$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$ on $\mathbb{C}$

$g \longmapsto g$

$f \longmapsto \exp(2\pi i f)$

$\log \longmapsto id \in \mathcal{O}^*(\mathbb{C}^*)$

GAGA

$0 \to H^0(\mathbb{C}^*, \mathbb{Z}) \to H^0(\mathbb{C}^*, \mathcal{O}) \to H^0(\mathbb{C}^*, \mathcal{O}^*) \to H^1(\mathbb{C}^*, \mathbb{Z}) \to 0$

$id \longmapsto (\neq 0)$

$\underline{X = \text{normal}}$

$$\mathrm{WDiv}\,X \supseteq \mathrm{CaDiv}\,X \supseteq \underline{\mathrm{PDiv}\,X} \twoheadleftarrow K(X)^*$$

$\downarrow \quad \downarrow$

$R \searrow \quad \mathrm{Cl}(X) \supseteq \mathrm{Pic}\,X$

$D \longleftrightarrow \mathcal{O}(D) \subseteq K(X)$

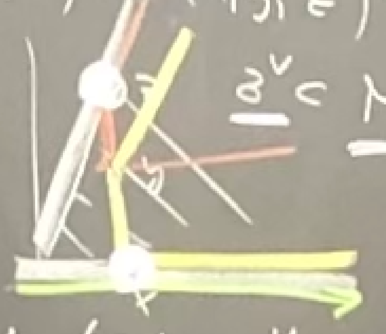
$D' = D + \mathrm{div}(f) \longleftrightarrow \mathcal{O}(D') = f \cdot \mathcal{O}(D) \subseteq K(X)$

- $\mathrm{Cl}(\mathrm{Spec}\,R) = 0 \Longleftrightarrow R$ = factorial
- $\mathrm{Pic}(\underset{\text{affine}}{TV(\sigma)}) = 0 \quad \forall \sigma$

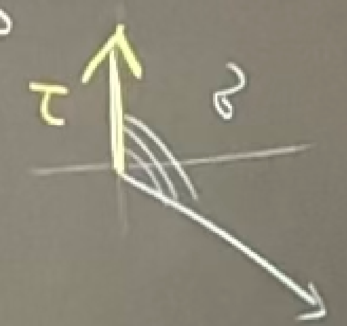
$\underline{\text{Example}}$: $X = \mathrm{Spec}\,\mathbb{C}[x,y,z]/_{xz-y^2}$ singular in $0 = (0,0,0) \hat{=} (x,y,z) =: m$; $\dim m/m^2 = 3$

$A = V(x,y) \leftrightarrow$ ($(x) \ni y$, but $(x) \ni y^2$) $(x,y) \subseteq R$

$B = V(y,z) \leftrightarrow (y,z)$



$\underline{\sigma^\vee} \subset \underline{M}$



If a „toric" divisor is ~~Cartier~~ Cartier $\Longleftarrow$ principal divisor $\mathrm{div}(f)$ with $f = x^r$ $(r \in \sigma^\vee \cap M)$

$\mathrm{div}(x) = \underline{2} \cdot V(x,y)$: $xz = y^2 \leadsto x \in (x,y)^2$ inside $\mathcal{O}_{\eta(A)}$

$\mathrm{div}(x) = 2 \cdot A \qquad \mathrm{div}(y) = A + B$

$\mathrm{div}(z) = 2 \cdot B$

(e.g. $\mathcal{O}_{\eta(A)} = \left(k[x,y,z]/(xz-y^2) \right)_{(x,y)} \ni m = (x,y) \cdot \mathcal{O}_{\eta(A)} \quad z \in \mathcal{O}^*_{\eta(A)}$

$k[y,z]_{(y)} = k[x,y,z]_{(x,y)} / (xz - y^2)$

$x = y^2/z \Longrightarrow (x,y) = (y)$

$\underline{Q}$: $x \sim y^k \leadsto k = \mathrm{ord}_A x$

$W\text{-Div} \supseteq \mathbb{Z} \cdot A + \mathbb{Z} B$

$\cup$

$(\text{toric})\ C\text{-Div} \quad 2 \mid$ (sum of the A-/B-coeffs.)

