X = variety / k, X = normal $\rightsquigarrow$ $K(X)$ = fct. field $\supset k$

Def. $\boxed{\text{Prime divisor}}$ $D \subset X$ closed, D = irred., reduced, $\boxed{\text{codim } 1}$

- $\dim X - \dim D = 1$
- $\eta(D)$ = generic pt. of D

$\boxed{P \hat{=} D = \text{prime div.}}$

$\boxed{f \in K(X) = \mathrm{Quot}\, A}$

$\boxed{A = \text{normal}}$ $\Rightarrow A_P =$ • normal • 1-dim. $\Rightarrow \boxed{P = \text{principal ideal in } A_P}$

$\rightsquigarrow \mathcal{O}_{X,\eta(D)} = A_P = $ 1-dim $\longleftrightarrow \boxed{\mathrm{ht}\, P = 1}$

$X \hat{=} A$, $D \hat{=} A/_P$

$X = \mathrm{Spec}\, A \ni P \hat{=} \eta_D$

$D = \mathrm{Spec}\, A/_P$

- $\mathrm{ord}_D f = K$ if $\boxed{f = A_P^* \cdot t^k \in A_P}$

$P \cdot A_P = (t) \rightsquigarrow a \in A_P$: $a = \text{unit} \cdot t^k$ $(k \in \mathbb{N})$

or · $\mathrm{ord}_P f :=$ length $A_P/(f)$ (e.g.: length $A_P/(t^k) = k$)

A = integral domain

- $\dim A$ = Krull dim ≙ ht of PI
- A = local, m = max. id, #gens(m-primary ids) take the min
- Hilbert pol. für $\mathrm{Gr}_m A$
- $A = k$-alg. $\rightsquigarrow \dim A = \mathrm{tr.deg}\, A$

max. ch.

P

(0)

ht ... ≤ Prime ch.

$X \supset \mathrm{Spec}\, A$, $f \in A \rightsquigarrow f: \mathrm{Spec}\, A \longrightarrow k$

$\cup$

D

$\left.\begin{array}{l} f_{|D} \overset{(?)}{=} 0 \quad Y/N \longleftrightarrow f \in P \\ \underline{\text{more:}}\ \mathrm{ord}_P f = \max k \text{ s.t. } f \in P^k. \end{array}\right\}$ in A_P.

Example. $X = \mathbb{A}^1_k$, i.e. $X = \mathrm{Spec}\, A$, $A = k[x]$; $P \hat{=} 0 \in X$, i.e. $P = (x) \subset k[x]$

$f = x^2 + 3x^5 \in k[x]$

$= x^2(1+3x^3) \sim x^2$ $(t = x,\ k = 2) \longrightarrow \mathrm{ord}_{0 \in \mathbb{A}^1}(f) = 2$

in $k[x]_{(x)} = (k[x] \setminus (x))^{-1} \cdot k[x]$

$\quad \| $

$\quad A_P$

$(D(x) = \mathrm{Spec}\, A_x = k[x, x^{-1}])$

Noeth. norm.

A = fin. gen. k-algebra $= k[x_1, \dots, x_n]/I \xleftarrow[\text{finite}]{\text{inject.}} k[y_1, \dots, y_\ell]$ $(\ell \le n)$

$\exists\, y_i$ = LK of $x_1, \dots, x_n$

$\mathrm{Spec} \longrightarrow \boxed{\text{fin. } \dots} \ \mathbb{A}^\ell$

$f \in K(X) = \mathrm{Quot}(A) = \mathrm{Quot}(A_P)$

1) $\mathrm{ord}_P f = k$ with $f = A_P^* \cdot t^k$

1') additivity: $\mathrm{ord}(fg) = \mathrm{ord} f + \mathrm{ord}\, g$

2) $f \in \mathrm{Quot}\, A_P \rightsquigarrow f = g/h \Rightarrow \mathrm{ord} f = \mathrm{ord}\, g - \mathrm{ord}\, h$

