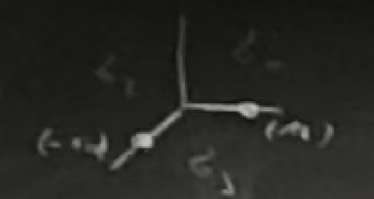
$X \longrightarrow Y \rightsquigarrow \Omega_{X/Y} = \mathcal{O}_X$-modules

Toric situation: Σ = smooth fan in $N_{\mathbb{R}} \cong \mathbb{R}^n$; $|\Sigma| = \mathbb{R}^n$

full-dim'l cones: $\cdot\ \sigma = \langle a^1, \dots, a^n \rangle$

$\cdot\ a^1, \dots, a^n \in N = \mathbb{Z}^n$: $\mathbb{Z}$-basis.

Ex:  $\{(1,0), (-1,-1)\} = \mathbb{Z}^2$-basis

Goal: Calculate $\Omega_{TV(\Sigma)}$; $X = TV(\Sigma)$

$$\boxed{0 \to M \to \mathbb{Z}^{\Sigma(1)} \to Cl(X) \to 0}$$

$r \longmapsto \mathrm{div}(x^r)$

$$\boxed{r \longmapsto (\dots, \langle r, a_\rho \rangle, \dots)}$$

$N \twoheadleftarrow \mathbb{Z}^{\Sigma(1)}$

$\Sigma(1) \ni a_\rho \longleftarrow e_\rho \quad (\rho \in \Sigma(1))$

(T-inv. prime divisors $= \overline{\mathrm{orb}(\rho)} = D_\rho = H_\rho$)

$\mathcal{O}_X(H) = \{ f \in K(X) \mid \mathrm{div}(f) + H \geq 0 \}$

$1 \in \mathcal{O}_X(H)$, $\mathrm{div}(1) = 0$, $i: H \hookrightarrow X$

$/ \otimes_{\mathbb{Z}} \mathcal{O}_X$; $\otimes_{\mathbb{Z}} \mathbb{C}$; $\otimes_{\mathbb{C}} \mathcal{O}_X$

$s \in \Gamma(\mathcal{L})$

$\bullet$ $H \subseteq X$ hypersurface (prime divisor)

$\mathcal{L} \rightsquigarrow \mathcal{O}_X(H)$; $\boxed{\mathcal{O}_X(-H) \subseteq \mathcal{O}_X} = \mathcal{O}_X(H)^{-1} \subseteq K(X)$

$\mathcal{O}_X \to \mathcal{L}$; $1 \mapsto \mathcal{O}_X$

i.e.: $\mathcal{O}_X(-H)$ is an ideal sheaf inside $\mathcal{O}_X$

$\bullet$ $\exists$ ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ defining H

$\boxed{\mathcal{J} = \mathcal{O}(-H)} \implies 0 \to \mathcal{J} \to \mathcal{O}_X \to i_* \mathcal{O}_H \to 0$

EULER - Sequence

$$\boxed{0 \to \Omega_X \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(-H_\rho) \longrightarrow Cl(X) \otimes \mathcal{O}_X \to 0}$$

$dx^r \longmapsto \sum_{\rho \in \Sigma(1)} \langle r, a_\rho \rangle x^r e_\rho$

$$0 \to \left[\Omega_X(\log H) = M \otimes_{\mathbb{Z}} \mathcal{O}_X \right] \longrightarrow \mathcal{O}_X^{\Sigma(1)} \longrightarrow Cl(X) \otimes \mathcal{O}_X \to 0$$

$\frac{dx^r}{x^r} \mapsto r \otimes 1 \longmapsto (\dots, \langle r, a_\rho \rangle, \dots) = \sum_{\rho \in \Sigma(1)} \langle r, a_\rho \rangle \cdot e_\rho$

$\bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_{(H_\rho)} \to 0$; $\bigoplus_{\rho \in \Sigma(1)} i_* \mathcal{O}_{(H_\rho)} \to 0$

