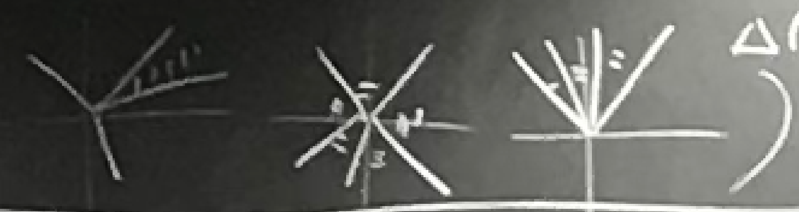
$\Delta \subseteq M_{\mathbb{R}}$ lattice polytope, Σ = fan in $N_{\mathbb{R}}$: $\Sigma \leq \mathcal{N}(\Delta)$

$\rightsquigarrow X = TV(\Sigma) \longrightarrow \mathbb{P}(\Delta) \subseteq \mathbb{P}^{\Delta \cap M}$: induced by $\mathcal{O}_X(\Delta)$ locally $= \chi^{\Delta(\sigma)} \cdot \mathcal{O}_{TV(\sigma)}$

$\Delta(\sigma)$ = vertex of Δ assoc. to σ (full-dim)

(„exotic" fans )

Prop: $\mathcal{O}(\Delta)$ is basepoint free, i.e. $\mathcal{O}(\Delta)$ is globally generated.

Cartier divisor

Proof: $\Gamma(X, \mathcal{O}(\Delta)) = \bigcap_{\sigma \in \Sigma} \Gamma(TV(\sigma), \mathcal{O}(\Delta)) \triangleq \bigcap_{\sigma \in \Sigma} \left[\Delta(\sigma) + (\sigma^{\vee} \cap M) \right] = \Delta \cap M$

$\cap \qquad\qquad \cap$

$k[M] \ni \chi^{r} \longleftrightarrow r \in M$

$\Delta(\sigma)$

in particular:

$\mathcal{O}_{TV(\sigma)} \otimes \Gamma(X, \mathcal{O}(\Delta)) \longrightarrow \Gamma(TV(\sigma), \mathcal{O}(\Delta)) = \mathcal{O}_{TV(\sigma)}$-module $\Rightarrow \Gamma(\mathcal{O}(\Delta))$ generate $\mathcal{O}(\Delta)$ in the points

$\chi^{\Delta(\sigma)} \in \Gamma(X, \mathcal{O}(\Delta))$; $\Gamma(TV(\sigma), \mathcal{O}(\Delta))$ generated by $\chi^{\Delta(\sigma)}$

T-fixed pt $= \operatorname{orb}(\sigma) \in X$.

$\Rightarrow$ all $\operatorname{orb}(\sigma)$ ($\sigma \in \Sigma$ full-dim) are not base points of $\mathcal{O}(\Delta)$

$Bp(\mathcal{O}(\Delta)) \subseteq X$ closed subset, T-invariant $\quad \left[Bp \neq \emptyset \Rightarrow \exists \text{ full-dim. } \sigma \in \Sigma \quad \operatorname{orb}(\sigma) \in Bp \right]$

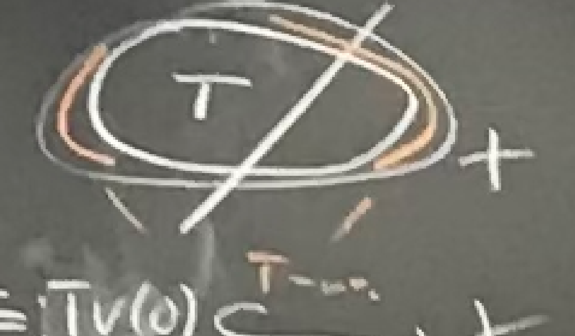
$\Delta \rightsquigarrow$ bp-free $\mathcal{O}(\Delta)$ $\quad \|$ Start with $\Sigma \rightsquigarrow \exists$ „all" polytopes ∇: $\mathcal{N}(\nabla) \geq \Sigma$