$X = TV(\sigma) \leadsto \tau \leq \sigma \Longleftrightarrow \underline{\sigma^\vee \cap \tau^\perp} \leq \sigma^\vee \leadsto \mathbb{C}[\sigma^\vee \cap M] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M]$

$\sigma \subset N_{\mathbb{R}}$

$x^r \longmapsto \begin{cases} x^r & (r \in \ldots) \\ 0 & (\text{otherwise}) \end{cases}$

$TV(\sigma) \qquad \mathrm{orb}(\tau) = TV(\bar{\sigma})$

$$TV(\sigma) = \coprod_{\tau \leq \sigma} \mathrm{orb}(\tau)$$

$\bar{\sigma}$: image of σ via $N \to N/_{\tau - \tau}$

$\dim \mathrm{orb}(\tau) = \dim \overline{\mathrm{orb}}(\tau) = \dim TV(\sigma) - \dim \tau$

$\underline{\text{Prime divisors}}$: „toric PD" $\Longleftrightarrow \overline{\mathrm{orb}(\tau)}$ with $\dim \tau = 1$

$\underline{\text{i.e.}}$: $\rho \in \sigma(1) \leadsto D_\rho := \overline{\mathrm{orb}}(\rho)$

$\underline{\text{general}}$: Σ = fan, $\rho \in \Sigma(1) \leadsto D_\rho = \overline{\mathrm{orb}}(\rho) \subset TV(\Sigma) \supseteq \underset{(\sigma \in \Sigma)}{TV(\sigma)}$

$0 \to \mathrm{PDiv}\,X \longrightarrow \mathrm{WDiv}\,X \longrightarrow \mathrm{Cl}\,X \to 0$

$\boxed{K(X)^* \longrightarrow \mathrm{WDiv}\,X \longrightarrow \mathrm{Cl}\,X \to 0}$

$\mathbb{C}(M) \longrightarrow \mathrm{WDiv}\,TV(\Sigma) \xrightarrow{\pi} \mathrm{Cl}\,TV(\Sigma) \to 0$

$\mathrm{Quot}\,\mathbb{C}[M]$

$0 \to \{\mathrm{PDiv}(\text{monomials})\} \to \left[\mathrm{WDiv}_T\, TV(\Sigma) = \mathbb{Z}^{\Sigma(1)} \right] \to \mathrm{Cl}(TV(\Sigma)) \to 0$

$\mathbb{Z}^n$

$M \quad r \mapsto \mathrm{div}(x^r)$

$\underline{\text{Prop}}$: $\pi|_{\mathrm{WDiv}_T} : \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}\,TV(\Sigma)$ is (still) surjective.

$\underline{\text{Proof}}$: • $\mathrm{Cl}(T) = 0$ (since $\mathbb{C}[M] = \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ – factorial

$TV(0) = \mathrm{Spec}\,\mathbb{C}[M] \qquad \mathbb{Z}^n \qquad$ (GauB: R = fact $\Longrightarrow R[y]$ = fact)

$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n]$]

• Let D = W-Div repr. some class $\bar{D} \in \mathrm{Cl}(X)$ $(X = TV(\Sigma))$

 +

$\leadsto D_{|T} \qquad D = \sum g_i D_i \leadsto D_{|T} := \sum g_i (D_i \cap T) \in \mathrm{WDiv}(T)$

T-inv. PD $\Longleftrightarrow \overline{\mathrm{orb}}(\rho)$

$T \times X \longrightarrow X$

$TV(\Sigma)$

$\cup$

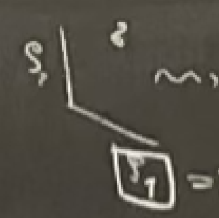
$T \times D \dashrightarrow D$

$\underline{\text{Ex}}$: $\Sigma = \sigma = \ldots \Longrightarrow \mathbb{Z}^{\Sigma(1)} = \mathbb{Z}^2$

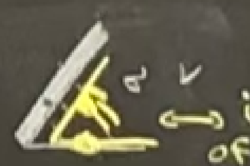
$(a,b) \hat{=} aA + bB$

$\underline{\text{wlll}}$: $\overline{\mathrm{orb}}(\rho_1) = A$

$\overline{\mathrm{orb}}(\rho_2) = B$.