$\mathcal{O}(H)$ $\mathcal{O}_H$

$x \cdot \mathbb{C}[x]$ $\mathbb{C}[x] / x \cdot \mathbb{C}[x]$

$\mathcal{O}_X$ = struct. sh. on X

$\mathcal{O}_H$ = – " – on H

locally on $\mathbb{C}^1 \subseteq TV(\Sigma)$, $H = \{0\} \hookrightarrow \mathbb{C}^1$

$x \cdot \mathbb{C}[x] = \mathcal{O}_X(-H) \to \mathbb{C}[x] \to \mathbb{C}[x] / x \cdot \mathbb{C}[x] = \mathbb{C}$

$\Omega_T = \Omega_{k[x_1^{\pm}, \dots, x_n^{\pm}] / k} = \boxed{\bigoplus_{i=1}^n k[\underline{x}^{\pm}] \cdot \frac{dx_i}{x_i}}$

$k[\underline{x}^{\pm}] = k[M] = \langle x^r \mid r \in M \rangle$

$d(fg) = f\,dg + g\,df$; $d(\log f) = \boxed{\frac{df}{f}}$

$\boxed{\frac{d(fg)}{fg}} = d\log(fg) = d(\log f + \log g) = d\log f + d\log g$ $= \boxed{\frac{df}{f} + \frac{dg}{g}}$

$\Omega_X(\log H) \supseteq \Omega_X$; still inside $j_* \Omega_T$

$\bullet$ $T \overset{j}{\hookrightarrow} X$ ($\hat{=}\ \mathbb{C}[\sigma^\vee \cap M] \subseteq \mathbb{C}[M]$)

$\mathbb{P}^2$?

Ω_T = generated by $\frac{dx^r}{x^r}$ $(r \in M)$, e.g.: we might take any $\mathbb{Z}$-basis $r^1, \dots, r^n$ $(M = \mathbb{Z}^n)$

$\Rightarrow \Omega_T = \bigoplus_{i=1}^n k[M] \cdot \frac{dx^{r^i}}{x^{r^i}}$

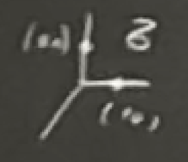
$r = g_1 r^1 + \dots + g_n r^n \implies \frac{dx^r}{x^r} = \sum g_i \frac{dx^{r^i}}{x^{r^i}}$ $(r^i, r \in M;\ g_i \in \mathbb{Z})$

x^r $(r \in M)$; $T = \mathrm{Spec}\, k[M]$

Goal: $\Omega_X \subseteq \Omega_X(\log H) \subseteq j_* \Omega_T$

locally on $TV(\sigma) = \mathrm{Spec}\, k[\sigma^\vee \cap M] \rightsquigarrow$ choose the M-basis $r^1, \dots, r^n$ s.t.: $\sigma^\vee = \langle r^1, \dots, r^n \rangle$

$\Downarrow$

(e.g.: $\mathbb{P}^2$:  $\sigma = \langle (1,0), (0,1) \rangle \subseteq N_{\mathbb{R}}$; $\sigma^\vee = \langle [1,0], [0,1] \rangle \subseteq M_{\mathbb{R}}$

w.l.o.g.: $\sigma^\vee = \langle e^1, \dots, e^n \rangle$, $M = \mathbb{Z}^n$

$dx_i = x_i \cdot \frac{dx_i}{x_i}$

$$\Omega_{TV(\sigma)} = \Omega_{k[x_1, \dots, x_n]} = \boxed{\bigoplus_{i=1}^n k[\underline{x}] \cdot dx_i \subsetneq \bigoplus_{i=1}^n k[\underline{x}] \cdot \frac{dx_i}{x_i}} = \Omega_{TV(\sigma)}(\log H)$$