$\Sigma = \mathcal{N}(\Delta)$; $\nabla \in C(\Delta)$

Σ = given fan $\rightsquigarrow X = TV(\Sigma)$

D = Cartier divisor $\triangleq \mathcal{L} := \mathcal{O}(D)$ = inv. sheaf $\subseteq \mathcal{K}(X)$

T-inv $\quad (D_{|T} = 0) \quad D = \sum_{\rho \in \Sigma(1)} \lambda_{\rho} \cdot \overline{\operatorname{orb}(\rho)} \iff D = \{ (TV(\sigma), \chi^{r_\sigma}) \}$, $\quad k[M] \ni \chi^{r_\sigma}$, $r_\sigma \in M$



$T = TV(0) \xhookrightarrow{T\text{-inv.}} X$

σ, σ' = adjacent full-dim. cones $\rightsquigarrow \cdot\; TV(\sigma \cap \sigma') = TV(\sigma) \cap TV(\sigma')$

$\tau := \sigma \cap \sigma'$ = codim-1-face of σ and of σ' = $(d-1)$-dim cone.

$\dim \Sigma = d$

$\cdot\; \chi^{r_\sigma}\big|_{\sigma \cap \sigma'} = \chi^{r_{\sigma'}}\big|_{\sigma \cap \sigma'} \iff r_\sigma - r_{\sigma'} \in (\sigma \cap \sigma')^{\perp} = \tau^{\perp}$ = 1-dim. subspace of $M_{\mathbb{R}}$.

M, $\tau^{\perp}$

Now: $\Delta \rightsquigarrow \Sigma = \mathcal{N}(\Delta)$. $\quad D \longleftrightarrow r_\sigma \in M$

$\Delta \longleftrightarrow \Delta(\sigma) \in M \quad \rightsquigarrow \Delta(\sigma) - \Delta(\sigma') \in \tau^{\perp}$

no surprise: $\tau^{\perp} \parallel$ edge $\overline{\Delta(\sigma)\Delta(\sigma')}$

$\Sigma \ni \tau$ (codim $=1$) $\longleftrightarrow$ edge in Δ (dim $=1$)

$\boxed{\tilde{D}}$ $\qquad \boxed{(r^{\sigma} - r^{\sigma'}) \in \underset{\lambda_D}{\mathbb{R}} \cdot (\Delta(\sigma) - \Delta(\sigma'))}$

$D + N \cdot \Delta \quad \rightsquigarrow$ new data $\tilde{r}^{\sigma} = r^{\sigma} + N \cdot \Delta(\sigma)$

Cart. div $\triangleq \mathcal{O}(\Delta)$

$(\tilde{r}^{\sigma} - \tilde{r}^{\sigma'}) = (r^{\sigma} - r^{\sigma'}) + N(\Delta(\sigma) - \Delta(\sigma')) \rightsquigarrow$ coeff. $\lambda_D + N \geq 0$, $N \geq 0$

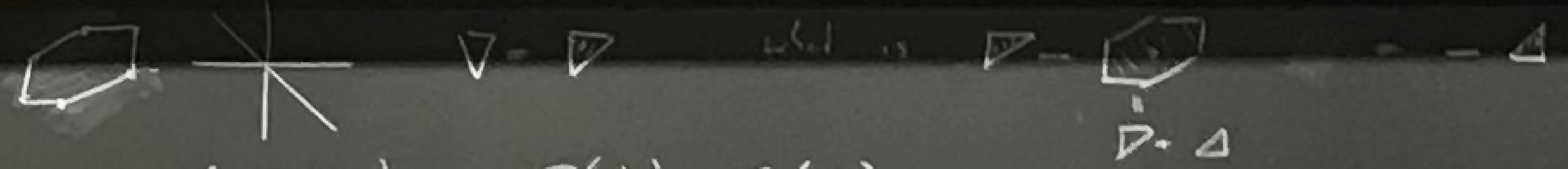
$\lambda_{\tilde{D}} = \lambda_D + N \geq 0$ is a dilation factor (for each $\sigma\sigma'$)

$\rightsquigarrow \{\lambda_{\tilde{D}}\}$ define an element $\nabla \in C(\Delta)$.