Ex. $X = \mathbb{A}^1_k$, $f(x) = \dfrac{x^2 + 7x^4 + 1}{x^2 + x^3}$

$\mathrm{ord}_{0 \in X} f = 0 - 2 = \underline{\underline{-2}}$

$k[x]_{(x)}$

$\dfrac{1}{x^2} \cdot \dfrac{x^2 + 7x^4 + 1}{1 + x}$

$(f \in \mathrm{Quot}\, A_P$: $f = t^{\mathrm{ord} f} \cdot (\text{elem. of } A_P^*))$

Def. „Weil-Divisor" := finite, integral lin. comb. $D = \sum_{i=1}^m g_i \cdot D_i$ $(g_i \in \mathbb{Z},\ D_i = \text{prime divisors})$

$\mathrm{Div}(X) := \{ \text{-"-} \}$ = free ab. gr. gen. by prime divisors.

Def „principal divisor": $f \in K(X)^* \rightsquigarrow \mathrm{div} f = (f) := \sum_{\substack{P = \text{prime div.} \\ \text{in } X}} \mathrm{ord}_P f \cdot P$

$(\mathrm{div}(f/g) = \mathrm{div} f - \mathrm{div}\, g)$

<u>Claim</u>: this is a <u>finite</u> sum!

w.l.o.g.:
- $X = \mathrm{Spec}\, A$
- $f \in \mathrm{Quot}\, A \rightsquigarrow$ w.l.o.g.: $\underline{\underline{f \in A}}$

$\Big\}$ $\mathrm{div}(f) = \sum_{\substack{P \in \mathrm{Spec}\, A \\ \mathrm{ht}(P) = 1}} \underset{\mathbb{N}}{\mathrm{ord}_P f} \cdot [P]$

$\forall P$: $f \in A_P$

$\mathrm{ord}_P f \ge 1 \iff f \in P \cdot A_P \iff \underline{f \in P}$ (i.e. $f \notin A_P^*$)

~~Q: $= 5[P_1] - 2[P_2] + 3[P_3]$~~

Need to show: $f \in A \Rightarrow \exists$ only finitely many $\boxed{P \in \mathrm{Spec}\, A \text{ of } \underline{\underline{\text{height } 1}}}$ $f \in P$.

$P \in \mathrm{Min}(A/(f)) \subseteq \mathrm{Ass}(A/(f))$ = finite set !!

(**Ex.** $A = k[x, y, z]$

$f = z \in (x - c, z)$ $\forall c \in k$)

$\square$

$X = \mathbb{A}^1$

(f)

‖ **Theorem.**
- $(f) \subseteq P = \mathrm{PI}$, P is minimal above (f) $\Rightarrow \mathrm{ht}\, P = 1$.
- $(f) \subseteq P$, $\mathrm{ht}\, P = 1 \Rightarrow P$ = minimal above (f), inside A (= height P in A_P) (= dim A_P)

<u>$X = \mathbb{A}^1$</u>: $f(x) = \dfrac{x^2 + 2x + 1}{x^2 + x^3}$ $\mathrm{div}(f) =$

$\deg(\mathrm{div} f) = -1$

- $x^2 + 2x + 1 = (x+1)^2$
- $x^2 + x^3 = x^2 \cdot (1 + x)$

$\mathrm{div} f = 2 \cdot [-1] - [1] - 2[0]$

$P_1 = (1 - x) \overset{\mathrm{ord}}{\Rightarrow} 1$

$P_2 = (x) \overset{\mathrm{ord}}{\Rightarrow} 2$

$k[x] \supset P \rightsquigarrow P = (x + 1)$

$\mathrm{ord}_{(x+1)} (x+1)^2 = 2$

<u>On curves</u>: $D = \sum g_i [P_i] \Rightarrow \deg D = \sum g_i$

Def. $\mathrm{Cl}(X) := \mathrm{Div}(X) \big/ \{\mathrm{div}(f) \mid f \in K(X)^*\} = \mathrm{PDiv}(X)$, i.e.

