Reflexive modules, Weil divisors

(A = normal int. domain ; X = normal variety)

Def. L = A-module is *reflexive* $:\Longleftrightarrow \exists$ A-module M : $L = \mathrm{Hom}_A(M, A) =: M^\vee$ (and M = f.g.)

(e.g. $M = \mathbb{Z}/2\mathbb{Z}$, $A = \mathbb{Z}$ $\rightsquigarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$)

Rmk. L = reflexive $\rightsquigarrow$ $L \xrightarrow{\sim} L^{\vee\vee} = \mathrm{Hom}(L^\vee, A)$ is an isomorphism. (homework)

e.g. $\mathbb{Z}/2\mathbb{Z}$ is not reflexive / $\mathbb{Z}$.

Property. L = reflexive $\Longrightarrow$ L = torsion free !

Example. $L = (x,y) \subseteq k[x,y] =: A$ is torsion free, but: $\mathrm{Hom}_{k[x,y]}((x,y), k[x,y]) = k[x,y]$

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Recall: $A = \bigcap_{\mathrm{ht}\, P = 1} A_P$

$\Longrightarrow$ for L = reflexive : $\boxed{L = \bigcap_{\mathrm{ht}\, P = 1} L_P}$ $(P \subset A)$ $\left(L = \mathrm{Hom}_A(M, A) \Longrightarrow L_P = \mathrm{Hom}_{A_P}(M_P, A_P) = \mathrm{Hom}_A(M, A_P) \right)$

$X = \mathrm{Spec}\, A \supseteq U$ open, s.t. $\boxed{\mathrm{codim}_X (X \setminus U) \geq 2}$. (e.g. $\mathbb{A}^2 \supset \mathbb{A}^2 \setminus \{0\}$)

L = torsion free $\rightsquigarrow$ $\tilde{L}(X) \xrightarrow{\text{inj}} \tilde{L}(U)$ $(\forall\, U \supseteq D(f))$

$\tilde{L}(X) = L \to \tilde{L}(D(f)) = L_f$

L = reflexive $\longrightarrow$ $\tilde{L}(X) \xrightarrow{\sim} \tilde{L}(U) \hookrightarrow L_P$, $\ell \longmapsto \ell_P$ $\forall P$

$P \in X$ s.t. $\mathrm{ht}\, P = 1 \Longrightarrow P \in U$ $\Longrightarrow$ use $L = \bigcap_{\mathrm{ht}=1} L_P$

$j : U \hookrightarrow X$ $(\mathrm{codim}(X \setminus U) \geq 2)$ $\rightsquigarrow$ $\tilde{L} \overset{a}{\hookrightarrow} j_* j^* \tilde{L}$

a = inj. for L = torsion free

a = isom. for L = reflexive.

Remark 2. If L = torsion free A-module s.t. $L = \bigcap_{\mathrm{ht}=1} L_P$ $\Longrightarrow$ L = reflexive.

Proof.

$$\begin{array}{ccc} L & \longrightarrow & L^{\vee\vee} \\ \wr\downarrow & & \downarrow\wr \\ \bigcap_{\mathrm{ht}=1} L_P & \xrightarrow{\sim} & \bigcap_{\mathrm{ht}=1} L_P^{\vee\vee} \end{array}$$

$L_P \xrightarrow{\sim} L_P^{\vee\vee}$ is an isom for all P

Remark 1: $P \in \mathrm{Spec}\, A$, $\mathrm{ht}\, P = 1$, L = torsion free $\Longrightarrow L_P$ = free $\boxed{A_P}$-module. (A_P = PID) $\Longrightarrow$ reflexive

(M = f.g. module over R = PID, torsion free $M = \cancel{(\mathrm{tors})} \oplus R^I$)

