X = normal variety / $k = \bar{k}$ $\rightsquigarrow$ [D = Cartier divisor $\rightsquigarrow$ sheaf $\mathcal{O}(D) \subseteq K(X)$]

$\Gamma(X, \mathcal{O}_X(D)) \longleftrightarrow \{ D' \geq 0$ and $D' \sim D$, i.e. $[D'] = [D]$ in $\mathrm{Pic}(X) \} = |D|$

$\Updownarrow$ $\exists f \in K(X)^* : D' = D + \mathrm{div}(f)$

- remove 0 on the LHS
- $f \in \Gamma(X, \mathcal{O}_X(D))$, $c \in \underset{\Gamma(X,\mathcal{O}_X)^*}{k^*} \Rightarrow f, c \cdot f$ define the same divisor

$f \longmapsto D' = D + \mathrm{div}(f)$

$(\mathcal{O}_X(D) = \{ f \in K(X)^* \mid \mathrm{div}(f) + D \geq 0 \} \cup \{0\})$

Def.
- $x \in X$ is a base point of $D :\Longleftrightarrow \forall D' \in |D| : x \in \mathrm{supp}\, D'$
- $\mathcal{L}$ = (qcoh) sheaf of $\mathcal{O}_X$-modules $\rightsquigarrow$ $\mathcal{L}$ is <u>in x</u> globally generated $\Longleftrightarrow \Gamma(X, \mathcal{L}) \otimes \mathcal{O}_{X,x} \to \mathcal{L}_x$ is surjective.

Prop. x is a base point for $D \Longleftrightarrow \mathcal{O}(D)$ is in x <u>not</u> globally generated.

Proof. $\Gamma(X, \mathcal{O}(D)) \otimes \mathcal{O}_{X,x} \longrightarrow \mathcal{O}(D)_x \xrightarrow{\frac{1}{g}} \boxed{\mathcal{O}_{X,x}} \subseteq K(X)$

$f \otimes 1 \longmapsto f = \boxed{fg} / g$

($D \hat{=} (U, g)$ with $x \in U$ = affine, $g \in K(X)^*$ $\Rightarrow \mathcal{O}_X(D)(U) = \frac{1}{g} \cdot \mathcal{O}_X(U)$)

$D' = D + \mathrm{div}(f) \overset{\text{locally}}{=} \mathrm{div}(g) - \mathrm{div}(f) = \mathrm{div}(fg)$

$x \in D' \Longleftrightarrow x \in \mathrm{div}(fg) \Longleftrightarrow \mathrm{ord}_x(fg) = 0 \Longleftrightarrow fg \in \mathcal{O}_{X,x}^* \Longleftrightarrow f$ generates $\frac{1}{g} \cdot \mathcal{O}_{X,x}$.

D = Cartier divisor on X, let $s_0, \ldots, s_m \in \Gamma(X, \mathcal{O}(D))$ = k-vs (e.g. $\{s_0, \ldots, s_m\}$ = basis of this vs)

$\Gamma(X, \mathcal{O}(D))$ is a $\Gamma(X, \mathcal{O}_X)$-module, $k \to \Gamma(X, \mathcal{O}_X)$ ($X \to \mathrm{Spec}\, k$ $\Leftrightarrow$ X = k-scheme, $k \to \mathcal{O}_X$)

try to build $X \longrightarrow \mathbb{P}^m$

(or, without coordinates: $X \to \mathbb{P}(\Gamma(X, \mathcal{O}(D))^\vee)$ if this is finite-dim.) (Alg 1: $X \subseteq \mathbb{P}^N$ closed, i.e. X = „projective" $\to k \cong \Gamma(\mathcal{O}_X)$)

($\mathbb{P}(V) = V \setminus \{0\} / k^*$, i.e. $x \in \mathbb{P}(V) \longleftrightarrow$ line in V)

$X \xrightarrow{\varphi_D} \mathbb{P}(\Gamma(X, \mathcal{O}(D))^\vee)$

$x \longmapsto$ line in $\Gamma(X, \mathcal{O}(D))^\vee$, given by some k-linear map $\Gamma(X, \mathcal{O}(D)) \longrightarrow k$ — has to be defined!!

