Blowing up of $I \subseteq A$ $\rightsquigarrow$ $\tilde{X} = Bl_A(I) \xrightarrow{\pi} X = \operatorname{Spec} A$ : $I = (g_1, \dots, g_k)$

$\mathcal{J} = \pi^{-1}\tilde{I} \cdot \mathcal{O}_{\tilde{X}}$ $\qquad$ $U_i = \operatorname{Spec} A\left[\frac{g_1}{g_i}, \dots, \frac{g_k}{g_i}\right] \subseteq Bl_A(I)$, $\quad A_{g_i} \supseteq A\left[\frac{g_1}{g_i}, \dots, \frac{g_k}{g_i}\right]$

$I \subseteq A \rightsquigarrow \tilde{I} \subseteq \tilde{A} = \mathcal{O}_{\operatorname{Spec} A} = \mathcal{O}_X$

$E = V(\mathcal{J}) \subseteq \tilde{X} \dashrightarrow V(I) \subseteq X$

How does $\mathcal{J}$ look like on $U_i = \operatorname{Spec} A[g/g_i]$ : $(g_1, \dots, g_k) \cdot A[g/g_i] = (g_i) \cdot A[g/g_i]$, $\quad (g_i) \cong A[g/g_i]$ : principal ideal gen by non-zero div

$\Longrightarrow$ $\mathcal{J} \subseteq \mathcal{O}_{\tilde{X}}$ is invertible ideal sheaf

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**Prop:** $Y \xrightarrow{f} \operatorname{Spec} A \supseteq V(I)$, $\quad f^{-1}\tilde{I} \cdot \mathcal{O}_Y = $ invertible (ideal sheaf) $\Longrightarrow$ $\exists$ unique factorization:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[d, dashed, "F"'] & & \operatorname{Spec} A \\ \tilde{X} \arrow[urr, "\pi"'] & & \end{tikzcd}$$

**Proof** wlog $Y = \operatorname{Spec} B \rightsquigarrow A \xrightarrow{\varphi} B$

$I = (g_1, \dots, g_k) \longmapsto \varphi(I) \cdot B = (\varphi(g_1), \dots, \varphi(g_k)) = (b)$ $\quad$ ($b \in B$ non-zero div)

$\longrightarrow$ $\cdot\ \exists\, s_i \in B : \varphi(g_i) = b \cdot s_i$

$\cdot\ \exists\, \lambda_i : b = \sum_{i=1}^{k} \lambda_i \cdot \varphi(g_i) = \sum \lambda_i \cdot b \cdot s_i$

$\Longrightarrow \boxed{\sum_{i=1}^{k} \lambda_i s_i = 1}$ i.e. $(s_1, \dots, s_k) = (1)$ $\rightsquigarrow$ $D(s_i) \subseteq \operatorname{Spec} B$ cover everything, i.e. $\bigcup_i D(s_i) = \operatorname{Spec} B$.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr] & B \arrow[r] & B_{s_i} \\ & A[g/g_i] \arrow[ur, dashed] & \end{tikzcd}$$

$\varphi(g_i) = b \cdot s_i \in B_{s_i}$

$\varphi(g_k/g_i) = s_k/s_i \in B_{s_i}$ //

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**(24) Weil & Cartier divisors**

$A =$ domain, noeth. **Def** $I \subseteq \operatorname{Quot}(A)$ is a „fractional ideal" $:\Longleftrightarrow \exists\, a \in A : a \cdot I \subseteq A$ $\quad (I \subseteq \frac{1}{a} \cdot A)$

$\Longleftrightarrow I \subseteq Q(A)$, $I =$ f.g. $A$-module.

$I^\vee := \operatorname{Hom}_A(I, A) = \{ q \in Q(A) \mid q I \subseteq A \}$ $\quad$ $\varphi : I \to A$ $A$-linear, $x \in I$, $q := \frac{\varphi(x)}{x} \in Q(A)$

**Lemma:** a) $a \in A \setminus \{0\}$, $P \in \operatorname{Ass}(A/a) \Longrightarrow \boxed{P^\vee \supsetneq A}$ ($A \subseteq P^\vee$ is trivial)

(i.e. $\exists\, A/P \overset{\iota}{\hookrightarrow} A/a$, $\quad 1 \longmapsto b$, $\quad P = \operatorname{Ann}_{A/a}(b) = ((a) : (b))$)

b) $A =$ normal, $\boxed{P^\vee \supsetneq A} \Longrightarrow A \supseteq \boxed{P \cdot P^\vee \supsetneq P}$ (clear: $P \subseteq P \cdot P^\vee$)

c) $(A, P) =$ local $(\neq 0)$ $\boxed{P \cdot P^\vee = A} \Longrightarrow P =$ principal $\quad$ ($\rightsquigarrow$ ht $P = 1$, $A =$ regular ring, $\dim A = 1$)

**Proof** (a) $a \in A$, $a \neq 0$; $P = ((a) : (b)) \Longrightarrow b \cdot P \subseteq (a) \Longrightarrow \frac{b}{a} \cdot P \subseteq A \Rightarrow \frac{b}{a} \in P^\vee$

check: $\frac{b}{a} \notin A$, i.e. $b \notin (a)$

if ... $\longrightarrow b = 0$ in $A/a \Longrightarrow \iota \neq$ injective ↯ //