$0 \to \mathrm{PDiv}(X) \to \mathrm{Div}(X) \longrightarrow \mathrm{Cl}(X) \to 0$

$K(X)^*$ $\xrightarrow{\mathrm{div}}$

Subgroup: $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$

<u>need</u>

Goal: $\mathrm{Cl}(\mathbb{P}^n) = ?$

$f \in K(X)^*$ has $\mathrm{div}(f) = 0 \iff$

$\forall$ prime div. $P \subset X$: $\mathrm{ord}_P f = 0$

<u>Example.</u> $X = \mathrm{Spec}\, k[x]$ $\hookrightarrow f \in k[x]^* = k$

$k^* \subseteq \ker \mathrm{div}$

$\longrightarrow \left[K(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \longrightarrow \mathrm{Cl}(X) \to 0 \right]$

$Cl(\mathbb{P}^1) = \mathbb{Z}$

$\mathrm{div}(f)$, $f \in K(\mathbb{P}^1) = k(t)$ $\quad (k = \bar{k}$, e.g. $k = \mathbb{C})$

$\hookrightarrow \boxed{\deg(\mathrm{div} f) = 0}$ $\quad t = y/x$

$f(t) = \frac{g(t)}{h(t)}$ $\quad g, h \in \mathbb{C}[t]$.

Instead use homogeneous coord: $\mathbb{P}^1 = \{(x:y) \mid x, y \in k, \dots\} \Rightarrow K(\mathbb{P}^1) \ni f(x,y) = \frac{\text{homog } g(x,y)}{\text{homog } h(x,y)}$ Same degree!

$g(x,y) = \prod_{j=1}^{m} (\alpha_j x - \beta_j y)$ $\quad (\alpha_j, \beta_j \in \mathbb{C}$ not both $= 0)$

$\mathrm{div}\, g(x,y) = \sum_{j=1}^{m} (\beta_j : \alpha_j) \rightsquigarrow \deg(\mathrm{div}(g)) = m$

$\mathrm{div}\, h \quad \rightsquigarrow \deg(\mathrm{div}(h)) = m$

$\mathrm{div}(f = \frac{g}{h}) = \mathrm{div}(g) - \mathrm{div}(h) \to \underline{\deg = 0}$.

Claim 2 $D = \sum n_i P_i \in \mathrm{Div}(\mathbb{P}^1)$, $\deg D = 0 \Rightarrow \exists f : D = \mathrm{div}(f)$

$D = D_+ - D_-$ (both have degree m)

$f = \prod (\alpha_i x - \beta_i y)^{n_i}$

$P_i = (\beta_i : \alpha_i)$

$$\begin{tikzcd} K(\mathbb{P}^1)^* \arrow[r] \arrow[dr, "0"'] & \mathrm{Div}(\mathbb{P}^1) \arrow[r] \arrow[d, "\deg"] & Cl(\mathbb{P}^1) \arrow[r] \arrow[dl, dashed, "\deg", "\sim"'] & 0 \\ & \mathbb{Z} & & \end{tikzcd}$$

$\boxed{Cl(\mathbb{P}^1) = \mathbb{Z}}$

$X = \mathrm{Spec}\, A$ $\quad (A = \text{noeth.})$

Prop $Cl\, X = 0 \iff A = \text{factorial}$.

Proof Recall
- factorial $\iff [f \in A$ irred. $\Rightarrow (f) = $ prime ideal$]$
- factorial $\iff$ every PI of height 1 is principal.