$\boxed{\rho_1} = \tau$



$\sigma^\vee \longleftrightarrow$ ideal of $\overline{\mathrm{orb}}(\rho_1)$

$\tau^\perp \cap \sigma^\vee$

(x,y)

$\Longrightarrow \exists f \in \mathbb{C}(M)$: $D_{|T} = \mathrm{div}(f)$ on T (goal/Q: $D = \mathrm{div}(f)$ on X ?) NO

$\Longrightarrow D - \mathrm{div}_X(f)$ = Weil divisor built from PD $\boxed{D_i \cap T = \emptyset}$

D' ($\underline{\text{recall}}$: $\bar{D}' = \bar{D}$ in $\mathrm{Cl}(X)$)

$\underline{\text{recall}}$: $X = TV(\Sigma) = \coprod_{\tau \in \Sigma} \mathrm{orb}(\tau)$; $\mathrm{orb}(0) = T$

$(\tau \leq \tau' \Longrightarrow \mathrm{orb}(\tau') \subseteq \overline{\mathrm{orb}}(\tau))$ $\quad X \setminus T = \coprod_{\tau \in \Sigma \setminus \{0\}} \mathrm{orb}(\tau) - U$

$\Longrightarrow \exists \rho \in \Sigma(1): D_i = \overline{\mathrm{orb}}(\rho) \qquad D_i \subseteq X \setminus T = \bigcup_{\rho \in \Sigma(1)} \overline{\mathrm{orb}}(\rho)$ (Since $\tau \ni \rho \Longrightarrow \mathrm{orb}(\tau) \subseteq \overline{\mathrm{orb}}(\rho)$)

codim. 1,

$\Longrightarrow D' \in \mathrm{WDiv}_T\, X$!! $\quad \square$

Let $D \in \mathrm{WDiv}_T X$, assume $D \overset{\pi}{\longmapsto} \bar{D} = 0 \in \mathrm{Cl}(X)$, i.e. D = principal $\Longrightarrow \exists f \in \mathbb{C}(M)$: $\underline{D = \mathrm{div}(f)}$

$\Downarrow$

$D = \sum_{\rho \in \Sigma(1)} g_\rho \cdot \overline{\mathrm{orb}}(\rho) \Longrightarrow 0 = D_{|T} = \mathrm{div}_X(f)_{|T} = \mathrm{div}_T(f)$ (div with respect to T)

$f \in \mathrm{Quot}\,\mathbb{C}[M]$; $\mathrm{div}_T(f) = 0 \quad \begin{matrix} \leq 0 \Longrightarrow \frac{1}{f} \in \mathbb{C}[M] \\ \geq \Longrightarrow f \in \mathbb{C}[M] \end{matrix} \Bigg\} \Longrightarrow f \in \mathbb{C}[M]^* = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^* = \{\text{monomials } x^r / r \in M\}$

$\Longrightarrow \boxed{f = x^r}$

$r \in M$, $X = TV(\Sigma)$ $\rightsquigarrow$ $\operatorname{div}(\chi^r)$: $\chi^r \in \mathbb{C}[M]^*$ $\rightsquigarrow$ $\operatorname{div}_T(\chi^r) = 0$

i.e. $\operatorname{div}(\chi^r)$ has support on $X \setminus T$

i.e. $\operatorname{div}(\chi^r) = \sum_{\rho \in \Sigma(1)} \lambda_{\rho,r} \cdot \overline{\operatorname{orb}}(\rho)$, $\chi^r \in \mathbb{C}[M]$

Q: $\lambda_{\rho,r} \in \mathbb{Z}$ is what ?? guess: $\langle r, \rho \rangle$

Prop: $\operatorname{ord}_{\overline{\operatorname{orb}}(\rho)}(\chi^r) = \langle r, \rho \rangle$.