$\left\langle \frac{dx_1}{x_1}, \dots, \frac{dx_n}{x_n} \right\rangle \not\ni \frac{dx_1}{t_2}$

$$\Omega_T = \Omega_{k[M]} = \bigoplus k[\underline{x}, \underline{x}^{-1}] \cdot dx_i = \bigoplus_{i=1}^n k[\underline{x}, \underline{x}^{-1}] \cdot \frac{dx_i}{t_i}$$

$0 \to 2\mathbb{Z} \subset \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

$\sigma, \sigma' \rightsquigarrow \tau = \sigma \cap \sigma' \implies \Omega_{TV(\sigma)}(\log)|_{TV(\tau)} = \Omega_{TV(\sigma')}(\log)|_{TV(\tau)}$

$\cdot$ coefficients: $k[\tau^\vee \cap M]$

$\cdot$ allowed diff.: $\boxed{\frac{dx^r}{x^r} \mid r \in M}$ (σ: $\boxed{\frac{dx^{r_1}}{x^{r_1}}, \dots, \frac{dx^{r_n}}{x^{r_n}}}$; σ': $\frac{dx^{r_1'}}{x^{r_1'}} \dots \frac{dx^{r_n'}}{x^{r_n'}}$)

$\Rightarrow$ we get $\Omega_X(\log) \supseteq \Omega_X$.

$0 \to \Omega_X \to \Omega_X(\log) \to (?) \to 0$

locally on $TV(\sigma)$: $0 \to k[\underline{x}] \cdot dx_i \hookrightarrow k[\underline{x}] \cdot \frac{dx_i}{x_i} \to k[\underline{x}] / x_i k[\underline{x}] \to 0$

$dx_i = x_i \frac{dx_i}{x_i}$

$k[\underline{x}] / x_i k[\underline{x}] = \mathcal{O}_{V(x_i)} = \mathcal{O}_{H_\rho}$; $V(x_i) \subseteq \mathbb{C}^n = TV(\sigma)$, $V(x_i) = \overline{\mathrm{orb}(\rho_i)}$

(*) $TV(\sigma)$:

$\Omega(\log) = \bigoplus_{i=1}^n k[\underline{x}] \cdot \frac{dx_i}{x_i}$ = generated by $\left\{ \frac{dx^r}{x^r} \right\}$ on $TV(\sigma)$

$\sim M \otimes k[\sigma^\vee \cap M]$, $k[\underline{x}] = k[\sigma^\vee \cap M]$

$\boxed{\frac{dx^r}{x^r} \longmapsto r \otimes 1}$

$dx^r \longmapsto r \otimes x^r$

Special case: $X = \mathbb{P}^n$; $H_\rho = V(z_\rho)$ = hyperplanes

$0 \to \Omega_X \to \mathcal{O}(-1)^{n+1} \to \mathcal{O}_X \to 0$ EULER sequence

$\bigoplus_{i=0}^n \mathcal{O}(-V(z_i))$ given by z_i

Ω_T = generated by $\frac{dx^r}{x^r}$ $(r \in M)$, e.g. we could take any $\mathbb{Z}$-basis $x^1, \dots, x^n$ $(M = \mathbb{Z}^n)$

$\rightsquigarrow \Omega_T = \bigoplus k[M] \frac{dx^i}{x^i}$ $\quad r = g_1 r^1 + \dots + g_n r^n \implies \frac{dx^r}{x^r} = \sum g_i \frac{dx^i}{x^i}$ $\quad x^r$ $(r \in M)$ $\quad (r^i, r \in M,\ g_i \in \mathbb{Z})$

Goal: $\Omega_X \subseteq \Omega_X(\log H) \subseteq j_* \Omega_T$ $\quad T = \operatorname{Spec} k[M]$

locally on $TV(\sigma) = \operatorname{Spec} k[\sigma^\vee \cap M]$ $\rightsquigarrow$ choose the M-basis $r^1, \dots, r^n$ s.t. $\sigma^\vee = \langle r^1, \dots, r^n \rangle$