Remark 3:
- $\mathrm{ht}\, P = 1 \Longrightarrow A_P$ = smooth $\rightsquigarrow U := \{ x \in X = \mathrm{Spec}\, A \mid x = \text{smooth} \} = X_{\text{smooth}}$ $\Longrightarrow U$ has $\mathrm{codim}(X \setminus U) \geq 2$
- L = reflexive $\rightsquigarrow$ $\mathrm{ht}\, P = 1$ gives L_P = locally free $\Longrightarrow U' := \{ x \in X \mid \tilde{L}_x = \text{free} \} \Longrightarrow \mathrm{codim} \geq 2$, $\tilde{L}_{|U'}$ = locally free $\Longrightarrow \tilde{L} = j^{!}_{*}$(locally free)

$X = \mathrm{Spec}\, A$ (or general normal X : $\mathcal{F}$ = coherent $\mathcal{O}_X$-module : $\mathcal{F}$ = "reflexive" $:\Longleftrightarrow \dots$ $\Longleftrightarrow \mathcal{F} \xrightarrow{\sim} \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_X))$)

Claim. ① D – Weil divisor in X $\Longrightarrow$ $\mathcal{O}_X(D)(V) := \{ f \in K(X) \mid D + \mathrm{div}\, f \geq 0 \text{ on } V \} \subseteq K(X) = \mathcal{O}_{X,\eta}$ is *reflexive* of rank 1.

② $L \subseteq \mathrm{Quot}\, A$ is a reflexive module $\Longrightarrow \exists\, D$ = Weil div. in $\mathrm{Spec}\, A$: $L = \mathcal{O}(D)$.

Proof. ① $U \subseteq X$ the set of pts. such that D = Cartier $(\Leftrightarrow \mathcal{O}(D)$ = locally free$)$

$(P \in X : \mathrm{ht} = 1$ $A_P = \mathcal{O}_{X,P}$ = regular $\Longrightarrow P \in U$ $\Longrightarrow \mathrm{codim}(X \setminus U) \geq 2)$

$\mathcal{O}_U(D_{|U}) = \mathcal{O}_X(D)_{|U}$ = invertible sheaf.

$V \subseteq X$ $\mathcal{O}_X(D)(V) \overset{\sim}{\hookrightarrow} \mathcal{O}_X(D)(V \cap U)$

$\Big\}$ $j : U \hookrightarrow X$ $\Longrightarrow j_* j^* \mathcal{O}(D) \xleftarrow{\sim} \mathcal{O}(D)$ $\Longrightarrow$ reflexive !

② L = reflexive $\subseteq \mathrm{Quot}\, A = K(X)$ $\rightsquigarrow \exists\, U$, $\mathrm{codim} \dots$: $\tilde{L}_{|U} \subseteq K(X)$ locally free $\Longrightarrow \exists\, D_U$ = Cartier div on U, $\tilde{L}_{|U} = \mathcal{O}_U(D_U)$

D_U = also Weil div $= \sum_i \lambda_i \cdot D_i$ (prime div $\subseteq U$) $\Longrightarrow D := \sum \lambda_i \overline{D_i}$ – Weil div. ($\overline{D_i} \subseteq X$) $\square$

Lemma. $X = \mathrm{Spec}\, A$, $D = \sum \lambda_i D_i$ $(\lambda_i \in \mathbb{Z}$, $D_i \subset \mathrm{Spec}\, A$ prime div$)$

$\mathcal{O}(D) \subseteq K(X) = \mathrm{Quot}\, A$ $\Longrightarrow \forall i\ \exists f_i \in \mathcal{O}(D)$: $\mathrm{ord}_{D_i} f_i = -\lambda_i$ (instead of just "$\geq$")

$\forall i$ $\lambda_i + \mathrm{ord}_{D_i}(f_i) \geq 0 \Longleftrightarrow \sum \lambda_i D_i + \mathrm{div}(f_i) \geq 0$, $\mathrm{div}(f_i) = \sum \mathrm{ord}_{D_i}(f_i) \cdot D_i$

Proof: $I(D_i)$ = P.I. defining $D_i \subset A$ (height = 1)

$f_i \in I(D_i) \Longrightarrow \mathrm{ord}_{D_i} f_i \geq 1$ (test in $A_{I(D_i)}$)