$s \longmapsto s(x)$, $s \in \Gamma(X, \mathcal{O}(D))$, $s(x) = \varphi(s_x)(x) \in k$

$\mathcal{O}(D)_x \xrightarrow[\sim]{\varphi} \mathcal{O}_{X,x}$, $\psi : \mathcal{O}(D)_x \xrightarrow{\sim} \mathcal{O}_{X,x}$, $\Phi = \psi \circ \varphi^{-1}$ = isom of $\mathcal{O}_{X,x}$-mod.

i.e. A = ring, $A \xrightarrow[\cdot a]{\sim} A$ isom of A-modules

$\begin{pmatrix} 1 \longmapsto a \\ b \longmapsto a \cdot b \end{pmatrix}$ $(\cdot a)$ = isom $\Longleftrightarrow a \in A^*$

Φ = mult with some $h \in \mathcal{O}_{X,x}^*$.

or: $h(x) \cdot s(x)$

Have to exclude: $\Gamma(X, \mathcal{O}(D)) \to k$, $s \longmapsto s(x)$ is the zero-map

i.e. $s(x) = 0 \ \forall s \in \Gamma(X, \mathcal{O}(D))$

What does $s(x) = 0 \ \forall s \in \Gamma(X, \mathcal{O}(D))$ mean?

$\Updownarrow$ $s_x \in \mathfrak{m}_x \cdot \mathcal{O}(D)_x$ ($\overset{\varphi}{\longleftrightarrow} \mathfrak{m}_x \cdot \mathcal{O}_{X,x} \longleftrightarrow$ value in x is 0)

$\Downarrow$ $\forall s \in \Gamma(X, \mathcal{O}(D)) : s_x \in \mathfrak{m}_x \mathcal{L}_x$, i.e. $\{ s_x \mid s \in \Gamma(X, \mathcal{O}(D)) \} \subseteq \mathcal{L}_x$ does <u>not</u> generate $\mathcal{L}_x$

φ_D is <u>not</u> defined in $x \Longleftrightarrow x$ = base point of D.

$\rightsquigarrow \varphi_D : X \setminus \mathrm{Bp}(D) \longrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}(D))^\vee)$

bp-free divisor $\rightsquigarrow \varphi_D : X \to \mathbb{P}^m$

otherwise: $\varphi_D : X \dashrightarrow \mathbb{P}^m$

$\mathcal{O}(1)$ = special sheaf on $\mathbb{P}(\Gamma(X, \mathcal{O}(D))^\vee) = \mathbb{P}^m$, $\Gamma(X, \mathcal{O}(D))^\vee \cong k^{m+1}$

$\varphi_D : X \dashrightarrow \mathbb{P}^m$

$\varphi_D^* \mathcal{O}(1) = \mathcal{O}_X(D)$ | $\Gamma(\mathcal{O}(1)) = \Gamma(X, \mathcal{O}(D)) \xrightarrow{id} \Gamma(X, \mathcal{O}(D))$

$\Gamma(\mathcal{O}_{\mathbb{P}^m}(1)) \longrightarrow \Gamma(X, \varphi_D^* \mathcal{O}(1))$

$\Gamma(X, \mathcal{O}(D)) \longmapsto \varphi_D^*$

$\varphi : X \longrightarrow \mathbb{P}^m$

Recall: $\Gamma(\mathbb{P}^m, \mathcal{O}(1)) = \mathrm{Span}_k \{ \underbrace{z_0, \ldots, z_m}_{\text{hom. coord. of } \mathbb{P}^m} \}$

$\Gamma(\mathbb{P}(\Gamma(\mathcal{O}(D))^\vee), \mathcal{O}(1)) = \Gamma(X, \mathcal{O}(D))$

$\Gamma(\mathbb{P}(V^\vee), \mathcal{O}(1)) = V$ or $V^\vee$?