(b) Assume $P \cdot P^\vee = P \Longrightarrow \forall k \geq 1 : P \cdot (P^\vee)^k = P \Longrightarrow p \cdot (P^\vee)^k \subseteq P \subseteq A \rightsquigarrow (P^\vee)^k \subseteq \frac{1}{p} A$. $\forall k$ $\quad$ ($p \in P$)

$x \in P^\vee \subseteq Q(A) \rightsquigarrow \forall k : x^k \in \frac{1}{p} A$

$\Longrightarrow A[x] \subseteq \frac{1}{p} A =$ f.g. $A$-module $\Longrightarrow x$ is integral over $A \Longrightarrow x \in A \Longrightarrow P^\vee \subseteq A$.

(c) Choose $a \in P \setminus P^2$ (hope $P = (a)$) $\quad aP^\vee \subseteq A$ ; claim $aP^\vee = A$

if not: $aP^\vee \subseteq P \Longrightarrow a \underbrace{PP^\vee}_{A} \subseteq P^2 \Longrightarrow a \in P^2$ ↯

$(a) = a \cdot P^\vee P = (aP^\vee) \cdot P = P$.

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**Prop** ① $(A, P) =$ local, normal, 1-dim $\Longrightarrow A =$ regular (i.e. $P =$ principal)

② $A =$ normal, $a \in A \setminus 0 \Longrightarrow P \in \operatorname{Ass}(A/a)$ are minimal, i.e. ht$(P) = 1$.

**Proof** ① $\dim A = 1 \Longrightarrow \forall a \in P : P \in \operatorname{Ass}(A/a) \xRightarrow{a, b, c} P =$ principal.

② $P \in \operatorname{Ass}(A/a) \Longrightarrow (A_P, P) =$ local $\Longrightarrow P = (p)$ inside $A_P \Longrightarrow$ ht $P = 1 \Longrightarrow$ minimal!

$A$ = domain, noeth. **Def** $I \subseteq \mathrm{Quot}(A)$ is a „fractional ideal" $:\Longleftrightarrow \exists a \in A \quad a \cdot I \subseteq A$ $\left( I \subseteq \frac{1}{a} A \right)$

$\Longleftrightarrow I \subseteq Q(A)$, $I$ = f.g. $A$-module.

$I^\vee = \mathrm{Hom}_A(I, A)$
$= \{ q \in Q(A) \mid q I \subseteq A \}$
$\cdot\, q : I \to A$ $A$-linear, $\tilde{q} \in Q(A)$, $x \in I$, $q = \frac{q(x)}{x}$

**Lemma** a) $a \in A \setminus \{0\}$, $P \in \mathrm{Ass}(A/a) \Longrightarrow \boxed{P^\vee \supsetneq A}$ ($A \subseteq P^\vee$ is trivial)

(i.e. $\exists\ A/P \overset{\iota}{\hookrightarrow} A/a$
$1 \longmapsto b$
$P = \mathrm{Ann}_{A/a}(b) = ((a):(b))$)

$a \in A$, $aP \subseteq P$ $\Downarrow P^\vee$

b) $A$ = normal, $\boxed{P^\vee \supsetneq A} \Longrightarrow A \supseteq \boxed{P \cdot P^\vee \supsetneq P}$ (clear: $P \subseteq P \cdot P^\vee$)

c) $(A, P)$ = local $\boxed{P \cdot P^\vee = A} \Longrightarrow P$ = principal ($\rightsquigarrow \mathrm{ht}\, P = 1$, $A$ = regular ring, $\dim A = 1$)

both $(\neq 0)$

**Proof** (a) $a \in A$, $a \neq 0$; $P = ((a):(b)) \Longrightarrow b \cdot P \subseteq (a) \Longrightarrow \frac{b}{a} \cdot P \subseteq A \Rightarrow \frac{b}{a} \in P^\vee$

check: $\frac{b}{a} \notin A$, i.e. $b \notin (a)$

if — " — $\Rightarrow b = 0$ in $A/a \Rightarrow \iota \neq$ injective ↯ //

(b) Assume $P \cdot P^\vee = P \Longrightarrow \forall k \geq 1$ $P \cdot (P^\vee)^k = P \Longrightarrow p \cdot (P^\vee)^k \subseteq P \subseteq A \Rightarrow (P^\vee)^k \subseteq \frac{1}{p} A$ $\forall k$ ($p \in P$)

$x \in P^\vee \rightsquigarrow \forall k$ $x^k \in \frac{1}{p} A$
$Q(A) \Longrightarrow A[x] \subseteq \frac{1}{p} A$ = f.g. $A$-module $\Longrightarrow x$ is integral over $A \Longrightarrow x \in A \Longrightarrow P^\vee \subseteq A$

(c) Choose $a \in P \setminus P^2$ (hope $P = (a)$) $aP^\vee \subseteq A$; claim $aP^\vee = A$

if not: $aP^\vee \subseteq P \Longrightarrow a \underbrace{PP^\vee}_{A} \subseteq P^2 \Longrightarrow a \in P^2$ ↯

$(a) = a \cdot P^\vee P = (aP^\vee) \cdot P = P$.

