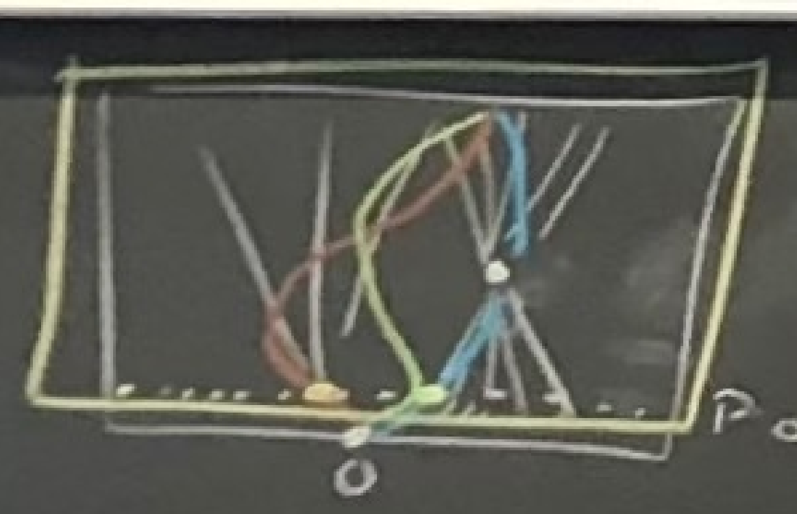
Theorem. A = f.g. k-algebra, A = domain.
$P \in \operatorname{Spec} A$ fixed. $\Rightarrow \dim A = \dim A_P + \dim A/P$.
($\geq$ is trivial)



Proof ① w.l.o.g.: $\operatorname{ht}(P) = 1$ ✓

② $k[y_1 \dots y_d] \xhookrightarrow{\text{finite}} A$

$\boxed{k[y] \cap P}$ $\vdots$ 0 $\boxed{P = PI \quad P}$ $\vdots$ (0) $\Longrightarrow$ w.l.o.g.: $A = k[y_1, \dots, y_d]$



Let $A = k[y_1, \dots, y_d]$ (= factorial ring). let P = PI of height 1. $\underset{\text{in } 5'}{\overset{\text{will come}}{\Longrightarrow}}$ $P = (f)$ principal (i.e. f = irreducible)

- $\dim A_P = \operatorname{ht} P = 1$
- $\dim A/P = \dim A/f = \dim A - 1$.

Prop: A = domain. noeth. A = factorial $\Longleftrightarrow$ PI of height 1 are principal.

$\Downarrow$ [f = irred $\Rightarrow (f)$ = prime ideal]

Proof:

(⇒) We know irred ⟺ prime. Let P = PI of height 1. Let $f \in P$, $f \neq 0 \Rightarrow$ can choose f = irred.

$\Rightarrow \underset{\text{prime}}{(f)} \subseteq P \Rightarrow (f) = P$ because of $\operatorname{ht} P = 1$.

(⇐) Let f = irred. element of A. Goal: $f \mid ab \Rightarrow f \mid a$ or $f \mid b$.

⇘ Let P = minimal PI over (f). $(P \supseteq (a_1, \dots, a_r) \Rightarrow \operatorname{ht} P \leq r)$

$\Rightarrow \operatorname{ht} P \leq 1 \Rightarrow \operatorname{ht} P = 1 \Rightarrow P = (g) \Rightarrow (f) \subseteq P = (g) \Rightarrow g \mid f$

$\Rightarrow (f) = (g)$ □

Remark: ① $a \in A$ non-zero div. $\Rightarrow$ min. prime ideals above a have ht = 1
„Krull's principal ideal theorem"

② A = factorial domain. $\Rightarrow$ max ideal in A_P are principal ($\operatorname{ht} P = 1 \Rightarrow \dim A_P = 1$) } $\operatorname{ht} P = 1$ in A-factorial ⇝ A_P = regular ‖ factorial ⇒ regularity in ht 1 (in codim 1)

G = finite, abelian group ($G = \mu_r$ = cyclic group with r elements)

$G \curvearrowright \mathbb{C}^n$ ↙ $g \in G \rightsquigarrow g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ automorphism

$\rightsquigarrow$ diagonizable i.e.: $\mathbb{C}^n = \oplus$ eigenspaces.

$\exists$ decomp. $\mathbb{C}^n = \bigoplus_\chi V_\chi$, χ = character, i.e. $\chi : G \longrightarrow \mathbb{C}^*$

$g \in G$ acts on V_χ via mult. with $\chi(g)$.