2nd version: $s_0, \ldots, s_m \in \Gamma(X, \mathcal{O}(D))$

$X \dashrightarrow \mathbb{P}^m$ $X \overset{\iota}{\hookrightarrow} \mathbb{P}^m$ $\iota^* \mathcal{O}(1) = \mathcal{O}(D)$, $D \hat{=}$ intersect. $X \cap$ hyperplanes

$x \longmapsto (s_0(x) : \ldots : s_m(x)) \in \mathbb{P}^m$

$\Rightarrow$ not all $s_0(x), \ldots, s_m(x)$ vanish

Assume x is not a bp of $s_0, \ldots, s_m$, i.e.

$\Gamma(X, \mathcal{O}(D)) \longrightarrow \mathcal{O}(D)_x$, $s \longmapsto s_x$ } $(s_0)_x, \ldots, (s_m)_x$ generate $\mathcal{O}(D)$ in x.

If $s_i(x) \neq 0 \rightsquigarrow U_i \subseteq \mathbb{P}^m$, $U_i = \mathrm{Spec}\, k\left[\frac{z_0}{z_i}, \ldots, \frac{z_m}{z_i}\right]$

$\rightsquigarrow X \dashrightarrow U_i$ $\hat{=}$ $k\left[\frac{z_0}{z_i}, \ldots, \frac{z_m}{z_i}\right] \longrightarrow B$

$X \supseteq V$ = affine = $\mathrm{Spec}\, B$, $V \to U_i$

$\frac{z_\nu}{z_i} \longmapsto \frac{s_\nu}{s_i}$

$s_0, \ldots, s_m \neq$ rat. fct., but $\frac{s_0}{s_i}, \ldots, \frac{s_m}{s_i}$ are $\in B$ $(\in \mathcal{O}_{X,x})$

Example. $\mathbb{P}^2 \overset{\pi}{\dashrightarrow} \mathbb{P}^1$ $\mathcal{O}(1) \mid \mathbb{P}^1 \rightsquigarrow \pi^* \mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^2}(1)$

$(z_0 : z_1 : z_2) \longmapsto (z_1 : z_2)$ $z_1, z_2 \in \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \longmapsto \boxed{z_1, z_2 \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))}$

not defined in $(1 : 0 : 0)$

C = normal curve $\Leftrightarrow$ smooth (regular) over $k = \bar{k}$

($D \subset X$ prime divisor $\rightsquigarrow \mathcal{O}_{X,\eta(D)}$ = 1-dim. reg. ring, $\mathfrak{m}_{X,\eta(D)} = (t)$)

$\rightsquigarrow \left[D \longmapsto \deg D : D = \sum_{p \in C} \lambda_p \cdot [p] \rightsquigarrow \deg D := \sum \lambda_p \right]$, $\lambda_p \in \mathbb{Z}$

Remark: $D \geq 0 \Rightarrow \deg D \geq 0$

D = Cartier divisor on some X, $f : Y \longrightarrow X$ map (Sedman schemes)

① • $f^*D = \sum \lambda_p \cdot (f^*P)$ if $D = \sum \lambda_p P$, P = prime divisor of X

• $f^*P := f^{-1}P$ (ex: $Y = \mathbb{C}^1 = \operatorname{Spec} \mathbb{C}[x]$; $f^{-1}P = V(x^2) = \operatorname{Spec} \mathbb{C}[x]/x^2 \rightsquigarrow 2 \cdot V(x)$)

BUT: $\mathbb{C}^1 \xrightarrow{f} V(y - x^2) \subseteq \mathbb{C}^2$, $f^{-1}(P) = \mathbb{C}^1$

② Pull back Cartier divisors:

$f : Y \longrightarrow X$, (U_i, g_i), $g_i \in K(X)$

$(f^{-1}U_i, f^*(g_i))$ if f = dominant!