$(\Rightarrow)$ Let $P = $ PI of height 1. $\Rightarrow 1 \cdot [P] = $ prime divisor $= \mathrm{div}(f)$ for some $f \in \mathrm{Quot}(A)$

$\Rightarrow \cdot\ \mathrm{ord}_Q f \geq 0$ $\quad (\forall Q = $ PI of ht 1$)$

$f \in \bigcap_{\mathrm{ht}\, Q = 1} A_Q = A$ and $\mathrm{ord}_P f = 1$. $f \in P \rightsquigarrow \underline{\text{claim}}\ (f) = P$ ok

$\cdot$ Let $p \in P \subset A \Rightarrow \mathrm{ord}_P p \geq 1$, $\mathrm{ord}_Q p \geq 0$ $\quad p/f \in \mathrm{Quot}\, A$ $\quad \mathrm{ord}_Q(p/f) \geq 0\ \forall Q$

$(\Leftarrow)$ Suffices show that all prime divisors $1 \cdot [P]$ are principal. $\Rightarrow p/f \in A \Rightarrow p \in (f) \subset A$.

$P \Rightarrow \exists f \in A : P = (f) \Rightarrow \mathrm{div} f = P$: $\cdot\ \mathrm{ord}_P f = 1$ $\quad f$ in $A_P \supset P = (t) \rightsquigarrow f = t^n$, $f = t$, $n = 1$

$\cdot\ Q \neq P \Rightarrow \mathrm{ord}_Q f = 0$, i.e. $f \notin Q$ otherwise $P \subset Q$ (ht 1) ↯

Cartier divisors $\quad X = $ noeth. scheme

$\mathcal{O}_X \hookrightarrow K(X) = $ fct field $\rightsquigarrow \mathcal{O}_X^* \hookrightarrow K(X)^*$ injection of sheaves of abelian grps (op $= \cdot$)

on $\underline{\mathrm{Spec}\, A}$ $\quad A \hookrightarrow \mathrm{Quot}\, A$ $\quad \rightsquigarrow 1 \to \mathcal{O}_X^* \to K(X)^* \xrightarrow{\pi} K(X)^*/\mathcal{O}_X^* \to 1$

Def Cartier divisor $:=$ element of $\Gamma(X, K(X)^*/\mathcal{O}_X^*)$.

Q Does it mean Cart. div $=$ el of $\Gamma(X, K(X)^*)/\Gamma(X, \mathcal{O}_X^*)$? $\underline{\text{NO}}$ $\quad \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k \supset \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^*) = k^*$

$\mathbb{P}^1$: $K(\mathbb{P}^1)^*/k^* = k(t)/k^* \rightsquigarrow$ "div"

BUT $s \in \Gamma(X, K(X)^*/\mathcal{O}_X^*)$

$\hookrightarrow \exists \{U_i\} = $ cov of X

$\cdot\ \boxed{f_i \in K(X)^*}$: $f_i \overset{\pi}{\longmapsto} s_i = s_{|U_i}$

$\cdot\ \pi(f_i) = \pi(f_j)$ on $U_{ij} := U_i \cap U_j$

$\iff \pi(f_i/f_j) = 1$ on $U_{ij} \iff f_i/f_j \in \mathcal{O}_X(U_{ij})^*$.

Re-Def of Cartier div:

Cartier div $s := \{(U_i, f_i)\}$ s.t.
- $\{U_i\} = $ cov.
- $f_i \in K(X)^*$
- $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$

$0 \to \mathrm{PDiv}\, X \to \mathrm{Div}\, X \to Cl\, X \to 0$

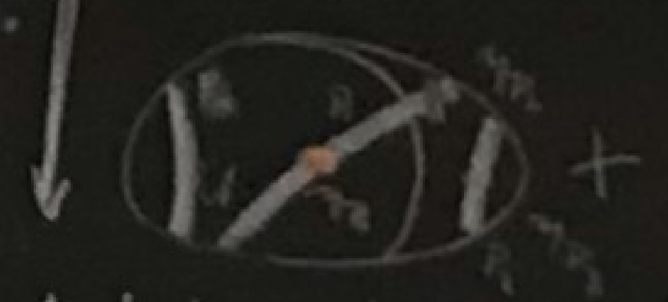
$K(X)^* \to K(X)^*/k^* \to \mathrm{PDiv}\, X$

Def P-Cartier div: represented by (X, f) for some $f \in K(X)^*$.