Assume that $G = \langle \zeta \rangle$, ζ = r-th root of unity $\in \mathbb{C}^*$

$G \curvearrowright \mathbb{C}^n$ via ζ acts via $(c_1, \dots, c_n) \longmapsto (\zeta^{a_1} c_1, \dots, \zeta^{a_n} c_n)$ $a_1, \dots, a_n \in \mathbb{Z}/r\mathbb{Z}$

$\gcd(n, a_1, \dots, a_n) = 1$

Example. μ_r acts on $\mathbb{C}^2$ via $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^q \end{pmatrix}$ $(a_1 = 1, a_2 = q)$

$\mathbb{C}^n/_G = (?)$ G = finite $\curvearrowright \operatorname{Spec} A \Longrightarrow \boxed{\operatorname{Spec} A/_G = \operatorname{Spec}(A^G)}$

$G \curvearrowright A$

$G \curvearrowright X$ $X/_G$

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left] & X/_G \arrow[r, dashed] & Y \end{tikzcd}$$

G

$\mu_r \curvearrowright \mathbb{C}^n$ via $\begin{pmatrix} \zeta^{a_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{a_n} \end{pmatrix}$, $\zeta \in \mu_r$ primitive r-th root of unity.

G ↳ $\mathbb{C}^n/_{\mu_r}$ = toric variety; affine $\rightsquigarrow$ **Q**: what is its cone?

$k[\underline{x}]^G = \langle \underline{x}^c \,/\, \langle c, a \rangle = 0 \text{ in } \mathbb{Z}/r\mathbb{Z} \rangle$ $k[\underline{x}] \mathrel{\hat{=}} \mathbb{N}^n \subset \mathbb{Z}^n$

$$0 \longrightarrow M \hookrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}/r\mathbb{Z} \longrightarrow 0$$

$\underline{c} \longmapsto \langle c, a \rangle$ lattices

$\mathbb{R}^n_{\geq 0} \xrightarrow{\sim} \mathbb{R}^n_{\geq 0} \cong x_1^{c_1} \cdots x_n^{c_n} \,/\, c_i \geq 0$ cones

$\Longrightarrow \mathbb{C}^n/_G = TV(\sigma, N) = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ with · M as above · $\sigma^\vee = \mathbb{R}^n_{\geq 0}$

$\Longrightarrow \sigma = \mathbb{R}^n_{\geq 0}$, lattice $= N = \mathbb{Z}^n + \frac{1}{r} \cdot \underline{a}$

$\mathbb{C}^n/_G = (\frac{1}{r} \cdot \underline{a}$ − sing.)

$A = \mathbb{C}[x_1, \dots, x_n]$ $a = (a_1, \dots, a_n)$

$x_i \longmapsto \zeta^{a_i} \cdot x_i$

$A^G \ni x_i^r$; $x_1^{c_1} \cdots x_n^{c_n} \longmapsto \zeta^{\langle c, a \rangle} \cdot \underline{x}^c$, $x_1^{c_1} \cdots x_n^{c_n} = \underline{x}^c$

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow N \longrightarrow \operatorname{Ext}^1(\mathbb{Z}/r\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$$

elements of $N \setminus \mathbb{Z}^n$?

$\underline{(a_1, \dots, a_n)} \in \mathbb{Z}^n$ evaluated on $c \in \mathbb{Z}^n \rightsquigarrow \langle c, a \rangle$

$\Longrightarrow$ evaluated on $\underline{c \in M} \Longrightarrow r \mid \langle c, a \rangle$

$\Longrightarrow \frac{1}{r} \cdot (a_1, \dots, a_n) \boxed{\in N}$ $\in \mathbb{Q}^n$

$\Longrightarrow N = \mathbb{Z}^n + \underline{\frac{1}{r} \cdot \underline{a}} \cdot \mathbb{Z}$

$\zeta \in \mu_r$ $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^q \end{pmatrix}$: $0 \longrightarrow M \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}/r\mathbb{Z} \longrightarrow 0$

$\underline{c} \longmapsto \langle (c_1, c_2), (1, q) \rangle$

$\mathbb{Z}^2$ ↔ choosing a $\mathbb{Z}$-basis of M: $(r,0)$, $(0,r)$ | $(-q,1)$ ||

$\{(-q,1), (r,0)\}$

$$M = \mathbb{Z}^2 \xrightarrow[\iota]{\begin{pmatrix} -q & r \\ 1 & 0 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow \mathbb{Z}/r\mathbb{Z} \longrightarrow 0$$