$\rightsquigarrow$ Prop: D = T-inv Cart. div in $TV(\Sigma)$, $\Sigma = \mathcal{N}(\Delta)$ $\Rightarrow$ for $N \gg 0$: $D + N \cdot \Delta \triangleq \nabla \in C(\Delta)$

$\Rightarrow D = \nabla - N \cdot \Delta$ — difference of 2 Cartier div. = diff. of these 2 polytopes.

Recall: $\nabla_1, \nabla_2 \rightsquigarrow \mathcal{O}(\nabla_1 + \nabla_2) = \mathcal{O}(\nabla_1) \otimes \mathcal{O}(\nabla_2)$ (Mink sum)

Ex: 

$\nabla = \nabla$ ... $\tilde{D} = \Delta$

Corollary. $\operatorname{Pic}(TV(\Sigma)) = C(\Delta) - C(\Delta)$

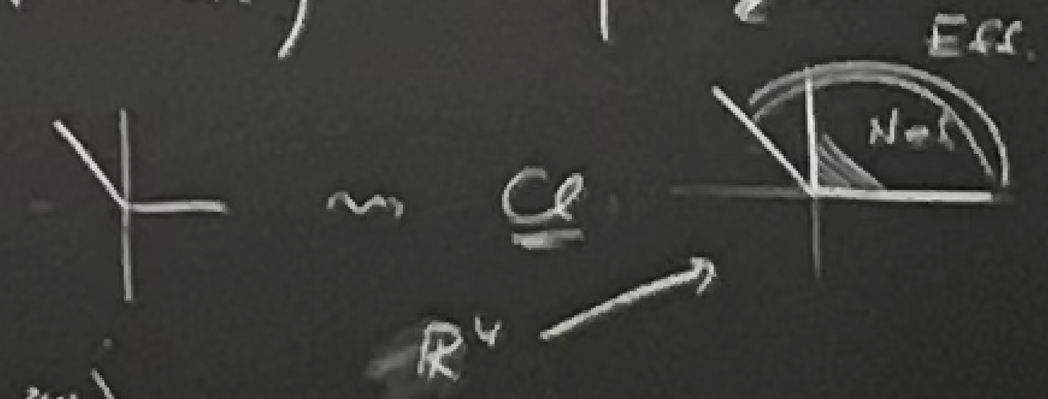
= abelian grp gen. by $C(\Delta) \supset C(\Delta)$ = „nef cone" = $\{ \mathcal{L} \in \operatorname{Pic} \mid \mathcal{L} = \text{nef} \}$

$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow Cl(X) = \operatorname{Pic}(X) \longrightarrow 0 \quad (X = \text{smooth}) \quad | \otimes_{\mathbb{Z}} \mathbb{R}$

$0 \longrightarrow M_{\mathbb{R}} \longrightarrow \mathbb{R}^{\Sigma(1)} \xrightarrow{\pi} \boxed{Cl(X) \otimes_{\mathbb{Z}} \mathbb{R}} \longrightarrow 0$

f.d. vs $\quad \mathbb{R}^{\Sigma(1)} \supseteq \mathbb{R}_{\geq 0}^{\Sigma(1)} \longrightarrow\!\!\!\!\rightarrow \operatorname{Eff}(X) \subseteq Cl(X)\otimes_{\mathbb{Z}}\mathbb{R}$

Prop: $\mathbb{R}_{\geq 0}^{\Sigma(1) \setminus \sigma(1)} \xrightarrow{\pi} \operatorname{Nef}(X) = \bigcap_{\sigma \in \Sigma} \pi(\mathbb{R}^{\Sigma(1) \setminus \sigma(1)}) \subseteq \operatorname{Eff}(X)$

$\mathbb{F}_1$:  $\rightsquigarrow Cl$ — Eff., Nef, $\mathbb{R}^4$

Careful: X = smooth $/\mathbb{C}$, non-toric $\quad \operatorname{Div}(X)$, $\operatorname{Curves}(X)$

$\rightsquigarrow \operatorname{Div}(X) \times \operatorname{Curves}(X) \longrightarrow \mathbb{Z}$

$Cl(X) \times A_1(X) \longrightarrow$

divide out kernel of pairing $\Rightarrow \boxed{N^1(X)} \times \boxed{N_1(X)} \longrightarrow \mathbb{Z}$