Corollary $1 \to \Gamma(X, \mathcal{O}_X^*) \to K(X)^* \to \Gamma(X, K(X)^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, K(X)^*) = 0$

$0 \to \mathrm{PDiv}\, X \to \mathrm{CaDiv}(X) \to Cl_{\text{Cartier}}(X) = H^1(X, \mathcal{O}_X^*)$

$s \rightsquigarrow \{(U_i, f_i)\}$ $\quad f_i \in K(X)^* \rightsquigarrow \mathrm{div}(f_i) = $ principal! Weil divisor on whole X: on U_i!



$\mathrm{div}(f_i)_{|U_i} = \mathrm{div}(f_i)$ with only all Prime div. P_i being disjoint to U_i.

$\mathrm{div}(f_i \in K(U_i) = K(X)) = $ p. div on $\underline{U_i}$.

$\mathrm{div}(s) := $ glueing of all $\mathrm{div}(f_i)_{|U_i}$.

i.e. $P = $ prime div on $X \Rightarrow \exists i : P \cap U_i \neq \emptyset$

$\Rightarrow n_P = $ coeff of $\mathrm{div}(f_i)_{|U_i}$ of P $\quad (\eta_P \in U_i)$

does $\underline{\text{not}}$ depend on i !! $\quad (\mathrm{div}(f_i) = \mathrm{div}(f_j)$ on $U_i \cap U_j)$

$\Rightarrow \mathrm{CaDiv}\, X \hookrightarrow \mathrm{WDiv}\, X$ $\quad (\mathrm{PDiv}$ maps into it$)$

$\mathrm{PDiv}(X) \hookrightarrow \underline{\mathrm{CaDiv}\,X} \hookrightarrow \mathrm{WDiv}\,X$

$D \in \mathrm{CaDiv}\,X : D = \{(U_i, f_i)\}$

($\mathrm{div}(f_i) = \ldots$ Weil on U_i $\hat{=}$ "$V(f)$")

__Sheaf__ $\mathcal{O}_X(D) \subseteq K(X)$

$\mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i} \cdot \mathcal{O}_{U_i} \subseteq K(X)$ (sheaf on U_i)

$\mathcal{O}_X(D)|_{U_i}|_{U_i \cap U_j} = \frac{1}{f_i} \cdot \mathcal{O}_{U_i \cap U_j} \overset{?}{\longleftrightarrow} \frac{1}{f_j} \mathcal{O}_{U_i \cap U_j} = \mathcal{O}(D)|_{U_j}|_{U_i \cap U_j}$

$\frac{1}{f_j} = \frac{1}{f_i} \cdot \left(\frac{f_i}{f_j}\right)$ $\qquad$ $f_i/f_j \in K(X)$, $\in \Gamma(U_i \cap U_j, \mathcal{O}^*)$

$\mathcal{O}_X \cong \mathcal{O}_X(D)$ __locally__, i.e. on U_i: $\mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i} \mathcal{O}_{U_i} \cong \mathcal{O}_{U_i}$

$\mathrm{CaDiv}\,X = \mathrm{WDiv}\,X \iff$ all (Weil) divisors are locally principal

$\iff \forall x \in X : \mathcal{O}_{X,x} =$ rings with $\mathrm{Cl}(\mathrm{Spec}\,\mathcal{O}_{X,x}) = 0$

$\iff \boxed{\forall x \in X : \mathcal{O}_{X,x} = \text{factorial}} \iff$: "$X$ = __factorial__"

$\leadsto$ • $\boxed{\mathcal{O}_X(D)} \subseteq K(X)$ sheaf.

• $\mathcal{O}_X$-module.