$\sigma^\vee = \iota(\mathbb{R}^2_{\geq 0})$ $\mathbb{R}^2_{\geq 0}$

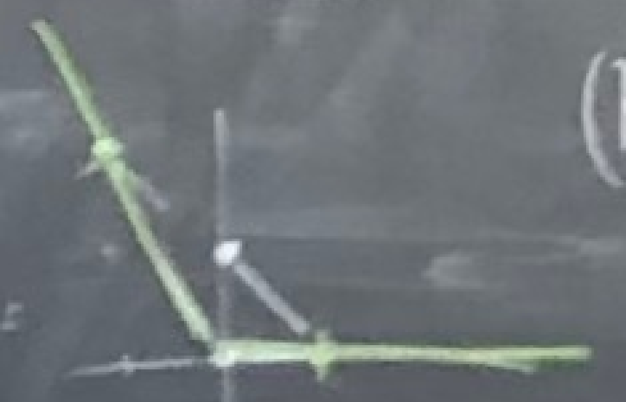
N-level: $\mathbb{Z}^2 \xhookrightarrow{\begin{pmatrix} -q & 1 \\ r & 0 \end{pmatrix}} (\mathbb{Z}^2 = N)$

$(\mathbb{R}^2_{\geq 0}) \xrightarrow{\sim} \sigma$ $\Longrightarrow \sigma = \langle (1,0); (-q, r) \rangle \subseteq \mathbb{R}^2$.

Example $r = 2$, $q = 1$

$\rightsquigarrow \sigma = \langle (1,0), (-1,2) \rangle$:

$TV(\sigma) = \mathbb{C}^2/_{\mu_r}$ with $q = 1$



2-dim cyclic quotient sing $\Longleftrightarrow$ 2-dim affine toric varieties

$\Updownarrow$

· $r \in \mathbb{N}_{\geq 2}$

· $q \in (\mathbb{Z}/r\mathbb{Z})^* \rightsquigarrow$ choose $q \in \{1, \dots, r-1\}$

topology, lens spaces

Q Is $TV(\sigma)$ smooth / singular? The most "interesting" pt. $\in TV(\sigma)$:

$e = \#\{HB\}$ $k[\sigma^\vee \cap M] \ni (x^r / r \in H$, (Hilbert basis of $\sigma^\vee$) $\rightsquigarrow \mathfrak{m}$ $(\Leftarrow r \in \sigma^\vee \cap M$ generate this s.g.)

$k[\mathbb{N}^e] \twoheadrightarrow k[\sigma^\vee \cap M]$ · # gens of $\mathfrak{m} = e = \dim \mathfrak{m}/\mathfrak{m}^2$

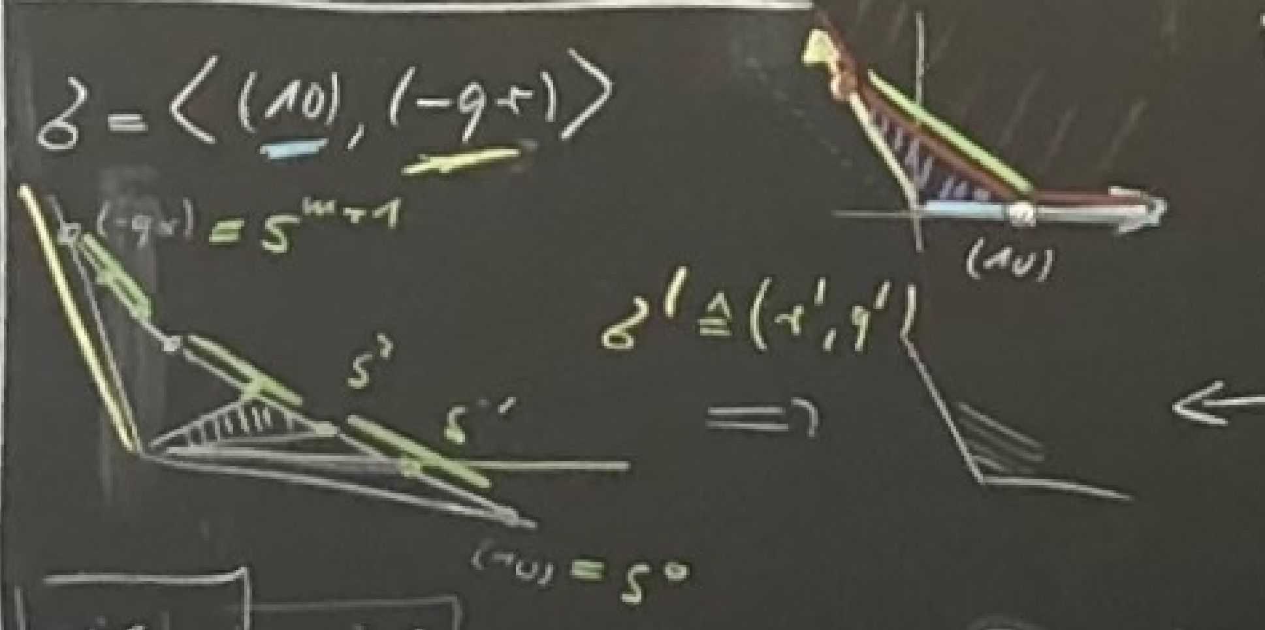
$\mathbb{C}^e \hookleftarrow TV(\sigma)$ · $\dim TV(\sigma) = \operatorname{rk} N = \operatorname{rk} M$ (= 2 in our ex-ple)