Proof: $\overline{\operatorname{orb}}(\rho)$ is the closure of

$$\boxed{\begin{matrix} \operatorname{orb}(\rho) & \overset{\text{closed}}{\subseteq} & TV(\rho) & \overset{\text{open}}{\subseteq} TV(\Sigma) \\ \cup & & \cup & \\ \eta_\rho & & TV(0) = T & \end{matrix}}$$

open subset: $\operatorname{orb} \subseteq \overline{\operatorname{orb}}$

- w.l.o.g.: $\Sigma = \rho$ (one ray)
- Choose coord: $\rho = (1,0,0,\dots,0) \in \mathbb{Z}^n (= N)$ $\Longrightarrow$ $\Sigma =$ 

dual: $r \in M = \mathbb{Z}^n$

$TV(\rho) = \operatorname{Spec} \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$

$f = \chi^r = x_1^{r_1} \cdots x_n^{r_n}$ $(r_i \in \mathbb{Z})$

localize in (x_1) $\Rightarrow$ $\mathfrak{m} \subset \mathcal{O}_\rho =$ loc.

$\mathfrak{m} = (x_1)$, ρ

$\Longrightarrow \operatorname{ord}_{(x_1)} f = r_1$

and: $r_1 = \langle r, e_1 \rangle$, $e_1 = \rho$ $\square$

$\Longrightarrow$ $M \xrightarrow{\operatorname{div}} \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathcal{Cl}(X) \longrightarrow 0$

$r \longmapsto \sum \langle \rho, r \rangle e_\rho$, $e_\rho \mathrel{\hat=} \overline{\operatorname{orb}}(\rho)$

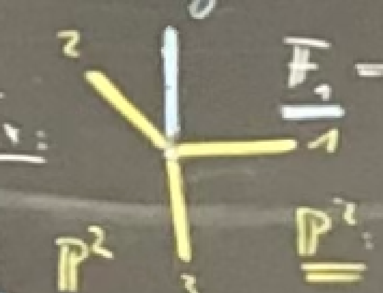
Observe: $\boxed{\begin{matrix} \mathbb{Z}^{\Sigma(1)} & \longrightarrow & N \\ e^\rho & \longmapsto & \rho \end{matrix}}$ $\longleftrightarrow$ encodes $\Sigma(1)$

often surjective; more often: surjective $\otimes \mathbb{Q}$ (e.g.: $\Sigma \ni$ full-dim cone)

$\Downarrow$

$0 \to M \xrightarrow{\operatorname{div}} \mathbb{Z}^{\Sigma(1)} \to \mathcal{Cl}(X) \to 0$

$N \longleftarrow \mathbb{Z}^{\Sigma(1)} \longleftarrow \mathcal{Cl}(X)^* \longleftarrow 0$

Ex:  $\mathbb{F}_1 = (\mathbb{P}^2)'$, $\downarrow \mathbb{P}^2$, $\mathbb{P}^2$: $\downarrow$ $\mathbb{P}^1$

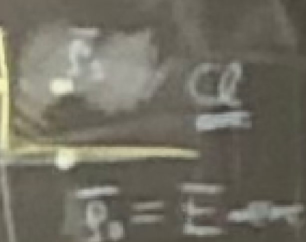
$\mathbb{Z}^2 \to \mathbb{Z}^4$

$0 \to \mathcal{Cl}^* \to \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$, $\mathcal{Cl}^* = \mathbb{Z}$, $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}$ N-Seq.

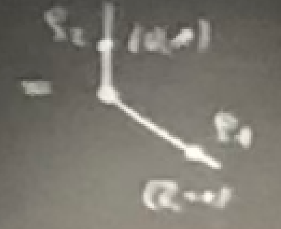
$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \xrightarrow{(1\,1\,1)} (\mathcal{Cl} = \mathbb{Z}) \to 0$

$\overline{\operatorname{orb}}(\rho) = e_\rho \longmapsto 1$, $= [x_0 = 0], [x_1 = 0], [x_2 = 0]$