$f_i \in I(D_i) \setminus I(D_i)^2 \Longrightarrow \mathrm{ord}_{D_i}(f_i) = 1$

$g_j \in I(D_j) \setminus I(D_i)$ $(j \neq i)$ $\Longrightarrow \mathrm{ord}_{D_j}(g_j) \geq 1$, $\mathrm{ord}_{D_i}(g_j) = 0$

$f_i := f_i^{-\lambda_i} \cdot \prod_{j \neq i} g_j^{k_j}$, $k_j \gg 0$ $\mathrm{ord}_{D_i}(f_i) = -\lambda_i$, $\mathrm{ord}_{D_j}(f_i)$ becomes $\geq -\lambda_j$ as large as we like

$\Longrightarrow f_i \in \mathcal{O}(D)$, $\mathrm{ord}_{D_i} f_i = -\lambda_i$. $\square$

Corollary. $\sum \lambda_i D_i = D$, $\sum \lambda_i' D_i = D'$

$D \leq D' \Longleftrightarrow \mathcal{O}(D) \subseteq \mathcal{O}(D')$ $(\subseteq K(X))$

Proof. $(\Rightarrow)$ ✓

$(\Leftarrow)$ Assume $\exists\, i$ s.t. $\lambda_i > \lambda_i'$ $\Longrightarrow \exists f_i$ (from Lemma) $\in \mathcal{O}(D)$, $\notin \mathcal{O}(D')$ $\square$

$\mathcal{O}(D) \cdot \mathcal{O}(D') = \mathcal{O}(D + D')$ (Cartier div)

$(\mathcal{O}(D) \cdot \mathcal{O}(D'))^{\vee\vee} = \mathcal{O}(D + D')$ (Weil div)

$X = \operatorname{Spec} A$ (or general normal X) $\mathcal{F}$ = coherent $\mathcal{O}_X$-module. $\mathcal{F}$ = reflexive $:\iff \dots$)
$\iff \mathcal{F} \xrightarrow{\sim} \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{F}, \mathcal{O}_X))$

Claim ① D - Weil divisor in $X \Rightarrow \mathcal{O}_X(D)(V) := \{ f \in K(X) \mid D + \operatorname{div} f \geq 0 \} \subseteq K(X) = \mathcal{O}_{X,\eta}$ on V
is reflexive of rank 1.

② $L \subseteq \operatorname{Quot} A$ is a reflexive module $\Rightarrow \exists\, D$ = Weil div. in $\operatorname{Spec} A$: $L = \mathcal{O}(D)$.

Proof. ① $U \subseteq X$ the set of pts. such that D = Cartier ($\leftrightarrow \mathcal{O}(D)$ = locally free)
($P \in X$, ht $= 1$ $A_P = \mathcal{O}_{X,P}$ = regular $\Rightarrow P \in U$ $\Rightarrow$ codim $(X \setminus U) \geq 2$)

$\mathcal{O}(D_{|U}) = \mathcal{O}_X(D)_{|U}$ = invertible sheaf.
$V \subseteq X$ $\mathcal{O}_X(D)(V) \xrightarrow{\sim} \mathcal{O}_X(D)(V \cap U)$
$\Big\}$ $j : U \hookrightarrow X \Rightarrow j_* j^* \mathcal{O}(D) \xleftarrow{\sim} \mathcal{O}(D)$
$\Rightarrow$ reflexive !

② L = reflexive $\rightsquigarrow \exists\, U$, codim ..... : $\tilde{L}_{|U} \subseteq K(X)$ locally free $\Rightarrow \exists\, D_U$ = Cartier div on U
$\tilde{L}_{|U} = \mathcal{O}_U(D_U)$
D_U = also Weil div $= \sum \lambda_i \cdot D_i \Rightarrow D := \sum \lambda_i \overline{D_i}$ = Weil ...