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**Prop**: ① $(A, P)$ = local, normal, 1-dim. $\Longrightarrow A$ = regular (i.e. $P$ = principal)

② $A$ = normal, $a \in A \setminus 0 \Longrightarrow P \in \mathrm{Ass}(A/a)$ are minimal, i.e. $\mathrm{ht}(P) = 1$.

**Proof**: ① $\dim A = 1 \Longrightarrow \forall a \in P$: $P \in \mathrm{Ass}(A/a) \xRightarrow{a, b, c} P$ = principal.

② $P \in \mathrm{Ass}(A/a) \Longrightarrow (A_P, P)$ = local $\Longrightarrow P = (p)$ inside $A_P \Longrightarrow \mathrm{ht}\, P = 1 \Longrightarrow$ minimal!

**Corollary** $A$ = normal $\Longrightarrow A = \bigcap_{\mathrm{ht}(P) = 1} A_P$ $(\subseteq Q(A))$ $(*)$

**Proof** Claim: $A = \bigcap_{b \in A \setminus 0} \bigcap_{\mathrm{Ass}(A/b) \ni P} A_P$ take $\frac{b}{a} \in Q(A)$

$I = \{ x \in A \mid x \cdot \frac{b}{a} \in A \} = \{ x \mid xa \in (b) \} = ((b) : (a)) \subseteq A$ ideal.

$1 \in I \Longleftrightarrow \frac{b}{a} \in A$. $\boxed{\text{Assume } \frac{b}{a} \notin A} \rightsquigarrow I \subsetneq A \rightsquigarrow \exists$ PI $Q \supseteq I$.

(IF $I = Q$ (PI) $\Rightarrow Q = ((b) : (a)) \Longrightarrow Q = \mathrm{Ann}_{A/b}\, a \Longrightarrow Q \in \mathrm{Ass}(A/b)$

$\Longrightarrow \frac{b}{a} \notin A_Q$ (if it were, then $\exists\, q \notin Q$ $\frac{b}{a} \cdot q \in A \Longrightarrow q \in I \Longrightarrow I \not\subseteq Q$ ↯ ))

If $I$ is not a prime $\Longrightarrow \exists\, x, y : xy \in I$, $x, y \notin I$.

$\Longrightarrow \frac{xb}{a} \notin A$: otherwise: $\exists a' \in A : xb = aa'$, i.e. $x \frac{b}{a} = a' \Longrightarrow x \in I$ ↯

$A \neq I' = \{ z \in A \,/\, z \cdot \frac{xb}{a} \in A \} \supsetneq I$. $I' \ni y$ $\rightsquigarrow$ eventually: $\exists\, I''' =$ prime. ↯

$X$ = scheme: • integral.
• normal (i.e. $X = \bigcup_\alpha \mathrm{Spec}\, A_\alpha$, $A_\alpha$ = normal)

**Def**: • $D$ = prime divisor on $X \Longleftrightarrow D \subset X$ integral subscheme (closed) of codim. 1

• $f \in Q(A) \rightsquigarrow \mathrm{ord}_D f$: (i) • $f \in A \rightsquigarrow \mathrm{ord}_D f$ = length $\underbrace{\mathcal{O}_{X, \eta_D}/f}_{0\text{-dim}} < \infty$ ($\mathcal{O}_{X, \eta_D}$ = localization of $A$ by $I(D)$)

$X \supseteq U = \mathrm{Spec}\, A \ni \eta_D \Longleftrightarrow U \cap D \neq \phi$

$0 \to \mathcal{O}/f \xrightarrow{g} \mathcal{O}/fg \to \mathcal{O}/g \to 0$

• $\mathcal{O}_{X, \eta_D}$ = normal $\Longrightarrow m_{\eta_D} = (t) \subseteq \mathcal{O}_{X, \eta_D}$ principal
$\Longrightarrow f = t^{\mathrm{ord}_D f}$

(ii) ord is additive, i.e. $\mathrm{ord}_D(fg) = \mathrm{ord}_D(f) + \mathrm{ord}_D(g)$

(iii) $f \in K(X) = \mathrm{Quot}(\mathcal{O}_{X, \eta_D}) \Longrightarrow f = \frac{g}{h}$ $\mathrm{ord}(f) := \mathrm{ord}(g) - \mathrm{ord}(h)$ //

$X = \mathrm{Spec}\, A$, $A$ = normal.
$f \in A$: $\mathrm{ord}_D(f) \geq 0$
($D$ = Prime div. of $X$)
$f \in Q(A)$, $\mathrm{ord}_D f \geq 0$ $\forall D \xRightarrow[\text{yes}]{?} f \in A$
$\Updownarrow$
$f \in \mathcal{O}_{X, \eta_D} = A_P$ $(*)$
($P = I(D)$, $\mathrm{ht}\, P = 1$)