$0 \to \mathbb{Z}^2 \to \mathbb{Z}^4 \xrightarrow{\begin{pmatrix}1&0&1&1\\0&1&1&1\end{pmatrix}} (\mathcal{Cl} = \mathbb{Z}^2) \to 0$



$\overline{\rho_0} = E$ [illegible]

$\sigma =$ 

$\sigma^\vee =$ 

$x = \chi^{(1,0)}$, $y = \chi^{(0,1)}$, $z = \chi^{[illegible]}$

$\operatorname{div}(x) = \operatorname{div} \chi^{(1,0)} = \langle (1,0), \rho_1 \rangle \overline{\operatorname{orb}}\rho_1 + \langle (1,0), \rho_2 \rangle \overline{\operatorname{orb}} \rho_2$

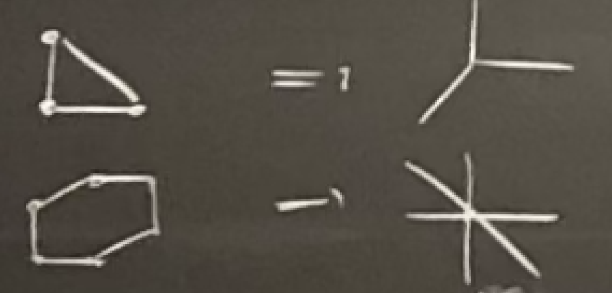
e.g. $\mathbb{Z}^2 \longrightarrow (\mathbb{Z}^2 = N)$

$e^{\rho_1} \longmapsto (2,-1)$

$e^{\rho_2} \longmapsto (0,1)$

$\mathrel{\hat=} \begin{pmatrix}2 & 0\\ -1 & 1\end{pmatrix}$, $\operatorname{div} = \begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}$

Let $X = TV(\Sigma)$. Let $\Delta =$ lattice polytope in M s.t.: $\mathcal{N}(\Delta) \geq \Sigma$ (Σ refines $\mathcal{N}(\Delta)$)

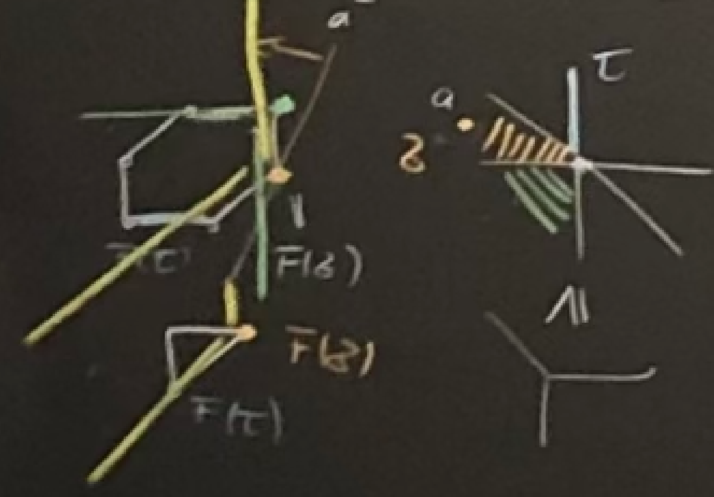
Ex $\Delta =$  $\Rightarrow$ $\mathbb{P}^2$

- $P(\Delta) \subseteq \mathbb{P}^{\#(\Delta \cap M) - 1}$
- $TV(\Sigma)$

$\Big\} \Rightarrow TV(\Sigma) \to \mathbb{P}(\Delta)$

isom. if: "=", $\Delta = (\geq 0)\Delta$

$\Delta \rightsquigarrow$ invertible sheaf "$\mathcal{O}(\Delta)$" $\subseteq K(X)$: $\boxed{\text{local pieces } \sigma \in \Sigma}$



vertex $\Delta(\sigma) \in \Delta$

face $F(\sigma) < \Delta$: $F(\sigma) := \{ r \in \Delta \mid \langle r, a \rangle = \min \langle \Delta, a \rangle \}$ for $a \in \operatorname{int} \sigma$

only need full-dim. σ $\Rightarrow$ $F(\sigma) =$ vertex $\Delta(\sigma) \in \Delta$

$$\mathcal{O}(\Delta)|_{TV(\sigma)} := \chi^{\Delta(\sigma)} \cdot \mathcal{O}_{TV(\sigma)} \subseteq \mathbb{C}[M]$$