Lemma. $X = \operatorname{Spec} A$, $D = \sum \lambda_i D_i$ ($\lambda_i \in \mathbb{Z}$, $D_i \subset \operatorname{Spec} A$ prime div)
$\mathcal{O}(D) \subseteq K(X) = \operatorname{Quot} A \Rightarrow \forall i\ \exists f_i \in \mathcal{O}(D)$: $\operatorname{ord}_{D_i} f_i = -\lambda_i$ (instead of just $\geq$)

$\forall i$ $\lambda_i + \operatorname{ord}_{D_i}(f) \geq 0 \iff \sum \lambda_i D_i + \operatorname{div}(f) \geq 0$
$\operatorname{div}(f) = \sum \operatorname{ord}_{D_i}(f) \cdot D_i$

Proof. $I(D_i)$ = PI defining $D_i \subset A$
(height $= 1$)
$f_i \in I(D_i) \Rightarrow \operatorname{ord}_{D_i} f_i \geq 1$ (test in $A_{I(D_i)}$)
$f_i \in I(D_i) \setminus I(D_i)^2 \Rightarrow \operatorname{ord}_{D_i}(f_i) = 1$
$g_j \in I(D_j) \setminus I(D_i) \Rightarrow \operatorname{ord}_{D_j}(g_j) \geq 1$, $\operatorname{ord}_{D_i}(g_j) = 0$
$(j \neq i)$
$f_i := f_i^{-\lambda_i} \cdot \prod_{j \neq i} g_j^{k_j}$, $k_j \gg 0$ $\operatorname{ord}_{D_i}(f_i) = -\lambda_i$; $\operatorname{ord}_{D_j}(f_i)$ becomes as large as we want

Corollary. $\sum \lambda_i D_i = D$
$\sum \lambda_i' D_i = D'$

$D \leq D' \iff \mathcal{O}(D) \subseteq \mathcal{O}(D')$ ($\subseteq K(X)$)

Proof. ($\Rightarrow$) ✓
($\Leftarrow$) Assume $\exists\, i$ s.t. $\lambda_i > \lambda_i'$
$\Rightarrow \exists f_i$ (from Lemma). $\in \mathcal{O}(D)$
$\notin \mathcal{O}(D')$ $\square$

$\mathcal{O}(D) \cdot \mathcal{O}(D') = \mathcal{O}(D + D')$ (Cartier div)
$(\dots)^{\vee\vee}$ ...

$L \hookrightarrow L_P \rightsquigarrow L^{\vee\vee} \hookrightarrow L_P^{\vee\vee}$ $\forall P \in \operatorname{Spec} A$
$P = (0) \doteq \eta_X$ $L^{\vee\vee} \hookrightarrow K(X)^{\vee\vee} = K(X)$

$L = A$-module, $L \subseteq \operatorname{Quot} A \rightsquigarrow \operatorname{Hom}_A(L, A) = \{ \varphi : L \to A \} \hookrightarrow K(X)$
$\operatorname{Quot} A \supseteq L$, $Q(A) = K(X)$
$f_\varphi := \varphi(\ell)/\ell$ for some/all $\ell \in L \setminus \{0\}$.
$\exists! f_\varphi \in Q(A)$: $\varphi(\ell) = f_\varphi\, \ell$
$\hookrightarrow L^\vee \hookrightarrow Q(A)$

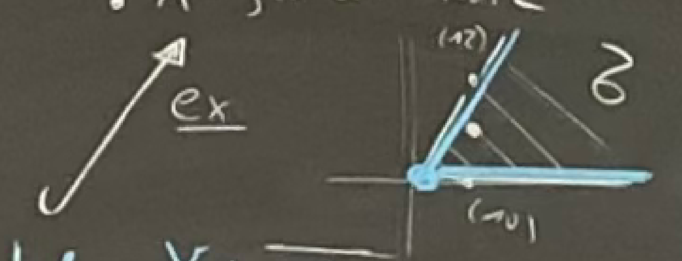
Last session. X = smooth $\rightsquigarrow \Omega_X$ = locally free of rank d $\rightsquigarrow \omega_X := \Lambda^d \Omega$ = invertible.
(dimension d)
(canonical sheaf / dualizing sheaf)
$\exists$ embedding $\boxed{\omega_X} \hookrightarrow K(X)$ $\rightsquigarrow$ „canonical divisor" K of X.
$\rightsquigarrow \exists$ the can. div. class $[K] = K$
(one meaning : $\mathcal{F}$ = coherent sheaf $\Rightarrow H^i(X, \mathcal{F}) = H^{d-i}(X, \mathcal{F}^\vee \otimes \omega_X)^*$ „Serre-duality")