(e.g. $\mathbb{P}^2$) $\quad \sigma = \langle (1,0), (0,1) \rangle$, $\sigma^\vee = \langle [1,0], [0,1] \rangle$ $\quad \Downarrow$ w.l.o.g.: $\sigma^\vee = \langle e^1, \dots, e^n \rangle$, $M = \mathbb{Z}^n$

$\Omega_{TV(\sigma)} = \Omega_{k[x_1, \dots, x_n]} = \boxed{\bigoplus_{i=1}^n k[x] \cdot dx_i \subsetneq \bigoplus_{i=1}^n k[x] \cdot \frac{dx_i}{x_i}} = \Omega_{TV(\sigma)}(\log H)$ $\quad dx_i = x_i \cdot \frac{dx_i}{x_i}$

$\langle \frac{dx_1}{x_1}, \dots, \frac{dx_n}{x_n} \rangle \not\ni \frac{dx_1}{x_2}$

$\Omega_T = \Omega_{k[M]}$ $\quad = \bigoplus k[x, x^{-1}] \cdot dx_i = \bigoplus_{i=1}^n k[x, x^{-1}] \cdot \frac{dx_i}{x_i}$

σ, σ' with $\tau = \sigma \cap \sigma' \implies \Omega_{TV(\sigma)}(\log) |_{TV(\tau)} = \Omega_{TV(\sigma')}(\log) |_{TV(\tau)}$

$\implies$ we get $\Omega_X(\log) \supseteq \Omega_X$

- coefficients: $k[\tau^\vee \cap M]$
- allowed diff.: $\boxed{\frac{dx^r}{x^r} \mid r \in M}$ $\quad$ (e.g. $\frac{dx^{r_1}}{x^{r_1}}, \dots, \frac{dx^{r_n}}{x^{r_n}}$; σ': $\frac{dx^{s_1}}{x^{s_1}}, \dots, \frac{dx^{s_n}}{x^{s_n}}$)

$0 \to \Omega_X \to \Omega_X(\log) \to (?) \to 0$

locally on $TV(\sigma)$: $0 \to k[x] \cdot dx_i \hookrightarrow k[x] \cdot \frac{dx_i}{x_i} \to k[x] / x_i k[x] \to 0$ $\quad x_i \cdot \frac{dx_i}{x_i}$

$\mathcal{O}_{H_g} = \mathcal{O}_{V(x_i)}$ $\quad V(x_i) \subseteq \mathbb{C}^n = TV(\sigma)$, $= \operatorname{orb}(\varrho_i)$

$\mathcal{O}(-H) \cong \mathcal{O}(-1)$ on $\mathbb{P}^n$

(*) $TV(\sigma)$: $\Omega(\log) = \bigoplus_{i=1}^n k[x] \cdot \frac{dx_i}{x_i}$ = generated by $\{ \frac{dx^r}{x^r} \}$ $k[x]$ on $TV(\sigma)$ $\quad \sim M \otimes k[\sigma^\vee \cap M]$

$\boxed{\frac{dx^r}{x^r} \longmapsto r \otimes 1}$

Special case: $X = \mathbb{P}^n$ $\quad H_g = V(z_g)$ = hyperplanes

$0 \to \Omega_X \to \mathcal{O}(-1)^{n+1} \to \mathcal{O}_X \to 0$ $\quad$ EULER sequence

Corollary: (on $\mathbb{P}^n$) $\quad 0 \to \mathcal{O}_X \to \mathcal{O}(+1)^{n+1} \to \mathcal{T}_X \to 0$

$0 \to \Gamma(X, \mathcal{O}_X) \to \Gamma(\mathcal{O}(1)^{n+1}) \to \Gamma(\mathcal{T}_X) \to \dots$

