Differentials

$A \to B$, $M = B$-module $\rightsquigarrow \mathrm{Der}_A(B,M) \ni d$

$d : B \to M$

$d(bc) = b\cdot d(c) + c\cdot d(b)$

$d = A$-linear $\Longleftrightarrow d_{|A} = 0$

$\exists!$ B-module $\Omega_{B|A}$ with A-derivation $\boxed{d : B \to \Omega_{B|A},\ b \mapsto db}$ (trivial)

$\longleftrightarrow$ $\mathrm{Hom}_B(\Omega_{B|A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B,M)$, $\varphi \longmapsto \varphi\circ d$

Proof: $\Omega_{B|A} := B^{\oplus\{\tilde{d}b \mid b\in B\}} \Big/ \begin{array}{l}\cdot\ \tilde{d}(b+b') - \tilde{d}b - \tilde{d}b' \\ \cdot\ \tilde{d}(bc) - b\tilde{d}(c) - c\tilde{d}(b) \\ \vdots \end{array}$

$\overline{\tilde{d}b} =: db$

$B^{\oplus B} = \{\text{finite } \sum_{b\in B} \lambda_b \cdot (b) \mid \lambda_b \in B\}$

Ex: $\left.\begin{array}{l} B = k[x,y] \\ A = k \end{array}\right\}$ $\Omega_{B|k} = k[x,y]\,dx \oplus k[x,y]\,dy$

$(\Omega_{\mathbb{P}^1\setminus\infty})_{|\mathbb{C}^*} \hat{=} k[x,x^{-1}]\cdot dx \;\|\; \Omega_{\mathbb{P}^1\setminus 0\,|\mathbb{C}^*} \hat{=} k[y,y^{-1}]\cdot dy$ $y = x^{-1}$

$\mathbb{P}^1 \supset \mathbb{P}^1\setminus\infty,\ \mathbb{P}^1\setminus 0 \supset \mathbb{C}^*$

$k[x] \to k[z,z^{-1}]$, $k[y] \to k[z,z^{-1}]$

Ex: ② E = ell. pl. c. $\Longrightarrow \Omega_E = \mathcal{O}_E$

① $\mathbb{P}^1$: $\Omega_{\mathbb{P}^1} = ?$ $\quad \Omega_{\mathbb{P}^1} = \mathcal{O}(d) = \mathcal{O}(-2)$, $\boxed{d=-2}$

$\mathbb{A}^1 = \mathbb{P}^1\setminus\{\infty\}$: $\mathrm{Spec}\, k[x]$

$\rightsquigarrow \Omega_{k[x]|k} = k[x]\,\boxed{dx}$

$\mathbb{P}^1\setminus\{0\}$: $\mathrm{Spec}\, k[y]$, $y = \frac{1}{x}$ $\quad \boxed{-\frac{1}{x^2}\cdot dx}$

$\Omega_{k[y]|k} = k[y]\cdot dy = k[\tfrac{1}{x}]\cdot d(\tfrac{1}{x})$

$k[x,\tfrac{1}{x}] \hat{=}$ intersect of both charts $\quad k[\tfrac{1}{x}]$

Remark: $A \to B \rightsquigarrow \boxed{B\otimes_A B} = B$-module

$B\otimes_A B \xrightarrow{\pi} B$, $(b,c) \longmapsto b\cdot c$

$\sum b_i\otimes c_i \ni b$

$\pi = B$-linear map. $\left(\begin{array}{l}\sum b_i\otimes c_i \longmapsto \sum b_i c_i \\ b\cdot \Downarrow \qquad\qquad \Downarrow \\ \sum (bb_i)\otimes c_i \longmapsto \sum (bb_i)c_i\end{array}\right.$

$bb_i\otimes c_i \overset{?}{\longleftrightarrow} b_i\otimes bc_i$ $\quad$ "=" only if $b\in A$ "

Vie: $b\cdot(\sum b_i\otimes c_i) := \sum (bb_i)\otimes c_i$.

$I := \mathrm{Ker}(\pi : B\otimes_A B \longrightarrow B)$

$\|\ A \to B$ $\quad$ ~~$c_1\otimes c_2 \to 0$, $c_1c_2 = 0$~~

$\boxed{\text{Claim: } \Omega_{B|A} \cong I/I^2}$ $\quad D(b) = b\otimes 1 - 1\otimes b$

$db \leftarrow b$, $b \mapsto Db$

Remark: Ω-construction fits well with localization: $A \to B$

- $a\in A$ s.t. $a\in B^* \Rightarrow A_a \to B \Rightarrow \Omega_{B|A_a} = \Omega_{B|A}$.
- $A \to B_b \Rightarrow \Omega_{B_b|A} = \Omega_{B|A}\otimes_B B_b$.