X = normal (not nec. smooth) $\rightsquigarrow \Omega_X$ does still exist, $\rightsquigarrow (\Lambda^d \Omega_X)^{\vee\vee} =: \omega_X$
(dim d)
instead: $U \hookrightarrow X$ $U = X_{sm} \Rightarrow X \setminus U$ has codim $\geq 2 \Rightarrow$
$j_* \omega_U =: \omega_X$ = reflexive $\boxed{K}$

Example $X = TV(\Sigma)$, $|\Sigma| = N_{\mathbb{R}}$ $\rightsquigarrow$ normal ; $\dim TV(\Sigma) = \operatorname{rk}(N)$ ($N \doteq \mathbb{Z}^d$)

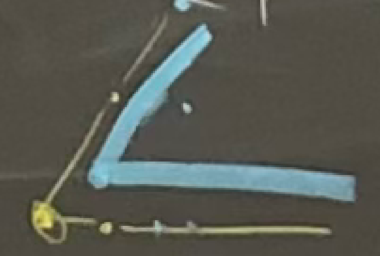
• If X = smooth $\Rightarrow$ Euler sequence : $0 \to \Omega_X \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}(-D_\rho) \to \mathcal{O}_X^{\oplus d(X)} \to 0$
$\Rightarrow \omega_X = \bigotimes_{\rho \in \Sigma(1)} \mathcal{O}_X(-D_\rho) = \mathcal{O}_X(-\sum_{\rho \in \Sigma(1)} D_\rho) \Longrightarrow \boxed{K = -\sum_{\rho \in \Sigma(1)} 1 \cdot D_\rho}$

• X = general toric



$TV(\sigma)$ = regular $\iff \sigma = \langle a^1, \dots, a^d \rangle$ and $\{a^1, \dots, a^d\}$ = basis of $\mathbb{Z}^d = N$
(lower dim σ $\rightsquigarrow$ = < part of a $\mathbb{Z}^d$-basis >)
$\det \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = 2 \neq \pm 1$

$U = X \setminus \overline{\operatorname{orb}(\sigma)}$
$U := TV(\bigcup \Sigma(1)) = $ smooth $\omega_U = -\sum_{\Sigma(1) \ni \rho} D_\rho |_U$
$X = TV(\Sigma)$



codim $(X \setminus U) \geq 2$ $\boxed{\omega_X = j_* \omega_U = -\sum_{\rho \in \Sigma(1)} D_\rho}$
$\sigma^\vee$ $k[\sigma^\vee \cap M]$

$K_X \leq 0 \Rightarrow \mathcal{O}(K_X) \subseteq \mathcal{O}_X$
$\omega_X \longleftarrow$ is generated by $\boxed{\frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_d}{x_d}}$ $x_1, \dots, x_d$ = loc. coord
($x_1, \dots, x_d \doteq$ basis of M : $r_1, \dots, r_d$ $x_i = \chi^{r_i}$)
Goal: understand the ideal $\mathcal{O}(K_X) \subseteq \mathcal{O}_X$ locally, for each $\sigma \in \Sigma$.
$\sigma \subseteq N$ $\sigma^\vee \subseteq M$ $\mathcal{O}_{X_\sigma} = k[\sigma^\vee \cap M]$ $\omega_X \longrightarrow K(X)$
$\frac{dx}{x} \longmapsto 1$
$\sigma = \langle a^1, \dots, a^n \rangle \Rightarrow \forall i$ $r \in \mathcal{O}(K_X) \iff \langle r, a^i \rangle \geq 1$ $\mathcal{O}(K_X) = (\operatorname{int} \sigma^\vee \cap M)$
dim $= d$