X, Y = affine schemes, $f: \operatorname{Spec} A \to \operatorname{Spec} B$ $\quad f_* \tilde{M} = \tilde{M}_B$

$\| \quad \|$

$\operatorname{Spec} A \quad \operatorname{Spec} B \qquad \tilde{M} \updownarrow \tilde{N} \qquad f^* \tilde{N} = \widetilde{N \otimes_B A}$

$M \quad N \qquad B \xrightarrow{\varphi} A \quad J \subseteq B \rightsquigarrow J \otimes_B A \longrightarrow A$, image $= \varphi(J) \cdot A$

Remark $f: X \to Y$ $\quad I \subseteq \mathcal{O}_X$ ideal sheaf $\longrightarrow f_* I \subseteq f_* \mathcal{O}_X$ ideal sheaf

$J \subseteq \mathcal{O}_Y$ $\quad$ – " – $\longrightarrow f^* J \to f^* \mathcal{O}_Y = \mathcal{O}_X$ $\quad$ ideal sheaf.

image $=: f^{-1} J \cdot \mathcal{O}_X$

Def. X = noeth. scheme, $\mathcal{F} = \mathcal{O}_X$-module. $\mathcal{F}$ = „quasi-coherent"

$\iff \exists\, X = \bigcup_\alpha U_\alpha$, $U_\alpha = \operatorname{Spec} A_\alpha$, $\mathcal{F}_{|U_\alpha} = \tilde{M}_\alpha$, M_α = some A_α-module.

(„coherent" $\iff M_\alpha$ = f.g. A_α-modules

$\left(\iff \text{it is the cokernel of some } \mathcal{O}_X^{r} \longrightarrow \mathcal{O}_X^{s} \to \mathcal{F} \to 0 \right)$ (locally))

Prop. ① $X = \operatorname{Spec} A \rightsquigarrow \mathcal{F}|X$ is q.c. $\iff \forall f \in A$ $\quad \Gamma(X, \mathcal{F})_f \xrightarrow{\sim} \Gamma(D(f), \mathcal{F})$ $\quad (*)$

② Kernels, images, cokernels of maps of q.c. sheaves are q.c.

③ $f: X \to Y \rightsquigarrow f_*, f^*$ respects „q.c."

④ $X = \operatorname{Spec} A$, $\mathcal{F}$ = q.c. $|X \implies \exists$ A-module M: $\mathcal{F} \cong \tilde{M}$.

Proof. ① $X = \operatorname{Spec} A$ $\quad \boxed{\widetilde{\Gamma(X, \mathcal{F})} \xrightarrow{\Phi} \mathcal{F}}$ $\qquad M \longmapsto \tilde{M}$ on $\operatorname{Spec} A$

A-module

$\sim$: A-modules $\longrightarrow \mathcal{O}_A$-sheaves

Φ = isom $\iff (*)$ $\qquad \mathcal{F}(X)$ (on $D(f)$. $\quad \Gamma(X, \mathcal{F})_f \longrightarrow \Gamma(D(f), \mathcal{F}) = A_f$-module

$\Gamma(X, \mathcal{F}) \xrightarrow{res} \Gamma(D(f), \mathcal{F})$

② $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ on X

w.l.o.g.: $X = \operatorname{Spec} A$, $\left.\begin{array}{l} \mathcal{F} = \tilde{M} \\ \mathcal{G} = \tilde{N} \end{array}\right\}$ $\tilde{M} \xrightarrow{\varphi} \tilde{N} \rightsquigarrow M \to N \rightsquigarrow 0 \to K \to M \to N \Rightarrow 0 \to \boxed{\tilde{K}} \to \tilde{M} \to \tilde{N}$

③ $f^*: X \to Y$ $\quad$ • w.l.o.g. $Y = \operatorname{Spec} B$, $\mathcal{G} = \tilde{N}$ $\qquad \tilde{M}_p = M_p$

$f^* \mathcal{G} \quad \mathcal{G}$ $\quad$ • $X \supset U = \operatorname{Spec} A \rightsquigarrow f_u: U \to Y$, $(f^* \mathcal{G})_{|U} = (f_u)^* \mathcal{G}$ $\to$ OK

f_* Assume $\mathcal{F}/X$ q.c., use covering $X = \bigcup_\alpha U_\alpha$, $U_\alpha = \operatorname{Spec} A_\alpha$, $\mathcal{F}_{|U_\alpha} = \widetilde{M_\alpha}$ ($M_\alpha = A_\alpha$-mod.)