$K(Y) \longleftarrow K(X)$

$f^*(g_i) \longleftarrow g_i$

$\operatorname{Spec} B = f^{-1}(U) \supseteq B \longleftarrow A$ ($= \operatorname{Spec} A \subset X$)

Pulling back of Cartier divisor classes works!

$D \hat{=} \boxed{\mathcal{O}(D) \subseteq K(X)}$ inv. sheaf. Sol: $[D] \hat{=} \mathcal{O}(D)$ as an abstract / invertible sheaf $\mathcal{L}$

$f : Y \longrightarrow X \rightsquigarrow f^*\mathcal{L}$ = invertible sheaf $\rightsquigarrow$ choose some $f^*\mathcal{L} \hookrightarrow K(Y) \rightsquigarrow$ divisor D'

($\mathcal{L} \to K(X) \Rightarrow \cancel{f^*\mathcal{L} \to K(Y)}$)

$f^*[D] = [D']$

Special case: $C \overset{\iota}{\hookrightarrow} X$ curve, $D | X$ Cartier divisor $\rightsquigarrow \iota^*[D]$ = class of divisors in C $= [D']$, D' = div. on C

$(C \cdot D) := \deg_C D' \Longleftarrow$ need: **Theorem:** $f \in K(C)^* \longrightarrow \deg(\operatorname{div}(f)) = 0$ st: C = complete, eg: C = projective $\cap$ (closed) $\mathbb{P}^N$

Proof: We know this for $\mathbb{P}^1$

• example: $C = \mathbb{A}^1 = \mathbb{C}^1 = \operatorname{Spec} \mathbb{C}[x]$, $f := x \rightsquigarrow \operatorname{div}(f) = 1 \cdot [x] \rightarrow \deg = 1$

Proof $f \neq$ const $\Rightarrow f : C \overset{\text{dom}}{\twoheadrightarrow} \mathbb{P}^1$

f = flat (almost every $M = k[t]$-module is flat.)

$\Downarrow$

$\forall p \in \mathbb{P}^1 : f^{-1}(p) \subseteq C$, length $\mathcal{O}_{f^{-1}(p)}$ does not depend on p; A (dom.) $\longleftrightarrow k[t]$ $\rightsquigarrow$ d

$p = \eta_{\mathbb{P}^1} \Rightarrow f^{-1}(\eta_{\mathbb{P}^1}) = \eta_C \longrightarrow K(\mathbb{P}^1) \hookrightarrow K(C)$ is a field ext. of degree d,

$p = 0 \Rightarrow f^{-1}(0) \longrightarrow f^*[0]$ = divisor of degree d on C | $f^*[\infty] \Rightarrow \deg d$.

$\operatorname{div}(f)$ on C $= f^*[0] - f^*[\infty] \rightsquigarrow \deg = 0$

Remark: D = Cartier div, $f : Y \to X$. If f^*D makes sense, then $[f^*D] = f^*[D]$ (via the RHS of the previous statement)

Proof $\mathcal{O}(D) = \frac{1}{g_i} \mathcal{O}_{U_i}$ on $U_i \subseteq X$ ($D \hat{=} (U_i, g_i)$ on this chart)

• Recipe of LHS: $\operatorname{Spec} B_i \xrightarrow{f} \operatorname{Spec} A_i \longleftrightarrow \varphi : A_i \longrightarrow B_i$, $g_i \longmapsto \varphi(g_i) \rightsquigarrow (f^*D) \hat{=} (\operatorname{Spec} B_i, \varphi(g_i))$ IF $\varphi(g_i) \neq 0$ in B_i = domain.

• Recipe of RHS: $\mathcal{O}_{U_i} \xleftarrow{\sim} \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{U_j}$; $\varepsilon_{ij} = g_i / g_j$

$\mathcal{O}_{U_i \cap U_j} \xrightarrow[\text{of } \mathcal{O}\text{-mod}]{\text{isom } \cdot \varepsilon_{ij}} \mathcal{O}_{U_i \cap U_j}$ $\hat{=}$ unit $\varepsilon_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ "1-cocycle"

($\varepsilon_{ij} \cdot \varepsilon_{jk} \cdot \varepsilon_{ki} = 1$ "cocycle cond")

$A_{ij} \longrightarrow B_{ij}$, $\varepsilon_{ij} \longmapsto \varphi(\varepsilon_{ij})$ units stay units!

