Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

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12. Aufgabenblatt zum 13.7.2022

Problem 101. Let F be a locally free sheaf on an integral, i.e. irreducible and reduced scheme X. Show that, for open subsets $U \subseteq X$, the restriction map $\Gamma(X, F) \to \Gamma(U, F)$ is injective.
Give counter examples for the cases when one of the assumptions is violated.

Problem 102. a) Show directly that the diagonal $\Delta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is a closed embedding. What is the homogeneous ideal of $\Delta(\mathbb{P}^1_{\mathbb{C}}) \subseteq \mathbb{P}^3_{\mathcal{C}}$ after additionally using the Segre embedding? Do you see the Veronese embedding within this picture?

b) Let $X := \mathbb{A}^1_{\mathbb{C}} \cup \mathbb{A}^1_{\mathbb{C}}$ glued along the common $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. Show directly that there are affine open $U_1, U_2 \subseteq X$ such that either $U_1 \cap U_2$ is not affine or that $U_1 \cap U_2 = U$ is affine with $U_i = \operatorname{Spec} A_i$ and $U = \operatorname{Spec} B$ such that $A_1 \otimes_{\mathbb{C}} A_2 \to B$ is not surjective.

c) In the situation of (b) show that $\Delta(X) \subseteq X \times_{\operatorname{Spec} \mathbb{C}} X$ is not a closed subset.

Problem 103. a) Show that d-dimensional k-varieties (with a perfect field k) are birational equivalent to hypersurfaces in $\mathbb{P}^{d+1}$.
(*Hint:* Use the theorem of the primitive element.)

b) Let $f, g \in k[x]$ be two different polynomials with simple roots. Construct a hypersurface of $\mathbb{C}^2$ that is birational equivalent to $V(y^2 - f(x),\, z^2 - g(x)) \subseteq \mathbb{C}^3$.

Problem 104. Assume that the ring A is factorial. Show that this implies $\operatorname{Pic}(\operatorname{Spec} A) = 0$, i.e. every invertible sheaf on $\operatorname{Spec} A$ is isomorphic to $\mathcal{O}_{\operatorname{Spec} A}$.

(*Hint:* For invertible sheaves $\mathcal{L}$ one is supposed to use the cocycle description on an open covering $\{D(g_i)\}$ with $\mathcal{L}|_{D(g_i)} \cong \mathcal{O}_{D(g_i)}$, cf. Problem 95. Via induction by the overall number of prime factors of the g_i, one can reduce the claim to the special case that all elements g_i are prime. Now, using again Problem 95, one can attain that $h_{ij} \in A^*$ for all i, j.)

Problem 105. Show (by using the toric language via polytopes in $M_{\mathbb{R}}$) that the blowing up of $\mathbb{P}^2$ in two points is isomorphic to the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ in one single point.