$\sigma, \sigma' \rightsquigarrow$ compare $\mathcal{O}(\Delta)|_{TV(\sigma)}$, $\mathcal{O}(\Delta)|_{TV(\sigma')}$ on $TV(\sigma \cap \sigma')$

$\chi^{\Delta(\sigma)}$, $\chi^{\Delta(\sigma')}$ $\rightsquigarrow$ $\chi^{\Delta(\sigma) - \Delta(\sigma')}$ unit $\Longleftarrow$ $\Delta(\sigma) - \Delta(\sigma') \in (\sigma \cap \sigma')^\perp$

- $\Delta \rightsquigarrow \mathcal{O}(\Delta) =$ inv. subsheaf of $K(X)$ $\rightsquigarrow$ divisor (Cartier)
- $\Gamma(X, \mathcal{O}(\Delta)) = \operatorname{span} \{ \chi^r \mid r \in \Delta \cap M \}$

ex $\Delta \mathrel{\hat=} \mathbb{P}^2$ $\rightarrow$ $\mathcal{O}(\Delta) = \mathcal{O}(1)$

$\Gamma(\cdots) = \operatorname{span}(\cdot \,\vdots)$ $\Gamma(\cdot) = \operatorname{span}(z_0, z_1, z_2)$

$\mathbb{C}$-vs.

Remark: $\Delta \rightsquigarrow \mathcal{O}(\Delta) =$ Cartier div. $\rightsquigarrow$ Weil divisor ? $\sum_{\rho \in \Sigma(1)} (?) \cdot \overline{\operatorname{orb}}(\rho)$

on $TV(\sigma)$: gen. $\chi^{\Delta(\sigma)}$ $\Longrightarrow$ Cartier data of Δ on $TV(\sigma)$: $(U = TV(\sigma), f = \chi^{-\Delta(\sigma)})$

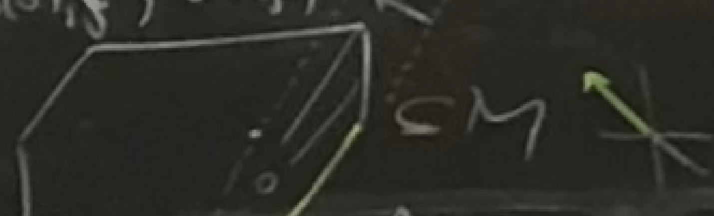
(Recall: $(U, f) \rightsquigarrow \mathcal{O}(D)|_U = \frac{1}{f} \mathcal{O}_U$, Cartier div. D)

$\operatorname{div}(\chi^{\Delta(\sigma)}) = \sum_\rho \langle \Delta(\sigma), \rho \rangle \cdot \overline{\operatorname{orb}}(\rho)$

$- \sum_\rho \langle \Delta(\sigma), \rho \rangle \cdot \overline{\operatorname{orb}}(\rho)$ on $TV(\sigma)$ $\rho \in \sigma(1)$

$$\Longrightarrow \boxed{\operatorname{Weil}(\Delta) = -\sum_{\rho \in \Sigma(1)} \min \langle \Delta, \rho \rangle \cdot \overline{\operatorname{orb}}(\rho)}$$

($\rho \in \sigma \Rightarrow \langle \Delta(\sigma), \rho \rangle$)

Cor: Δ  $\subset M$

$\Delta + r$

- D = Cart. on X $\longleftrightarrow$ $\mathcal{O}(D) \subseteq K(X)$ invertible sheaf.

Property: $\mathcal{O}(D+D') = \mathcal{O}(D) \otimes \mathcal{O}(D') = \mathcal{O}(D) \cdot \mathcal{O}(D')$

- D = Weil div $\rightsquigarrow$ $\mathcal{O}(D)$ exists, too?