$\mathcal{L} \cong \mathcal{O}_X \Longleftrightarrow \exists\, \eta_i \in \Gamma(U_i, \mathcal{O}^*) : \varepsilon_{ij} = \eta_j \cdot \eta_i^{-1}$

Remark: $f \in K(C)^*$, $\operatorname{div}(f) = P - Q$ (for some pts $P, Q \in C$)

$\Rightarrow f : C \to \mathbb{P}^1 \rightsquigarrow f^{-1}(0) = P$, $f^{-1}(\infty) = Q$ with $\underline{d = 1}$ $\rightsquigarrow K(\mathbb{P}^1) \xhookrightarrow{\deg = 1} K(C)$, i.e. $K(\mathbb{P}^1) \xrightarrow{\sim} K(C)$, i.e. f = birational.

$\boxed{f : C \xrightarrow{\sim} \mathbb{P}^1}$

$U \overset{\text{op}}{\hookrightarrow} Z$, Z' ② – this does not always exist!

$\widetilde{\mathbb{P}}^2$ = blow up of $\mathbb{P}^2$ in $(0:0:1)$

$\mathbb{P}^2 \setminus \{(0:0:1)\} \longrightarrow \mathbb{P}^2$, $\mathbb{P}^1$

Example: $\mathbb{P}^1 = E \overset{\iota}{\hookrightarrow} \widetilde{\mathbb{A}}^2$ ($\xrightarrow{\pi} \mathbb{A}^2$), $E = \pi^{-1}(0)$

$(E \cdot E) = -1$ (curve $\iota : \mathbb{P}^1 \to \widetilde{\mathbb{A}}^2$; Cartier div.)

$\iota^*\mathcal{O}(E)$: $k\left[\frac{x_2}{x_1}\right] \xleftarrow{\iota^*} k\left[x_1, \frac{x_2}{x_1}\right] \longleftarrow k[x_1, x_2]$

$\boxed{\frac{1}{x_1} \cdot k\left[x_1, \frac{x_2}{x_1}\right]}$; $(x_1, x_2) \rightsquigarrow (x_1, x_2)$; $(x_1, x_1 \frac{x_2}{x_1}) = (x_1)$ = eqn. of E

$k\left[\frac{x_1}{x_2}\right] \xleftarrow{\iota^*} k\left[x_2, \frac{x_1}{x_2}\right]$

$\mathcal{O}(E)$: $\boxed{\frac{1}{x_2} \cdot k\left[x_2, \frac{x_1}{x_2}\right]}$

$\Rightarrow \mathcal{O}(E)$ has $\frac{1}{x_1} / \frac{1}{x_2} = \frac{x_2}{x_1}$ as 1-cocycle (unit in $k\left[x_1, x_2, \frac{x_1}{x_2}, \frac{x_2}{x_1}\right]$) $\longrightarrow k\left[\frac{x_1}{x_2}, \frac{x_2}{x_1}\right]$, $\longmapsto \boxed{\frac{x_2}{x_1}}$

$C \overset{\iota}{\hookrightarrow} X$ curve, $D \hookrightarrow X$ effective Cartier divisor $\Big\}$ $(C \cdot D) = \deg \iota^*D \geq 0$ (exists IF $C \not\subset D$)

$E = \mathbb{P}^1$, $\mathcal{O}(1)$

$k\left[\frac{x_2}{x_1}\right] \cdot \{x_1, x_2\} = k\left[\frac{x_2}{x_1}\right] \cdot x_1 \xrightarrow[\sim]{\cdot \frac{1}{x_1}} k\left[\frac{x_2}{x_1}\right]$

$k\left[\frac{x_1}{x_2}\right] \cdot \{x_1, x_2\} = k\left[\frac{x_1}{x_2}\right] \cdot x_2 \xrightarrow{\sim} k\left[\frac{x_1}{x_2}\right] \cdot \frac{x_2}{x_1}$

cocycle $\frac{x_1}{x_2}$

$\Rightarrow \iota^*(\mathcal{O}(E)) = \mathcal{O}_{\mathbb{P}^1}(-1)$

D = Sp-free on X ; $C \overset{i}{\hookrightarrow} X$ curve $\Rightarrow (D \cdot C) \geq 0$.