$\to Y \to X$ schemes

eg: $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ $\quad \Longrightarrow \Omega_{Y|X} = \mathcal{O}_Y$-module on Y (i.e. sheaf): $\Omega_{\mathrm{Spec} B|\mathrm{Spec} A} := \widetilde{\Omega_{B|A}}$

Ex: $X = k$-scheme $\Longrightarrow \Omega_X := \Omega_{X|\mathrm{Spec}\, k}$ ($X \to \mathrm{Spec}\, k$)

Hint for Ω_E: $E_0 = V(y^2 - f_3(x))$ = one affine chart $\quad$ f_3 = pol. of degree 3 $\quad$ ex: $y^2 = x(x^2-1)$ $\|$ s.t. f_3 has no double roots.

locally $y^2 = f_3(x) \Longrightarrow d(y^2) = d(f_3(x))$

$\Rightarrow 2\cdot y\cdot dy = f_3'(x)\cdot dx$

$d : k[x,y]/_{y^2-f_3(x)} \longrightarrow \Omega_{E_0|k}$

$x \longmapsto dx$

$y \longmapsto dy$

this is an local gen. for Ω_{E_0} $\Longleftarrow$ $\boxed{\dfrac{dy}{f_3'(x)}} = \boxed{\dfrac{dx}{2\cdot y}}$

look for $(\alpha,\beta)\in E_0$ (i.e. $x=\alpha$, $y=\beta$)

$y = f_3'(x) = 0$

(on E_0: $y^2 = f_3(x)$)

$f_3(x) = f_3'(x) = 0$ $\quad$ ↯

Q: where are both fcts not defined? for which α, β?

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2 exact sequences:

- $A \to B \to C$, $b \mapsto b$

$$\begin{tikzcd} \Omega_{B|A}\otimes_B C \arrow[r] & \Omega_{C|A} \arrow[r] & \Omega_{C|B} \arrow[r] & 0 \\ B \arrow[u, "d"] \arrow[r] & C \arrow[u, "d"] \arrow[r] & C \arrow[u, "d"] \end{tikzcd}$$

$db\otimes c \longmapsto c\cdot db$

$dc \longmapsto dc$

Claim: this sequence is exact

Proof: Apply functor $\mathrm{Hom}_C(\cdot, N)$, N = C-module.

$\mathrm{Hom}_C((B\text{-module } M)\otimes_B C, N) = \mathrm{Hom}_B(M, N_B)$

$\mathrm{Der}_A(B,N) \longleftarrow \mathrm{Der}_A(C,N) \longleftarrow \mathrm{Der}_B(C,N) \longleftarrow 0$

$\boxed{A \to B \to C}$, $\to N$ $\quad$ $\boxed{\begin{array}{l} C \to N \\ B \dashrightarrow 0 \end{array}}$ $\quad\square$

$A \to B \to C$

$X \xleftarrow{f} Y \xleftarrow{g} Z$ schemes $\quad g^*\Omega_{Y|X} \longrightarrow \Omega_{Z|X} \longrightarrow \Omega_{Z|Y} \to 0$

$\Omega_{Z|Y}$ = $\mathcal{O}_Z$-module ($\rightsquigarrow$ sheaf on Z)

Example: $Z \xrightarrow{g} Y$, $\searrow \swarrow$ $\mathrm{Spec}\, k$ $\quad g^*\Omega_Y \to \Omega_Z \to \Omega_{Z|Y} \to 0$

Special situation: $A \to B \to\!\!\to C = B/I$ $\longleftrightarrow$ $X \leftarrow Y \leftarrow Z = V(I)$

$I/I^2 \longrightarrow \Omega_{B|A}\otimes_B B/I \longrightarrow \Omega_{B/I|A} \to 0$

$f \longmapsto df\otimes 1$ $\quad\square$

Hand for Ω_E

$E_0 = V(y^2 - f_3(x))$ = one affine chart ‖ s.t. f_3 has no double roots.

