A = domain, noeth. **Def** $I \subseteq \mathrm{Quot}(A)$ is a „fractional ideal" $:\Longleftrightarrow \exists a \in A \quad a \cdot I \subseteq A$ $\left(I \subseteq \frac{1}{a} A\right)$

$\Longleftrightarrow I \subseteq Q(A)$, I = f.g. A-module.

$I^\vee = \mathrm{Hom}_A(I, A)$

$= \{ q \in Q(A) \mid q I \subseteq A \}$

$I \xrightarrow{q} A$ A-linear, $\widehat{Q(A)}$ $x \in I$ $q = \frac{q(x)}{x}$

Lemma a) $a \in A \setminus \{0\}$, $P \in \mathrm{Ass}(A/a) \Longrightarrow \boxed{P^\vee \supsetneq A}$ ($A \subseteq P^\vee$ is trivial)

(i.e. $\exists\, A/P \overset{\iota}{\hookrightarrow} A/a$
$1 \longmapsto b$
$P = \mathrm{Ann}_{A/a}(b) = ((a):(b))$)

$a \in A$, $aP \subseteq P$, $\Downarrow P^\vee$

b) A = normal, $\boxed{P^\vee \supsetneq A} \Longrightarrow A \supseteq \boxed{P \cdot P^\vee \supsetneq P}$ (clear: $P \subseteq P \cdot P^\vee$)

c) (A, P) = local $\boxed{P \cdot P^\vee = A} \Longrightarrow P$ = principal ($\rightsquigarrow$ ht $P = 1$, A = regular ring, dim $A = 1$)

both $(\neq 0)$

Proof (a) $a \in A$, $a \neq 0$; $P = ((a):(b)) \Longrightarrow b \cdot P \subseteq (a) \Longrightarrow \frac{b}{a} \cdot P \subseteq A \Rightarrow \underline{\frac{b}{a} \in P^\vee}$

check: $\frac{b}{a} \notin A$, i.e. $b \notin (a)$

if - " - $\Rightarrow b = 0$ in $A/a \Rightarrow \iota \neq$ injective ↯ //

(b) Assume $P \cdot P^\vee = P \Rightarrow \forall k \geq 1$ $\underset{\substack{\cup\\ p}}{P} \cdot (P^\vee)^k = P \Longrightarrow p \cdot (P^\vee)^k \subseteq P \subseteq A \Rightarrow (P^\vee)^k \subseteq \frac{1}{p} A$ $\forall k$

$\underset{Q(A)}{x \in P^\vee} \rightsquigarrow \forall k$ $x^k \in \frac{1}{p} A$
$\Longrightarrow A[x] \subseteq \frac{1}{p} A$ - f.g. A-module $\Longrightarrow x$ is integral over $A \Longrightarrow x \in A \Longrightarrow P^\vee \subseteq A$.

(c) Choose $a \in P \setminus P^2$ (hope: $P = (a)$) $aP^\vee \subseteq A$; claim: $aP^\vee = A$

if not: $aP^\vee \subseteq P \Longrightarrow a \underbrace{PP^\vee}_{A} \subseteq P^2 \Longrightarrow a \in P^2$ ↯

$(a) = a \cdot P^\vee P = (aP^\vee) \cdot P = P$.

Prop ① (A, P) = local, normal, 1-dim. $\Longrightarrow A$ = regular (i.e. P = principal)

② A = normal, $a \in A \setminus 0 \longrightarrow P \in \mathrm{Ass}(A/a)$ are minimal, i.e. ht$(P) = 1$.

Proof: ① dim $A = 1 \Longrightarrow \forall a \in P$: $P \in \mathrm{Ass}(A/a) \overset{a,b,c}{\Longrightarrow} P$ = principal.

② $P \in \mathrm{Ass}(A/a) \Longrightarrow (A_P, P)$ = local $\Longrightarrow P = (p)$ inside A_P $\Longrightarrow$ ht $P = 1 \Longrightarrow$ minimal!