$N^1(X) = \operatorname{Div}(X) / \equiv \quad (D \equiv D' \iff (D.C) = (D'.C)\ \forall C)$

Rem. $\Delta \rightsquigarrow \Sigma = \mathcal{N}(\Delta)$. $\quad D \longleftrightarrow r_\sigma \in M$

$\Delta \longleftrightarrow \Delta(\sigma) \in M \rightsquigarrow \Delta(\sigma) - \Delta(\sigma') \in \tau^\perp$

$\boxed{\tilde{D}}$ $\quad \boxed{(r^\sigma - r^{\sigma'}) \in \mathbb{R} \cdot (\Delta(\sigma) - \Delta(\sigma'))}$

no surprise: $\tau^\perp \parallel$ edge $\Delta(\sigma)\Delta(\sigma')$

$\Sigma \ni \tau \longleftrightarrow$ edge in Δ
(codim $=1$) $\quad$ (dim $=1$)

$D + N \cdot \Delta \rightsquigarrow$ new data $\tilde{r}^\sigma = r^\sigma + N \cdot \Delta(\sigma)$

Cart. div $= \mathcal{O}(\Delta)$ $\quad (\tilde{r}^\sigma - \tilde{r}^{\sigma'}) = (r^\sigma - r^{\sigma'}) + N(\Delta(\sigma) - \Delta(\sigma')) \rightsquigarrow$ coeff. $\lambda_{\tilde{D}} + N \geq 0$

$\lambda_{\tilde{D}} = \lambda_D + N \geq 0$ is a dilation factor (for each $\sigma\sigma'$) $\quad \geq 0$

$\rightsquigarrow \{\lambda_{\tilde{D}}\}$ define an element $\nabla \in C(\Delta)$.

$\rightsquigarrow$ Prop: $D = T$-inv. Cart. div in $TV(\Sigma)$ $\Rightarrow$ for $N \gg 0$: $D + N \cdot \Delta^{\mathcal{L}} \nabla \in C(\Delta)$

$\mathcal{N}(\Delta)$ $\quad \Rightarrow D = \nabla - N \cdot \Delta$ difference of 2 Cartier div.

Recall: $\nabla_1, \nabla_2 \rightsquigarrow \mathcal{O}(\nabla_1 + \nabla_2) = \mathcal{O}(\nabla_1) \otimes \mathcal{O}(\nabla_2)$

Ex. $\quad \nabla = \nabla \quad$ what is $\nabla -$? $\quad - = - \Delta$

$\nabla + \Delta$

Corollary: $\mathrm{Pic}(TV(\Sigma)) = C(\Delta) - C(\Delta)$
$=$ abelian grp gen. by $C(\Delta) \supset C(\Delta) =$ „nef cone"
$= \{ \mathcal{L} \in \mathrm{Pic} \mid \mathcal{L} = \text{nef} \}$

$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to Cl(X) = \mathrm{Pic}(X) \to 0 \quad (X = \text{smooth}) \quad | \otimes_{\mathbb{Z}} \mathbb{R}$

$0 \to M_{\mathbb{R}} \to \mathbb{R}^{\Sigma(1)} \xrightarrow{\pi} \boxed{Cl(X) \otimes_{\mathbb{Z}} \mathbb{R}} \to 0$

d.d. vs $\quad \mathbb{R}^{\Sigma(1)}_{\geq 0} \twoheadrightarrow \mathrm{Eff}(X)$

$\mathbb{F}_1$: $\rightsquigarrow Cl$ $\quad$ Eff, Nef

Prop: $\mathbb{R}_{\geq 0}^{\Sigma(1) \setminus \sigma(1)} \xrightarrow{\pi}$ $\quad \mathrm{Nef}(X) = \bigcap_{\sigma \in \Sigma} \pi(\mathbb{R}_{\geq 0}^{\Sigma(1) \setminus \sigma(1)})$