↳ pol. of degree 3 ex: $y^2 = x(x^2-1)$

locally $y^2 = f_3(x) \Rightarrow d(y^2) = d(f_3(x))$

$d: k[x,y]/(y^2 - f_3(x)) \longrightarrow \Omega_{E_0/k}$

$x \longmapsto dx$

$y \longmapsto dy$

$2\cdot y \cdot dy = f_3'(x)\cdot dx$

this is an local gen. for Ω_{E_0} $\Leftarrow$ $\boxed{\dfrac{dy}{f_3'(x)}} = \boxed{\dfrac{dx}{2\cdot y}}$

look for $(\alpha,\beta) \in E_0$ (i.e. $x = \alpha$, $y = \beta$)

$y = f_3'(x) = 0$

(on E_0: $y^2 = f_3(x)$)

$f_3(x) = f_3'(x) = 0$ ↯

Q: when are both fcts not defined? for which α, β?

2 exact sequences:

• $A \to B \to C$

$b \mapsto b$

$$\begin{tikzcd}
\Omega_{B/A}\otimes_B C \arrow[r] & \Omega_{C/A} \arrow[r] & \Omega_{C/B} \arrow[r] & 0 \\
B \arrow[u, "d"] \arrow[r] & C \arrow[u, "d"] \arrow[r] & C \arrow[u, "d"] &
\end{tikzcd}$$

$db \otimes c \longmapsto c\cdot db$

$dc \longmapsto dc$

Claim: this sequence is exact

Proof: Apply functor $\mathrm{Hom}_C(\cdot, N)$, N = C-module.

$\mathrm{Der}_A(B,N) \longleftarrow \mathrm{Der}_A(C,N) \longleftarrow \mathrm{Der}_B(C,N) \longleftarrow 0$

$\mathrm{Hom}_C((B\text{-module } M)\otimes_B C, N) = \mathrm{Hom}_B(M, N_B)$

$\boxed{A \to B \to C}$, $C \to N$, 0

$C \to N$, $B \to 0$ □

$A \to B \to C$

$X \xleftarrow{f} Y \xleftarrow{g} Z$ schemes

$g^*\Omega_{Y/X} \longrightarrow \Omega_{Z/X} \longrightarrow \Omega_{Z/Y} \to 0$

$\Omega_{Z/Y}$ = $\mathcal{O}_Z$-module (~ sheaf on Z)

Example: $Z \xrightarrow{g} Y$, $\searrow \swarrow$ Spec k

$g^*\Omega_Y \to \Omega_Z \to \Omega_{Z/Y} \to 0$

Special situation: $A \to B \twoheadrightarrow C = B/I$ $\longleftrightarrow$ $X \leftarrow Y \hookleftarrow Z = V(I)$

$I/I^2 \longrightarrow \Omega_{B/A}\otimes_B B/I \longrightarrow \Omega_{(B/I)/A} \to 0$

$f \longmapsto df \otimes 1$ □

$Z = V(I) \subseteq Y$ over $X \Rightarrow I/I^2 \to \Omega_{Y/X}|_Z \longrightarrow \Omega_{Z/X} \to 0$

(restriction on Z = applying ι^*, $\iota: Z \hookrightarrow Y$)

Example · $X = \mathrm{Spec}\, k$

· $Y = \mathbb{A}^2 = \mathrm{Spec}\, k[x,y]$

$Z = V(f)$ ($f \in k[x,y]$)

$A = k[x,y]/f = \mathcal{O}_Z$

$\Rightarrow \Omega_{Z/k} = A^2 / \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (k[x,y]/(f))^2 / \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$(f)/(f^2) \longrightarrow \frac{k[x,y]}{f}\, dx \oplus \frac{k[x,y]}{f}\, dy \to \Omega_{Z/k} \to 0$

$\mathcal{O}_Z$; $\mathcal{O}_Z$-module

$f \longmapsto df(x,y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y}\cdot dy$

Ex: $d(\cancel{x^2y + y^2})$

$d(x^2y) = y\cdot d(x^2) + x^2\, dy$, $y \cdot d(x^2) = 2xy\, dx$

$\frac{\partial x^2 y}{\partial x} = 2xy$

$\frac{\partial x^2 y}{\partial y} = x^2$

· X, Y = schemes over k ⇝ $\Omega_X, \Omega_{X/Y}$ — $\mathcal{O}_X$-modules

$X \to Y$

new: · $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) =: T_X$ ($\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$)

„cotangent sheaf" „tangent sheaf on X"

· $T_{X/Y} := \mathcal{H}om(\Omega_{X/Y}, \mathcal{O}_X)$.

Corollary. $X \xrightarrow{f} Y \to S \Rightarrow 0 \to T_{X/Y} \to T_{X/S} \to f^*T_{Y/S}$

$Z = V(I) \Rightarrow 0 \to T_{Z/S} \to T_{Y/S} \otimes \mathcal{O}_Z \to \mathcal{H}om(I/I^2, \mathcal{O}_Z)$