Def. D = effective ("$D \geq 0$") $:\Longleftrightarrow$ all coeff. of D are ≥ 0.
(Weil div)

ex. $\mathbb{P}^1$: $\mathrm{div}\left(\frac{x_1}{x_0}\right) = [0] - [\infty] \sim 0$, $\not\geq 0$, ≥ 0

Def. D = Weil div $\rightsquigarrow$ $\mathcal{O}(D) := \{ f \in K(X) \mid \mathrm{div}(f) + D \geq 0 \}$

i.e. $\mathcal{O}(D)(U \subseteq X) := \{ f \in K(X) \mid \mathrm{div}(f)_{|U} + D_{|U} \geq 0 \} \subseteq K(X)$ subsheaf.

$\Gamma(X, \mathcal{O}(D)) = \{ f \in K(X)^* \mid \mathrm{div}(f) + D \geq 0 \text{ on } \underline{\text{whole } X} \} \cup \{0\}$

$f \in \Gamma(X, \mathcal{O}(D))$ $(f \neq 0)$ $\longmapsto$ $D_f := \mathrm{div}(f) + D$ new divisor:
- $\overline{D_f} = \overline{D}$ in $\mathcal{Cl}(X)$
- $D_f \geq 0$

Def. "local system" of a Weil divisor D: $|D| = \{ D' \geq 0 \text{ Weil div} \mid D' \sim D, \text{ i.e. } \overline{D'} = \overline{D} \text{ in } \mathcal{Cl}(X) \}$

Corollary: $|D| = \Gamma(X, \mathcal{O}(D)) \setminus \{0\} / k^* = \mathbb{P}(\Gamma(X, \mathcal{O}(D)))$

$\mathbb{P}(V) = V \setminus 0 / k^*$

$\mathbb{P}(V) = V^* \setminus 0 / k^*$

Eff. div. in $\mathcal{Cl}$:

$\mathrm{WDiv}(X) \xrightarrow{\mathrm{cl}} \mathcal{Cl}(X) \to 0$ (dim jump)

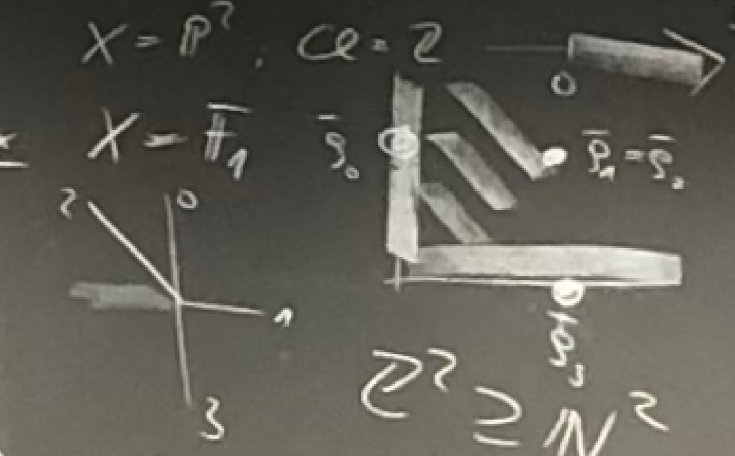
$\cup$

Eff WDiv, image = cone = "Eff(X)" $\supseteq$ "pseudoeffective cone"

$\cap$

$\mathcal{Cl} \otimes_{\mathbb{Z}} \mathbb{R}$

Ex $X = \mathbb{F}_1$, $X = \mathbb{P}^2$, $\mathcal{Cl} = \mathbb{Z}$



$\mathrm{Eff}(\mathbb{F}_1) \subseteq$

$\mathbb{Z}^2 \geq \mathbb{N}^2$

Prop: $X = TV(\Sigma) \longrightarrow$ $0 \to M \to \mathbb{Z}^{\Sigma(1)} \xrightarrow{\mathrm{cl}} \mathcal{Cl}(X) \to 0$

$\mathrm{Eff}(X) = \mathrm{cl}\left(\mathbb{R}^{\Sigma(1)}_{\geq 0}\right)$

Proof: "$\supseteq$" $D \in \mathbb{N}^{\Sigma(1)}$ effective $\Rightarrow$ $\mathrm{cl}(D) \in \mathrm{Eff}$.

