$A \to B \to C$: $\Omega_{B|A} \otimes_B C \to \Omega_{C|A} \to \Omega_{C|B} \to 0$ (C-linear)

$A \to B \to B/I$: $I/I^2 \to \Omega_{B|A} \otimes_B B/I \to \Omega_{B/I|A} \to 0$

Euler sequence on $\mathbb{P}^n$: $0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{n+1} \xrightarrow{(z_0, \dots, z_n)} \mathcal{O} \to 0$

on $TV(\Sigma)$: $0 \to \Omega_{TV(\Sigma)} \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}(-D_\rho) \to \mathcal{O} \otimes_{\mathbb{Z}} Cl(TV(\Sigma)) \to 0$

(A, m) = local k-algebra, assume: · A = domain, f.t.
· $k \hookrightarrow A \to A/m$ ($\sim$)

$\Downarrow$

$m/m^2 = \Omega_{A|k} \otimes_A k$: $0 \to \boxed{m/m^2 \xrightarrow{\sim} \Omega_{A|k} \otimes_A k} \to \cancel{\Omega_{k|k}} \to 0$

$k \to A \to A/m = k$

Proof: $N = A/m$-module ⇝ $\mathrm{Hom}_k(m/m^2, N) \leftrightarrow \mathrm{Hom}_k(\Omega_{A|k} \otimes k, N)$

$\mathrm{Der}_k(A, N)$ (N = A-module)

$d|_m \leftarrow d$

$a \in A \Rightarrow \exists!\ a = m + \bar{a}$ ⇝ $d(a) = \varphi(m) = \varphi(a - \bar{a})$

Surjectivity: $\varphi: m/m^2 \to N$ k-lin.
$m \subset A$, $k \hookrightarrow A$ (?)

Prop. A = f.t. over k, A = domain, local as before ‖ Let k = perfect field (ex. $k = \bar{k}$)

$\Omega_{A|k}$ = (locally) free A-module with rank = $\dim A$ ⟺ A = regular (i.e. $\dim_k m/m^2 = \dim A$)

Proof · M = f.g. A-module (A = noeth, local): M = free over A ⟸ $M \otimes_A k$ = k-vs, $M \otimes_A Q(A) = Q(A)$-vs — same dim

(dims: $\dim_k M \otimes k \geq \dim_{Q(A)} M \otimes_A Q(A)$) — f.g. in m

$p \in \mathrm{Spec} A$: similarly $M_p ⇝ M_p/p M_p = M_p \otimes A_p = M \otimes_A A_p/p A_p$

· $\Omega_{A|k} \otimes_A k = m/m^2$ $\dim_k \geq d = \dim A$

· $\Omega_{A|k} \otimes Q(A) = \Omega_{Q(A)|k}$.

$k \hookrightarrow Q(A) = L$ field ext.
$k(s_1, \dots, s_d) = K$ — fin. sep. ext. ($d = \dim A$)

$k \hookrightarrow K = k(s_1, \dots, s_d) \hookrightarrow L = K(t)$

(t = separable over K ⟺ $m_t(x) \in K[x]$ minimal polynomial of $t \in L$ over K (irred.), i.e. $m(t) = 0$ has no multiple roots, i.e. $\gcd(m, m') = 1$.)

$m(x)$ minimal pol: $m(t) = 0$, $m'(t) \neq 0$

‖ char $k = p$: $K = k(t^p) \subset L = K(t)$, $t^p = s \in K$, $t = \sqrt[p]{s}$

$m(x) = x^p - s \in K[x]$, over L: $(x^p - t^p) = (x - t)^p$

$\Omega_{K|k}$: $K = Q(k[s_1, \dots, s_d])$ ⇝ $\Omega_{k[\underline{s}]|k} = \bigoplus_{i=1}^{d} k[\underline{s}] \cdot ds_i$ ⇝ $\Omega_{K|k} = K\, ds_1 \oplus \dots \oplus K\, ds_d$

$\Omega_{K|k} \otimes_K L \to \Omega_{L|k} \to \Omega_{L|K} \to 0$ L-vs.

