X = smooth $\rightsquigarrow$ $\omega_X^{?} = \det \Omega_X$ ($\hat{=}$ Cartier div)

X = normal $\rightsquigarrow$ $X_{sm} \xhookrightarrow{i} X$ $\omega_X := i_* \omega_{X_{sm}}$ $\rightsquigarrow$ reflexive sheaf $\hat{=}$ Weil div („canonical divisor")

Def: X is Gorenstein $:\Longleftrightarrow$ ω_X = invertible sheaf.

$X = TV(\Sigma)$, $|\Sigma| = N_{\mathbb{R}}$ $\rightsquigarrow$ $\omega_X = \mathcal{O}(-\sum_{\rho \in \Sigma(1)} \mathrm{orb}(\rho)) \subseteq \mathcal{O}_X$, $D_\rho = H_\rho = \mathrm{orb}(\rho)$

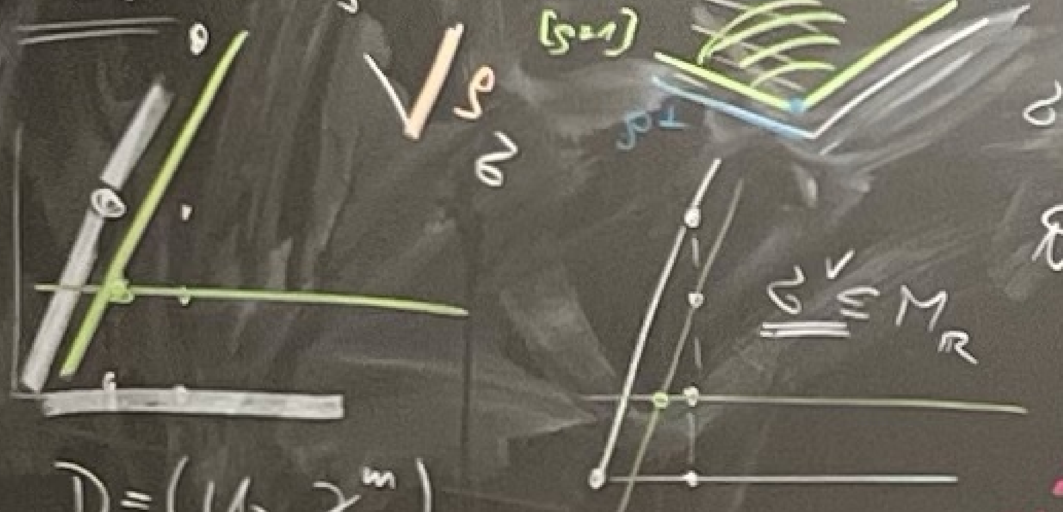
$U_\sigma = TV(\sigma)$ = affine

$\mathrm{div}(\chi^r) = \sum_{\rho \in \Sigma(1)} \langle r, \rho \rangle \cdot D_\rho$ $(r \in M)$ $\Longrightarrow$ $\boxed{\langle \chi^r \mid \langle r, \rho \rangle \geq 1 \rangle} \subseteq k[\sigma^\vee \cap M]$

$= \langle \chi^r \mid r \in (\mathrm{int}\, \sigma^\vee) \cap M \rangle$

$\mathcal{O}_{U_\sigma} \hat{=} \sigma^\vee$

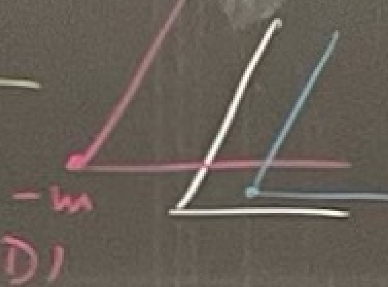
$\omega_{U_\sigma} \hat{=} (\mathrm{int}\, \sigma^\vee)$ T-inv



$\mathcal{O}(D)(U) = \{ f \mid \mathrm{div}(f) + D \geq 0$ on $U \}$

$\sigma^\vee \subseteq M_{\mathbb{R}}$

Let D = some Cartier div on U_σ $\Longrightarrow$ principal div. $D = (U_\sigma, \chi^m)$



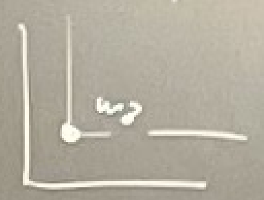
ω_X is, in gen, not Cartier!