$j_\alpha : U_\alpha \hookrightarrow X \rightsquigarrow \mathcal{F} \longrightarrow \underbrace{(j_\alpha)_* (j_\alpha^*)}\mathcal{F}$

$V \subseteq X$ $\quad \mathcal{F}(V) \longrightarrow \mathcal{F}(U_\alpha \cap V)^{\mathcal{F}_{|U_\alpha}}$

$\Longrightarrow \quad j_{\alpha\beta} : U_\alpha \cap U_\beta \hookrightarrow X$

$0 \to \mathcal{F} \to \bigoplus_\alpha (j_\alpha)_* (j_\alpha^* \mathcal{F}) \to \bigoplus_{\alpha,\beta} (j_{\alpha\beta})_* (j_{\alpha\beta}^* \mathcal{F})$

$0 \to \mathcal{F}(V) \to \bigoplus_\alpha \mathcal{F}(U_\alpha \cap V) \to \bigoplus_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta \cap V)$

Correction. $U_\alpha \cap U_\beta = \bigcup_\nu W_{\alpha\beta,\nu}$; $j_{\alpha\beta,\nu} : W_{\alpha\beta,\nu} \hookrightarrow X$

$\Rightarrow 0 \to \mathcal{F} \to \bigoplus_\alpha (j_\alpha)_* (j_\alpha^*) \mathcal{F} \longrightarrow \bigoplus_{\alpha,\beta,\nu} (j_{\alpha\beta,\nu})_* (j_{\alpha\beta,\nu}^*) \mathcal{F} \quad \Big| \; f_*$

$\Rightarrow 0 \to f_* \mathcal{F} \longrightarrow \bigoplus_\alpha (f_\alpha)_* (j_\alpha^* \mathcal{F}) \longrightarrow \bigoplus_{\alpha,\beta,\nu} (f_{\alpha\beta,\nu})_* (j_{\alpha\beta,\nu}^*) \mathcal{F}$

$f_\alpha : U_\alpha \hookrightarrow X \longrightarrow Y$

$\operatorname{Spec} A_\alpha \longrightarrow \operatorname{Spec} B$

$f_{\alpha\beta\nu} : \operatorname{Spec} A_{\alpha\beta\nu} \longrightarrow \operatorname{Spec} B$

is of the form $\widetilde{M}$ for some M $\quad$ similar $\quad \Longrightarrow f_* \mathcal{F} = \widetilde{L}$ for some L. (on $Y = \operatorname{Spec} B$)

④ $id : X \to X = \operatorname{Spec} A$

Closed embedding

Def $Z \to X$ is cl. emb. $\Leftrightarrow$ locally (on the target): $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$.

Prop $Z \xrightarrow{f} X$ morphism of schemes. f = cl. emb. $\Leftrightarrow$ • $Z \hookrightarrow X$ is, i.e., cl. emb. on the top. level,

• $\mathcal{O}_X \longrightarrow f_* \mathcal{O}_Z$ is surjective.

($X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/I$: $\mathcal{O}_X = \widetilde{A}$, $\mathcal{O}_Z = \widetilde{A/I}$, $f_* \mathcal{O}_Z = \widetilde{A/I}$)

Proof. ($\Rightarrow$) $\quad$ (✓)

($\Leftarrow$) $Z \overset{f}{\hookrightarrow} X$; $\mathcal{J} := \operatorname{Ker}(\mathcal{O}_X \to f_* \mathcal{O}_Z)$ q.c. ideal sheaf $\rightsquigarrow$ if $X = \operatorname{Spec} A \Rightarrow \mathcal{J} = \widetilde{I}$ for an ideal $I \subseteq A$.

$\Rightarrow f_* \mathcal{O}_Z = \widetilde{A/I} \Rightarrow Z' := \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A = X$

$\theta \circlearrowright Z$

$P \in X \to Z = \{P \in X \mid (f_* \mathcal{O}_Z)_P \neq 0\} = Z'$

$i : Z \hookrightarrow X$ cl. emb. $\rightsquigarrow \mathcal{J} = \operatorname{Ker}(\mathcal{O}_X \to i_* \mathcal{O}_Z)$ $\quad \mathcal{J} =: \mathcal{J}_{Z \hookrightarrow X}$; qu. coh.!!