Corollary A = normal $\Longrightarrow A = \bigcap_{\mathrm{ht}(P)=1} A_P$ $(\subseteq Q(A))$ $(*)$

Proof claim: $A = \bigcap_{b \in A \setminus 0} \bigcap_{\mathrm{Ass}(A/b) \ni P} A_P$ take $\frac{b}{a} \in Q(A)$

$I = \{ x \in A \mid x \cdot \frac{b}{a} \in A \} = \{ x \mid xa \in (b) \} = ((b):(a)) \subseteq A$ ideal.

$1 \in I \Longleftrightarrow \frac{b}{a} \in A$. $\boxed{\text{Assume } \frac{b}{a} \notin A} \rightsquigarrow I \subsetneq A \rightsquigarrow \exists$ PI $Q \supseteq I$.

(IF $I = Q$ (PI) $\Rightarrow Q = ((b):(a)) \Longrightarrow Q = \mathrm{Ann}_{A/b}\, a \Longrightarrow Q \in \mathrm{Ass}(A/b)$

$\Longrightarrow \frac{b}{a} \notin A_Q$ (if it were, then $\exists q \notin Q$ $\frac{b}{a} \cdot q \in A \Rightarrow q \in I \Rightarrow I \nsubseteq Q$ ↯)

If I is not a prime $\Longrightarrow \exists x, y : xy \in I$, $x, y \notin I$.

$\Longrightarrow \frac{xb}{a} \notin A$: otherwise: $\exists a' \in A : xb = aa'$, i.e. $x \frac{b}{a} = a' \Longrightarrow x \in I$ ↯

$A \neq \underline{I'} = \{ z \in A \mid z \cdot \frac{xb}{a} \in A \} \supsetneq I$. $I' \ni y$ $\rightsquigarrow$ eventually: $\exists I''' =$ prime. ↯

X = scheme: • integral.
• normal (i.e. $X = \bigcup_\alpha \mathrm{Spec}\, A_\alpha$, A_α = normal)

Def: • D = prime divisor on $X \Longleftrightarrow D \subset X$ integral subscheme (closed) of codim. 1

• $f \in Q(A) \rightsquigarrow \mathrm{ord}_D f$: (i) • $f \in A \rightsquigarrow \mathrm{ord}_D f$ = length $\underbrace{\mathcal{O}_{X,\eta_D}/f}_{0\text{-dim}} < \infty$ ($\mathcal{O}_{X,\eta_D}$ = localization of A by $I(D)$)

$X \supseteq U = \mathrm{Spec}\, A \ni \eta_D \Longleftrightarrow U \cap D \neq \emptyset$

$0 \to \mathcal{O}/f \xrightarrow{g} \mathcal{O}/fg \to \mathcal{O}/g \to 0$

• $\mathcal{O}_{X,\eta_D}$ = normal $\Longrightarrow \mathfrak{m}_{\eta_D} = (t) \subseteq \mathcal{O}_{X,\eta_D}$ principal
$\Longrightarrow f = t^{\mathrm{ord}_D f}$

(ii) ord is additive, i.e. $\mathrm{ord}_D(fg) = \mathrm{ord}_D(f) + \mathrm{ord}_D(g)$

(iii) $f \in K(X) = \mathrm{Quot}(\mathcal{O}_{X,\eta_D}) \Longrightarrow f = \frac{g}{h}$ $\mathrm{ord}(f) := \mathrm{ord}(g) - \mathrm{ord}(h)$ //

$X = \mathrm{Spec}\, A$, A = normal.
$f \in A$: $\mathrm{ord}_D(f) \geq 0$
(D = prime divisor of X)
$f \in Q(A)$, $\mathrm{ord}_D f \geq 0$ $\forall D \underset{\text{yes}}{\overset{?}{\Longrightarrow}} f \in A$
$\Updownarrow$
$f \in \mathcal{O}_{X,\eta_D} = A_P$ $(*)$
($P = I(D)$, ht $P = 1$)