$L ds_1 \oplus \dots \oplus L ds_d$ $\dim = d$ goal: $\dim_L \Omega_{L|k} = d$

· ⇒ $0 = d\, m(t) = m'(t) \cdot dt$ ⇒ $dt = 0$. (because $m'(t) \neq 0$)

· injectivity of 1st map: $N = L$-vs ⇝ $\mathrm{Der}_k(L, N) \xrightarrow{(?)} \mathrm{Der}_k(K, N_K)$

$0 = d\, m(t) = m'(t)\, dt + (dm)(t)$

($m(x) = \sum \alpha_i x^i$ ⇝ $(dm)(t) = \sum (d\alpha_i) \cdot t^i \in N$)

⇒ $dt = -(dm)(t) / m'(t)$.

‖ $d: L \to N$ derivation ($N = L$-vs), K, $d|_K = 0$ goal $d = 0$, i.e. $d(t) = 0$

$t \in L$ $m(x) \in K[x]$, $m(t) = 0$ $\alpha_i x^i$ $+ d\alpha_i \cdot t^i$ $d(\alpha_i \cdot t^i) = \alpha_i\, d(t^i)$

$t \in L \xrightarrow{(?)} N$, $K \xrightarrow{d} N$ (not 0) Q: extendable?

$\dim_k \Omega \otimes A/m \geq d$ regular ⟺ " = "

$\dim \Omega \otimes Q(A) = d$

Ex. $Y^2 = [\mathrm{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & x_1 & x_2 \end{pmatrix} \leq 1] \subseteq \mathbb{A}^4$ locally in 0

Corollary X = variety $/k$ ⇝ X = regular (i.e. $\forall x \in X$ $\mathcal{O}_{X,x}$ = reg.) ⟺ $\Omega_{X|k}$ = locally free

Example. $X = \mathbb{P}^1$ ⇝ $\Omega = \mathcal{O}(-2)$

X = smooth EC ⇒ $\Omega = \mathcal{O}$

sing EC ⇒ $\Omega \neq$ loc free.

Corollary: $X_{smooth} \hookrightarrow X$ is open + dense.

Subvarieties $Y \xhookrightarrow{i} X$ = variety, smooth ($\dim X = d$), I = ideal sheaf, i.e. $\mathcal{O}_Y = \mathcal{O}_X / I$

‖ ex. Y = effective Cartier divisor D ⇝ $I = \mathcal{O}_X(-D)$

$0 \to I/I^2 \xrightarrow{(*)} \Omega_{X|k} \otimes \mathcal{O}_Y \to \Omega_{Y|k} \to 0$

coh. $\mathcal{O}_Y$-module; locally free of rank d

I = loc gen by r elements

$\Uparrow$

Prop. Y = smooth ⟺ $\Omega_{Y|k}$ is locally free and (*) is injective. Then I/I^2 = locally free of rk = codim(Y) = r

Proof. (⟸) rk $\Omega_{Y|k} = q$ ⇒ I/I^2 = locally free of rank $r = n - q$

⇒ $\boxed{\dim m_y/m_y^2 = q}$ $\forall y \in Y$ ⇒ $\dim Y \leq q$ | ⇒ I = loc gen by r elements ⇒ $\dim Y \geq n - r = q$ ⇒ $\boxed{\dim Y = q}$ ⇒ Y = smooth.

$\Omega_{K|k}$ $K = \mathrm{Quot}\ k[s_1, \dots, s_d] \rightsquigarrow \Omega_{k[\underline{s}]|k} = \bigoplus_{i=1}^{d} k[\underline{s}]\, ds_i \rightsquigarrow \Omega_{K|k} = K\, ds_1 \oplus \dots \oplus K\, ds_d$

$\Omega_{K|k} \otimes_K L \to \Omega_{L|k} \to \Omega_{L|K} \to 0$ $\quad L$-vs.

(K,k alg) $\underline{\dim = d}$ $\quad$ goal $\dim_L \Omega_{L|k} = d$

$\Rightarrow 0 = d\, m(t) = m'(t) \cdot dt \Rightarrow dt = 0$. (because $m'(t) \neq 0$)

• injectivity of 1st map: $N = L$-vs $\rightsquigarrow \mathrm{Der}_k(L, N) \xrightarrow[(?)]{\text{surj}} \mathrm{Der}_k(K, N_K)$

$0 = d\, m(t) = \underline{m'(t)\, dt} + (dm)(t)$

$\left(m(x) = \sum \alpha_i x^i \rightsquigarrow (dm)(t) = \sum (d\alpha_i) \cdot t^i \in N \right)$

$\Rightarrow dt = -(dm)(t) / m'(t)$.