"$\subseteq$". Let $D \geq 0$ eff. div. on $X = TV(\Sigma)$

(recall: $D_{|T} = \mathrm{div}_T f \rightsquigarrow D' = D - \mathrm{div}(f) \in \mathrm{WDiv}_T$)

$f \in K(T)$ $\rightsquigarrow$ $\forall r \in M$: $f \cdot \chi^r$ does the same job

$\Downarrow$ may assume

$\mathrm{div}(\chi^r) \cap T = \emptyset$

- $f \in \mathbb{C}[M]$: $\mathrm{div}(f)_{|T} = D_{|T} \geq 0$
- may suppose: $0 \in \mathrm{supp} f$ $\left(f = \sum_{r \in M} \lambda_r \cdot \chi^r ;\ \mathrm{supp} f = \{ r \mid \lambda_r \neq 0 \}\right)$, $0 \in M$, i.e. $f = 1 + \ldots$, $1 = \chi^0$

$\Rightarrow$ $\mathrm{ord}_{\overline{\mathrm{orb}(\rho)}}(f) = \min_{r \in \mathrm{supp} f} \langle r, \rho \rangle \leq 0$

$\Rightarrow$

$$\begin{array}{ccc} \mathrm{WDiv} & \longrightarrow & \mathcal{Cl} \\ \cup & & \\ \text{eff. WDiv} & & \end{array}$$

$D' = D - \mathrm{div}(f) \geq 0$, $D \geq 0$, $-\mathrm{div}(f) \geq 0$

Def. ① $\mathcal{F}$ = sheaf of $\mathcal{O}_X$-modules on X (variety / k). $\mathcal{F}$ is "globally generated" $:\Longleftrightarrow$

$\Gamma(X, \mathcal{F}) \otimes_k \mathcal{O}_X(U) \xrightarrow{\pi} \mathcal{F}(U)$ is surjective, i.e. $\forall x \in X$:

$\mathcal{O}_{X,x} \otimes \Gamma(X, \mathcal{F}) \twoheadrightarrow \mathcal{F}_x$ is surjective.

$k^2 \otimes_k \mathcal{O}_X = \mathcal{O}_X^2$

② D = Cartier divisor. D is base point free $\Longleftrightarrow$ $\forall x \in X$: $\exists$
- $D' \sim D$ ($\overline{D'} = \overline{D}$ in $\mathcal{Cl}(X)$)
- $D' \geq 0$
- $x \notin \mathrm{supp}\, D'$

Prop: D is bp-free $\Longleftrightarrow$ $\mathcal{O}(D)$ is globally generated.

Proof. Recall $f \in \Gamma(X, \mathcal{O}(D)) \longleftrightarrow D' := D + \mathrm{div}(f) \geq 0$

$(\Rightarrow)$ $\Gamma(X, \mathcal{O}(D)) \otimes \mathcal{O}_{X,x} \overset{??}{\underset{\pi}{\twoheadrightarrow}} \mathcal{O}(D)_x$; choose $\underline{D'}$ as above.

f (s.t. $\underline{\underline{D'}} = D + \mathrm{div}(\underline{f})$) $\longmapsto f_x$

U = open neighborhood of x $\rightsquigarrow$ $D \doteq (U, g)$ $\Rightarrow$ $\mathcal{O}(D)_x = \frac{1}{g} \cdot \mathcal{O}_{X,x}$ $\Rightarrow$ $D' \doteq \frac{f}{g} \cdot \mathcal{O}_{X,x}$ $\rightarrow$

$x \notin \mathrm{supp}\, D'$ $\hat{=}$ $f \mapsto$ generator of $\mathcal{O}(D)_x \subseteq K(X)$, $\mathcal{O}(D)_x \cong \mathcal{O}_{X,x}$, $h \cdot \mathcal{O}_{X,x}$

next week or in the problems.

remains to show:

$x \notin \mathrm{supp}\, D' \Longleftrightarrow \pi(f) = \text{gen of } \mathcal{O}(D)_x$.