<u>Proof</u> $x \in C$ (some pt) $\rightsquigarrow \exists D' \in |D|$ ($D' \sim D$, $D' \geq 0$) s.t. $x \notin \operatorname{Supp} D'$, $x \notin D' \Rightarrow C \not\subseteq D'$

$\Rightarrow (D \cdot C) = (D' \cdot C) = \deg(i^* D') \geq 0$

$D' \geq 0 \Rightarrow i^* D' \geq 0$

<u>Cor</u> $E \subseteq \tilde{\mathbb{A}}^2$ is not Sp-free!

<u>Def</u> $[D]$ is <u>nef</u> $\Leftrightarrow \forall C$ = curve in X : $(C \cdot D) \geq 0$

"~~numerically~~ effective"

"numerically eventually <u>free</u>"

Spfree $\rightarrow$ nef

<u>Def</u> $[D]$ = "Semi ample" $\Leftrightarrow \exists k \geq 1$: (kD) = Sp-free.

<u>Ex</u> If C = curve, $\underbrace{P - Q}_{D :=}$ $\neq$ principal divisor (exists, e.g. on all curves $\neq \mathbb{P}^1$)

$\cdot$ D = nef (because of $\deg D = 0$)

$\cdot$ Let $D' \in |D| \rightsquigarrow \left.\begin{array}{l} D' \sim D \Rightarrow \deg D' = 0 \\ D' \geq 0 \end{array}\right\}$ $D' = 0 \Rightarrow D$ = principal $\lightning$

Δ = lattice polytope in $M \cong \mathbb{Z}^d \rightsquigarrow \mathcal{N}(\Delta) = \Sigma \Rightarrow TV(\Sigma) \to \mathbb{P}(\Delta) \subseteq \mathbb{P}^{\Delta \cap M}$

∇ = another lattice polytope : <u>Prop</u>. $\exists \nabla'$: $\nabla + \nabla' = \ell \cdot \Delta$ (some $\ell \in \mathbb{Q}_{\geq 0}$) $\Leftrightarrow \Sigma < \mathcal{N}(\nabla)$

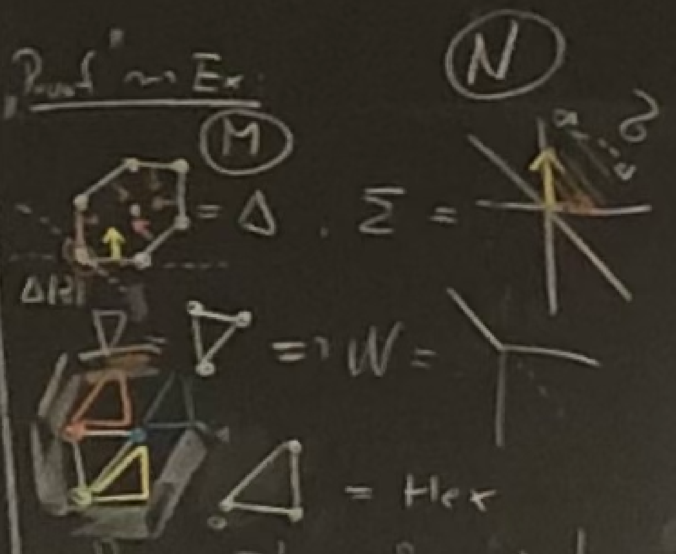
tail $\Delta := \{ r \in M_{\mathbb{R}} \mid r + \Delta \subseteq \Delta \}$ = V = tail(Δ)

$\rightarrow$ for a tail cone (ex: tail $= 0$)

$\hookrightarrow$ $\cdot$ $\nabla + \nabla' = \nabla'' + \nabla' \Rightarrow \nabla = \nabla''$

$\cdot$ s.s. of pol. with tail $= \sigma \rightarrow \sigma$ = weakly dual.