$d : L \longrightarrow N$ derivation ($N = L$-vs)

$K \hookrightarrow L$, $d_{|K} = 0$ $\quad$ goal $d = 0$, i.e. $d(t) = 0$

$t \in L$ $\quad m(x) \in K[x]$, $m(t) = 0$

$\alpha_i x^i \quad + d\alpha_i\, t^i$

$d(\alpha_i t^i) = \alpha_i\, d(t^i)$, K, $t^i\, dt$

$t \in L \xrightarrow{(?)} N$, $K \hookrightarrow L$, $K \xrightarrow{d\ (\text{not } 0)} N$

$\underline{Q}$ extendable?

$\dim_k \Omega \otimes A/m \geq d$ $\quad$ regular $\Leftrightarrow$ „ = "

m/m^2

$\dim \Omega \otimes Q(A) = d$

$\underline{\text{Ex}}$ $Y^2 = \left[\mathrm{rk} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \leq 1 \right] \subseteq \mathbb{A}^4$

$\underline{\text{locally in } 0}$

$\underline{\text{Corollary}}$ X = variety $/k \rightsquigarrow X$ = regular (i.e. $\forall x \in X$ $\mathcal{O}_{X,x}$ = reg.) $\Leftrightarrow \Omega_{X|k}$ = locally free

$\underline{\text{Example}}$ $X = \mathbb{P}^1 \rightsquigarrow \Omega = \mathcal{O}(-2)$

X = smooth EC $\Rightarrow \Omega = \mathcal{O}$

sing EC $\Rightarrow \Omega \neq$ loc free.

$\underline{\text{Corollary}}$: $X_{\text{smooth}} \hookrightarrow X$ is open + dense.

$\underline{\text{Subvarieties}}$ $Y \overset{i}{\hookrightarrow} X$ = variety, smooth, ($\dim X = d$) $\quad I$ = ideal sheaf, $\underline{\text{i.e.}}\ \mathcal{O}_Y = \mathcal{O}_X / I$

$0 \to I/I^2 \xrightarrow{(*)} \Omega_{X|k} \otimes \mathcal{O}_Y \to \Omega_{Y|k} \to 0$

coh. $\mathcal{O}_Y$-module $\quad$ locally free of rank d

$\underline{\text{ex}}$ Y = effective Cartier divisor D $\rightsquigarrow I = \mathcal{O}_X(-D)$

I = loc gen by r elements

$\Uparrow$

$\underline{\text{Prop}}$ Y = smooth $\Leftrightarrow \Omega_{Y|k}$ is locally free and $\underline{(*)\text{ is injective}}$. Then, I/I^2 = locally free of rk = codim(Y) $= r$

$\underline{\text{Proof}}$ ($\Leftarrow$) rk $\Omega_{Y|k} =: q \Rightarrow I/I^2$ = locally free of rank $r = n - q$

$\Rightarrow \boxed{\dim m_y/m_y^2 = q}\ \forall y \in Y$ $\Rightarrow \dim Y \leq q$

$\Rightarrow I$ = loc gen. by r elements $\Rightarrow \dim Y \geq n - r = q$

$\Rightarrow \boxed{\dim Y = q} \Rightarrow Y$ = smooth.

($\Rightarrow$) goal $\underline{Y = \text{smooth}}$

$I/I^2 \xrightarrow{\text{injective?}} \Omega_X \otimes \mathcal{O}_Y \xrightarrow{\pi} \Omega_Y \to 0$

$0 \to K = \ker(\pi)$ $\quad \Rightarrow K$ = locally free of rank r.

locally free of rank $q = \dim Y$ $\quad r = n - q$

Choose $\boxed{\alpha_1, \dots, \alpha_r} \in K\ (\to K_x)$ small, K_x

basis $\rightsquigarrow \exists \boxed{\alpha_{r+1}, \dots, \alpha_n} \in (\Omega_X \otimes \mathcal{O}_Y)_x$:

$\underline{\text{i.e.}}$ $K_x \overset{i}{\hookrightarrow} (\Omega_X \otimes \mathcal{O}_Y)_x$