$\mathcal{T}_X$ = dual sheaf, Ω_X = cotangent ~~sheaf~~ sheaf

X = scheme $\rightsquigarrow$ $X(k) = \{ \operatorname{Spec} k \to X \}$ $\quad$ k-valued 'points' on X (over $k = \bar{k}$)

$\operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X$ $\implies$ get $\cdot P \in X(k)$ $\quad \cdot \mathcal{O}_{X,P} \to k[\varepsilon]/\varepsilon^2 = k \oplus \varepsilon \cdot k$ $\quad \supseteq m_P$

tangent vector t in P $\quad \Omega \rightsquigarrow \Lambda^2 \Omega$

$\Downarrow$ $\quad$ elements of $(m_P/m_P^2)^\vee$ $\longleftrightarrow$ $m_P/m_P^2 \to k$ $\longleftrightarrow$ $\mathcal{O}_{X,P} \to k$ derivation

$m/m^2 \cong$ dual to the tangent space. $\quad \cong \Omega_{X,P} \to k$ linear map

Claim: $A = k$-algebra (local) $\rightsquigarrow$ $\Omega_{A|k} \otimes_A k = m/m^2$ $\quad (m \subseteq A)$ $\quad k \to A \to A/m = k$ $\quad m \cong m = k$-valued pt.

EULER - Sequence

$0 \to \Omega_X \xrightarrow{dx^r \longmapsto \sum_{g \in \Sigma(1)} \langle r, a_g \rangle x^r e_g} \bigoplus_{g \in \Sigma(1)} \mathcal{O}_X(-H_g) \to Cl(X) \otimes \mathcal{O}_X \to 0$

$\to [\Omega_X(\log H) \overset{(*)}{=} M \otimes_{\mathbb{Z}} \mathcal{O}_X] \to \mathcal{O}_X^{\Sigma(1)} \to Cl(X) \otimes \mathcal{O}_X \to 0$

$\frac{dx^r}{x^r} \mapsto r \otimes 1 \mapsto (\dots, \langle r, a_g \rangle, \dots)$ $\quad \sum_{g \in \Sigma(1)} \langle r, a_g \rangle \cdot e_g$

$\bigoplus_{g \in \Sigma(1)} \mathcal{O}_{(H_g)}$ $\quad \bigoplus_{g \in \Sigma(1)} \mathcal{O}_{(H_g)}$

locally on $\mathbb{C}^n \subset TV(\Sigma)$: $H = \{0\} \times \mathbb{C}^{n-1}$, $x \cdot \mathbb{C}[x] = \mathcal{O}_X(-H)$, $\mathbb{C}[x]$, $\mathbb{C}[x]/x \cdot \mathbb{C}[x] = \mathbb{C}$

$\mathcal{O}(H)$ $\quad \mathcal{O}_H$ $\quad \frac{1}{x} \cdot \mathbb{C}[x]$ $\quad \mathbb{C}[x]/x \cdot \mathbb{C}[x]$ $\quad \mathcal{O}_X$ = structure sheaf on X $\quad \mathcal{O}_H$ = ... on H

$\Omega_X(\log H) \supseteq \Omega_X$, still inside $j_* \Omega_T$

$\cdot T \subset X$ $\quad (\doteq \mathbb{C}[\sigma^\vee \cap M] \subseteq \mathbb{C}[M])$

$\Omega_T = \Omega_{k[x_1^{\pm}, \dots, x_n^{\pm}] | k} = \boxed{\bigoplus_{i=1}^n k[x^{\pm}] \cdot \frac{dx_i}{x_i}}$ $\quad k[M] = \langle x^r \mid r \in M \rangle$

$d(fg) = f\,dg + g\,df$ $\quad d(\log f) = \boxed{\frac{df}{f}}$ $\quad \boxed{\frac{d(fg)}{fg}} = d\log(fg) = d(\log f + \log g) = d\log f + d\log g$ $\quad \boxed{\frac{df}{f} + \frac{dg}{g}}$