"Proof" $\rightsquigarrow$ Ex:



(M) $= \Delta$, $\Sigma =$ (N)

$\Delta(k)$

$\nabla \Rightarrow W =$

Δ = Hex

$\nabla + \nabla' = \{ r + r' \mid r \in \nabla, r' \in \nabla' \}$

<u>Ex</u>: $\longmapsto + | = \square$

$a \in N \longmapsto \min \langle \Delta, a \rangle$ = pw. linear. linear $= \langle \Delta(\sigma), a \rangle$ for $a \in \sigma$

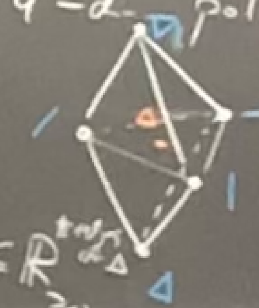
Σ = fan of linearity regions

<u>Def</u>. $C(\Delta) := \{ \nabla \mid \exists \nabla' : \nabla + \nabla' = \mathbb{Q}_{\geq 0} \cdot \Delta \}$ "Minkowski summands of Δ"

$\nabla_1 \quad \nabla_1' \quad \ldots \Rightarrow$ polyhedral cone.

<u>Ex</u> $C(\text{Hex})$ — 4-dim polyh. cone

crosscut =



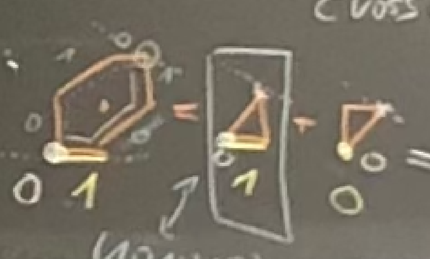
$\cdot$ $0 \in C(\Delta)$

$\cdot$ $\Delta \in C(\Delta)$ $\quad \mathbb{Q}_{\geq 0} \cdot \Delta$

$C(\text{Hex}) \subseteq \mathbb{R}^6_{\geq 0}$

<u>Alternative way</u> to describe $C(\Delta)$:

$\nabla \in C(\Delta)$, i.e. $\nabla + \nabla' = \Delta$



$0 \; 1 \quad 1 \quad 0$

$(1,0,1,0,1,0)$

$\Rightarrow C(\Delta) \subseteq \mathbb{R}^{\#\text{edges of } \Delta}_{\geq 0}$

$\nabla \in C(\Delta) \Rightarrow \exists \mathcal{O}(\nabla)$ = inv. sheaf on $X = TV(\Sigma)$ ($\sigma \in \Sigma \rightsquigarrow$ vertex $\nabla(\sigma) \in \nabla$, $\mathcal{O}(\nabla)|_{TV(\sigma)} = x^{\nabla(\sigma)} \cdot \mathcal{O}_{TV(\sigma)}$)

$\Downarrow$

$TV(\Sigma) \xrightarrow{\pi} \mathbb{P}(\nabla) \hookrightarrow \mathbb{P}^{\nabla \cap M}$ (special case: $\nabla = \Delta$)

$\cdot$ $\mathcal{O}(\nabla)$ = the inv. sheaf

$\cdot$ $\Gamma(TV(\Sigma), \mathcal{O}(\nabla))$ = k-vs, base = $\nabla \cap M$ $\quad$ <u>ex</u>: $\mathbb{P}^2 \rightsquigarrow \Delta \cong \mathcal{O}(1)$

Sp-free $\Rightarrow TV(\Sigma) \to \mathbb{P}(\Gamma(\mathcal{O}(\nabla))^*)$