X = scheme ; $\mathcal{J} \subseteq \mathcal{O}_X$ ideal sh. $\mathcal{O}_X/\mathcal{J}$ – does not come from a closed emb.!

Example: $X = \mathbb{P}^n_{\mathbb{C}}$; $z_0, \ldots, z_n$ $\quad Z = V(\underline{z_0})$ ($\cong \mathbb{P}^{n-1}$) $\rightsquigarrow$ Locally: $\mathbb{C}[\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}] = A_i$ ($i = 0, \ldots, n$)

equ. of $Z \subseteq X$ in U_i : $Z \cap U_i = V(\frac{z_0}{z_i})$ on $U_i = \operatorname{Spec} A_i$ ($\cong \mathbb{C}^n$)

$\Rightarrow \mathcal{J}$:= ideal sheaf : $\mathcal{J}_{|U_i} = \widetilde{\left(\frac{z_0}{z_i}\right)}$ $\quad (\mathcal{J}_{|U_i})_{|U_j} = \widetilde{\left(\frac{z_0}{z_i}\right)}$ $\quad (\mathcal{J}_{|U_j})_{|U_i} = \widetilde{\left(\frac{z_0}{z_j}\right)}$

i.e. $\boxed{\mathcal{J}_{|U_i} = \frac{z_0}{z_i} \cdot \mathcal{O}_{U_i}} \hookrightarrow \mathcal{O}_{U_i}$ $\quad \cong \mathcal{O}^1_{U_i}$

$\left(\frac{z_0}{z_i}\right) = \left(\frac{z_0}{z_j}\right)$ inside $A_{ij} = \mathbb{C}[\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}, \frac{z_0}{z_j}, \ldots, \frac{z_n}{z_j}]$

$\frac{z_0}{z_j} \cdot \boxed{\frac{z_j}{z_i}}$ = unit in

$\mathcal{J}$ = invertible sheaf

$\frac{1}{z_0}$

On the other hand: $\{$inv. sheaves on $\mathbb{P}^n\}/_{\sim} =: \operatorname{Pic} \mathbb{P}^n = \mathbb{Z}$

$\mathbb{C}[z_0, \ldots, z_n] \supseteq \mathcal{O}(d) \longleftarrow d$

$d = -1$

$z_i^d \cdot \mathbb{C}[\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}] = \mathcal{O}(d)_{|U_i}$

$(z_0, \ldots, z_n)^d$ (if $d > 0$)

$\Rightarrow \mathcal{J}_{V(z_0) \hookrightarrow \mathbb{P}^n} = \mathcal{O}(-1)$

$F(z_0, \ldots, z_n) \in \mathbb{C}[z_0, \ldots, z_n]$

homog., degree $= d \geq 1$

$Z := V(F)$

$\mathcal{J} := \mathcal{J}_{Z \hookrightarrow X} \cong \mathcal{O}(-d)$

$f : X \longrightarrow Y$ morph. of schemes

U_i

ideal sheaf I $\quad Z$ closed subscheme. $\rightsquigarrow \overline{f(Z)} \subseteq Y$ carries a natural scheme structure.

ideal sheaf $\mathcal{J}$ $\quad (i_* \mathcal{O}_Z = \mathcal{O}_X/I)$

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{\varphi} & f_* \mathcal{O}_X \\ \downarrow & & \downarrow \\ \mathcal{O}_Y/\mathcal{J} & \dashrightarrow & f_* \mathcal{O}_X / f_* I \hookrightarrow f_*(\mathcal{O}_X/I) \\ \overline{f(Z)} & \longleftarrow & Z \end{array}$$

$0 \to I \to \mathcal{O}_X \to \mathcal{O}_X/I \to 0 \quad | \; f_*$

$0 \to f_* I \to f_* \mathcal{O}_X \to f_*(\mathcal{O}_X/I)$

$f_* \mathcal{O}_X / f_* I$

$\mathcal{J}$ wants to be as large as possible $\Rightarrow \mathcal{J} := \varphi^{-1}(f_* I)$