$(0, \dots, 0) \longleftarrow \mathfrak{m}$

σ = pol. cone in N $\Downarrow$ $\sigma^\vee$ has $\geq (\operatorname{rk} N)$-many rays $\cap$ HB

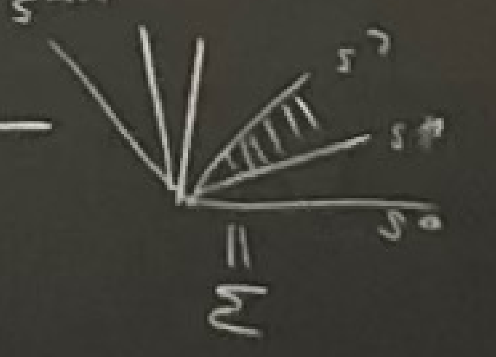
Remark: $TV(\sigma)$ = smooth $\Longleftrightarrow (\sigma, N) \cong (\mathbb{R}^n_{\geq 0}, \mathbb{Z}^n)$  $\rightsquigarrow TV = \mathbb{C}^n$

$\sigma = \langle (1,0), (-q, r) \rangle$



$\nabla := \operatorname{conv}\{(\sigma \cap N) \setminus 0\} \rightsquigarrow$ compact part of $\partial\nabla = \partial_c \nabla$

$\rightsquigarrow$ sequence $s^0, s^1, \dots, s^{m+1}$ on $\partial_c \nabla \cap N \Longrightarrow$ triangle $\operatorname{conv}\{0, s^i, s^{i+1}\}$ has only $0, s^i, s^{i+1}$ as lattice pts.



$\longleftarrow$ Σ $\Longrightarrow \boxed{TV(\sigma) \xleftarrow{\text{proper}} TV(\Sigma)} = \bigcup_i TV(s^i, s^{i+1}) \Longrightarrow \{s^i, s^{i+1}\} = \mathbb{Z}$-basis of $\mathbb{Z}^2$

$TV(\Sigma)$ = smooth variety ↳ $TV(s^i, s^{i+1}) \cong \mathbb{C}^2 \rightarrow$ smooth!

$\underline{TV(\sigma) \setminus \mathfrak{m}} \supseteq T = \underline{TV(0)}$

$\boxed{s^{i-1}, \boxed{s^i}, s^{i+1}} \rightsquigarrow$ unique $\lambda_{i-1} s^{i-1} + \lambda_i s^i + \lambda_{i+1} s^{i+1} = 0$, $\lambda_{i-1} = \pm 1$, $\lambda_{i+1} = \pm 1$ $\rightsquigarrow \boxed{s^{i-1} + s^{i+1} = b_i \cdot s^i}$ $b_i \in \mathbb{N}$, $b_i \geq 2$. $\sigma \longleftrightarrow (b_1, \dots, b_m)$

Theorem: $[b_1, \dots, b_m] := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots}} = \frac{r}{q}$

$= b_1 - \frac{1}{[b_2, \dots, b_m]}$

Ex: $r = 2$, $q = 1 \rightsquigarrow \frac{r}{q} = \frac{2}{1} = 2 = [2]$

$\underline{b_1 = 2}$

Proof.

$\sigma^\vee \rightsquigarrow [a_1, \dots, a_s] = \frac{r}{r-q}$

$\boxed{z_{i-1} z_{i+1} = z_i^{\boxed{a_i}}}$