Careful: $X =$ smooth $/ \mathbb{C}$, non-toric $\quad \mathrm{Div}(X)$, $\mathrm{Curves}(X) \rightsquigarrow \mathrm{Div}(X) \times \mathrm{Curves}(X) \to \mathbb{Z}$

divide out kernel of pairing $\Rightarrow N^1(X) \times N_1(X) \to \mathbb{Z}$

$N^1(X) = \mathrm{Div}(X)/\ldots \quad (D \equiv D' \Leftrightarrow (D.C) = (D'.C))$

$$\begin{array}{ccccc} \mathrm{Div}(X) & \times & \mathrm{Curves}(X) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \| \\ Cl(X) & \times & A^1(X) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \| \\ N^1(X) & \times & N_1(X) & \longrightarrow & \mathbb{Z} \end{array}$$

$\Delta \rightsquigarrow \Sigma = \mathcal{N}(\Delta) \rightsquigarrow X = TV(\Sigma)$

$\nabla_1, \nabla_2 \in C(\Delta) \iff \Sigma \leq \mathcal{N}(\nabla_i) \rightsquigarrow \mathcal{O}(\nabla_i) = \mathcal{O}_X$-mod.

$X =$ scheme $\rightsquigarrow \boxed{\mathcal{O}_X} =$ ring sheaf.

diff. geom.: $X =$ mf $\rightsquigarrow$ vb $Y \longrightarrow X$, $s =$ section of $Y \to$ sheaf
over $\mathbb{C}$

ex. $X \times \mathbb{C} \longrightarrow X$ $\quad$ regular sect $\iff \mathcal{O}_X$

Goal: Re-understand: Tangent-Bundle, Cotangent-Bundle, in the language of sheaves.

Def • $R =$ ring, $M = R$-module $\rightsquigarrow d: R \to M$ is a derivation $:\iff$ ① $d =$ additive
② $d(f \cdot g) = f\, d(g) + g \cdot d(f)$ („Leibniz rule")

Ex. $R = \mathbb{k}[x]$, $d = \frac{\partial}{\partial x}$: $\mathbb{k}[x] \to \mathbb{k}[x]$, $f(x) \mapsto f'(x)$ $\quad \mathbb{k}[x]$-module $\quad \mathbb{k}[x]/(x-2) \xrightarrow{\sim} \mathbb{k}$

$d = \frac{\partial}{\partial x}(2)$: $\mathbb{k}[x] \to \boxed{\mathbb{k}}$, $f(x) \mapsto \frac{\partial f}{\partial x}(2)$

• $A \xrightarrow{\varphi} R$ algebra, $M = R$-module $\rightsquigarrow d: R \to M$ is an „A-derivation" $:\iff$ • $d|_A = 0$ $\iff$ $d(\varphi(a)) = 0$
• $d = A$-linear.

Proof (of equiv.) $\quad \underbrace{d(a \cdot f)}_{\Leftarrow A\text{-linearity}} = a \cdot d(f) + \underbrace{f \cdot d(a)}_{=0 \Leftarrow d|_A = 0}$

Prop: $\exists!$ R-module $\Omega_{R|A}$ s.t.: $\mathrm{Hom}_R(\Omega_{R|A}, M) = \boxed{\mathrm{Der}_A(R, M)}$
d = univ. A-derivation

Proof: $\Omega_{R|A} :=$ free R-module, basis $= \{ „D_r" \mid r \in R \} / \cdot D_{r+r'} = D_r + D_{r'}$, $\cdot D_{r r'} = r\, D_{r'} + r'\, D_r$ $\quad \rightsquigarrow \overline{D_r} =: dr$

$\boxed{+} + \boxed{s} + \boxed{rs}$, $d = \boxed{+}$, $(r \in M \rightsquigarrow \mathbb{Z}^r \in \mathbb{Z}[M])$

Example: $\Omega_{\mathbb{k}[x]|\mathbb{k}} = \mathbb{k}[x] \cdot dx \cong \mathbb{k}[x]$ (free mod of rk 1) $\quad D: \mathbb{k}[x] \to M$ = derivation, $x \mapsto D(x) \in M$, $\mathbb{k}[x]\,dx$, dx

$\Omega_{\mathbb{k}[x,y]|\mathbb{k}} = \mathbb{k}[x,y] \cdot dx \oplus \mathbb{k}[x,y]\, dy$

Prop: $A \to B \to C$ $\quad \Omega_{B|A} \otimes C \to \Omega_{C|A} \to \Omega_{C|B} \to 0$