• $\boxed{\text{invertible sheaf}}$ (locally free, rank 1)

$\mathcal{L}$ = inv. sheaf $\hookrightarrow K(X)$

$\Downarrow$

$\exists U_i : \mathcal{O}_{U_i} \xrightarrow[\varphi_i]{\sim} \mathcal{L}_{|U_i} \hookrightarrow K(X)$

$1 \longmapsto$ "$\frac{1}{f_i}$" $\Rightarrow \mathcal{L}_{U_i} = \frac{1}{f_i} \cdot \mathcal{O}_{U_i}$

• $\mathrm{CaDiv} \longleftrightarrow$ invertible subsheaves $\mathcal{L} \hookrightarrow K(X)$

$\cup \qquad \cup$

• $\mathrm{PDiv} \longleftrightarrow \mathcal{L} = \frac{1}{f} \cdot \mathcal{O}_X$ for some $f \in K(X)$

$\updownarrow$

$\mathcal{L} \subset K(X)$ with $\mathcal{L} \cong \mathcal{O}_X$

$\Rightarrow \mathrm{CaDiv}/\mathrm{PDiv} \cong \{\text{inv. sheaves } \cancel{\subseteq K(X)}\}/\cong = \mathrm{Pic}\,X$

$\mathrm{WDiv}/\mathrm{PDiv} = \mathrm{Cl}(X)$

$\mathcal{O}(D+D') = \mathcal{O}(D) \otimes_{\mathcal{O}_X} \mathcal{O}(D') \subset K(X)$

$\mathcal{L} \cong \mathcal{L}'$ as $\mathcal{O}_X$-mod.

$\mathcal{L} \cong \mathcal{O}_X$ — " —

$\Rightarrow \exists \varphi : \mathcal{O}_X \xrightarrow{\sim} \mathcal{L} \subset K(X)$

$1 \longmapsto \frac{1}{f} \Rightarrow \mathcal{L} = \frac{1}{f} \mathcal{O}_X$

__Lemma__: X = integral

$\Rightarrow$ Every $\mathcal{L}$ = inv. can be embedded $\hookrightarrow K(X)$

(i.e. $\exists\, i : \mathcal{L} \hookrightarrow K(X)$)

__Proof__: • $\mathcal{L} \otimes_{\mathcal{O}_X} K(X)$ = const. $\mathcal{L}_\eta \otimes_{\mathcal{O}_{X,\eta}} K(X) = \mathcal{L}_\eta$

$\mathcal{L}_\eta \cong K(X)$ (take $U \subset X$: $\mathcal{L}_{|U} \cong \mathcal{O}_U$) $\overset{?}{\Longrightarrow}$ $\mathcal{L} \hookrightarrow \mathcal{L} \otimes K(X) \cong K(X)$ $\square$

$\mathcal{F} \otimes_{\mathcal{O}_X} K(X) = \mathcal{F}_{\eta(X)}$ (const. sh.)

$\mathcal{L}$ = inv $\longrightarrow$ 1-cocycle encoding $\mathcal{L}$:

U_i: $\mathcal{O}_{U_i} \xrightarrow[\varphi_i]{\sim} \mathcal{L}_{|U_i}$, $1 \longmapsto s_i$

$\mathcal{O}_{U_j} \xrightarrow[\varphi_j]{\sim} \mathcal{L}_{|U_j}$, $1 \longmapsto s_j$

$\mathcal{L}_{|U_i \cap U_j} = s_i \mathcal{O}_{U_{ij}} = s_j \mathcal{O}_{U_{ij}} \longrightarrow s_i = \varepsilon_{ij} s_j$, $\varepsilon_{ij} \in \mathcal{O}^*_{U_i \cap U_j}$

$\varepsilon_{ij}\, \varepsilon_{jk}\, \varepsilon_{ki} = 1$ in $\mathcal{O}^*_{U_{ijk}}$

$\mathcal{L} \underset{\psi}{\cong} \mathcal{O}_X \iff$ 1-cocycle = coboundary: $\exists \eta_i : \varepsilon_{ij} = \eta_i/\eta_j$

$\Downarrow$

$H^1(\mathcal{O}^*) = \mathrm{Pic}\,X$

$\varepsilon_{ij} = s_i/s_j = \eta_i/\eta_j$

$\varepsilon_{ij} \longleftrightarrow f_i/f_j$