$(A/I)^r \hookrightarrow (A/I)^n$

$\rightsquigarrow \underline{\text{assume}}$: $\alpha_1, \dots, \alpha_r \in I \subseteq A$ (ideal)

$Y \overset{\text{closed}}{\hookrightarrow}$

$I' = (\alpha_1, \dots, \alpha_r) \subseteq I \subseteq A$ $\quad \rightsquigarrow Y' = V(I') \subseteq X$ (locally)

$A = \mathcal{O}_{X,x}$ (some $x \in Y$)

$A/I = \mathcal{O}_{Y,x}$

$I/I^2 \twoheadrightarrow I/mI = A/m$-vs.

$(A/I')^r \to I'/I'^2 \longrightarrow \Omega_X \otimes \mathcal{O}_{Y'} \to \Omega_{Y'} \to 0 \Rightarrow$ RHS for $Y' \Rightarrow Y'$ = smooth of dim q

$(A/I')^r \hookrightarrow (A/I')^n \to (A/I')^q$

$\Rightarrow Y \overset{\text{closed}}{\hookrightarrow} Y'$, both smooth, dim q.

$\Rightarrow Y = Y'$. $\quad \square$

$\underset{(q)}{Y} \hookrightarrow \underset{(n)}{X} \rightsquigarrow 0 \to I/I^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \to 0$.

$\Lambda^a \mathcal{F}$ = loc free of rank $\binom{\mathrm{rk}\,\mathcal{F}}{a}$

$\underline{\text{Application}}$. $\mathcal{F}$ = locally free of rank m $\rightsquigarrow \det \mathcal{F} = \Lambda^m \mathcal{F}$ = invertible sheaf ($=$ quot. of $\mathcal{F}^{\otimes m}$)

(A = local ring $\Rightarrow \Lambda^m A^m = A \cdot e_1 \wedge \dots \wedge e_m$ (e_i = basis of A^m))

$\underline{\text{Def}}$ X = smooth, $\rightsquigarrow \det \Omega_X =: \omega_X$ „canonical sheaf".

$\underline{\text{Ex}}$: • $\omega_{\mathbb{P}^1} = \Lambda^1 \Omega_{\mathbb{P}^1} = \mathcal{O}(-2)$.

• $\underline{\omega_{\mathbb{P}^n}}$: $\quad 0 \to \Omega \longrightarrow \mathcal{O}(-1)^{n+1} \to \mathcal{O} \to 0$

$\omega = \det \Omega = \det \left(\mathcal{O}(-1)^{\oplus(n+1)} \right) \otimes \det \mathcal{O}$

$= \mathcal{O}(-1)^{\otimes(n+1)} = \boxed{\mathcal{O}(-n-1) = \omega_{\mathbb{P}^n}}$

$0 \to \overset{k}{\mathcal{F}} \to \overset{l}{\mathcal{G}} \to \overset{m}{\mathcal{H}} \to 0$ $\quad l = k + m$

$\Lambda^l \mathcal{G} = \Lambda^k \mathcal{F} \otimes \Lambda^m \mathcal{H}$ (Problem session)

$\det \mathcal{G} = \det \mathcal{F} \otimes \det \mathcal{H}$

$\underline{\text{further appl.}}$ Y = eff. divisor $D \subseteq X$, assume $\underline{\text{smooth}}$

$I/I^2 = I \otimes A/I$

$\Rightarrow 0 \to \mathcal{O}(-D) \otimes \mathcal{O}_X/\mathcal{O}(-D) \to \Omega_X \otimes \mathcal{O}_D \to \Omega_D \to 0 \Rightarrow \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_D = \omega_D \otimes \mathcal{O}_D(-D) \Rightarrow \omega_D = \omega_X|_D \otimes \mathcal{O}(D) = \omega_X(D)|_D$

$\boxed{\mathcal{O}_D(-D)} = \mathcal{O}(-D) \otimes \mathcal{O}_D$ $\quad \boxed{\omega_D = \omega_X(D)|_D}$ adjunction $\underline{\text{Ex}}$ $E \hookrightarrow \mathbb{P}^2$ $\Omega_E = \omega_E = \mathcal{O}(-3)(3)|_E = \mathcal{O}_X|_E = \mathcal{O}_E$. //