$X = TV(\Sigma)$ is Gorenstein $\Longleftrightarrow$ $\forall \sigma \in \Sigma$: $\omega_\sigma = \chi^{m_\sigma} \cdot \mathcal{O}_{U_\sigma}$ $\Longleftrightarrow$ $\mathrm{int}\, \sigma^\vee = m_\sigma + \sigma^\vee$, $m_\sigma \in M$

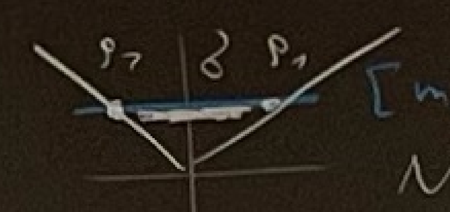
$\Longleftrightarrow$ $\forall \rho \in \sigma(1)$: $\langle m_\sigma, \rho \rangle = 1$

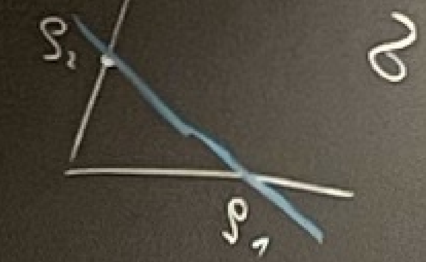
Example: σ = smooth $\Longleftrightarrow$ $\sigma = \langle a^1, \ldots, a^d \rangle$ ($a^i \in N$, $d = \dim TV(\sigma)$) and $\rho^1, \ldots, \rho^d$ = $\mathbb{Z}$-basis of $N = \mathbb{Z}^d$.

$\Longrightarrow m_\sigma \in M$.

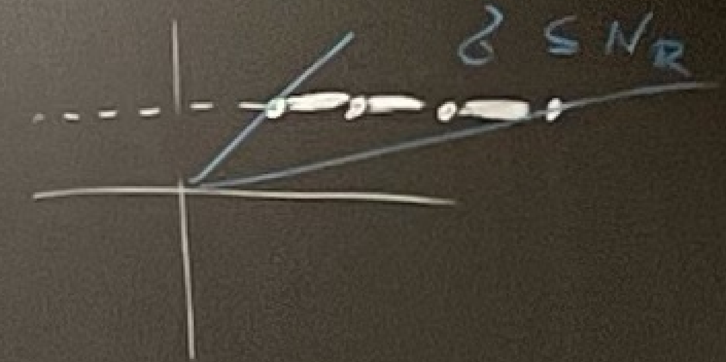


„Gorenstein" $\not\Rightarrow$ smooth:





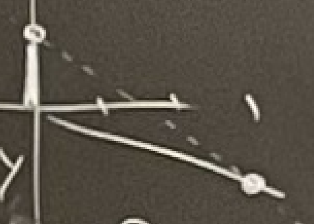
$m_\sigma \in M$ $\rightsquigarrow$ affine plane $[m_\sigma = 1] = \{ a \in N_{\mathbb{R}} \mid \langle m_\sigma, a \rangle = 1 \}$



σ = Gorenstein $\Longleftrightarrow$ σ = cone (lattice polytope Q in height 1), $Q = \sigma \cap [m_\sigma = 1]$

Ex: $\sigma^\vee = \langle (1,0), (1,3) \rangle$

$\sigma = \langle (0,1), (3,-1) \rangle$



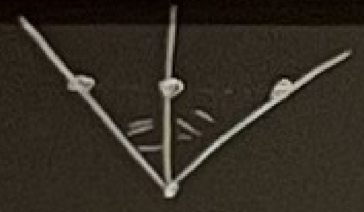
Ex: $\mathbb{P}^n$: $\omega = \mathcal{O}(-n-1) \not\cong \mathcal{O}$ $\rightarrow$ not CY

$E = EC = V(F_3) \subseteq \mathbb{P}^2 \Longrightarrow \omega_E \cong \mathcal{O}_E$.

Def: X = variety, normal.

X = „Calabi-Yau-variety" $:\Longleftrightarrow$ $\omega_X \cong \mathcal{O}_X$ (i.e.: $\omega_X \hat{=}$ principal div)

Example • Σ =



$\rightsquigarrow TV(\Sigma)$: $\omega_X \cong \mathcal{O}_X$ (homework)

• $|\Sigma| = N_{\mathbb{R}}$ $\rightsquigarrow$ $TV(\Sigma)$ cannot be CY. (– " –)

ω_X – can it be ample?

Def: D = Cartier div. on X, let $f_0, \ldots, f_n \in \Gamma(X, \mathcal{O}(D))$ generate $\mathcal{O}(D)$

($\rightsquigarrow \exists$ $X \xrightarrow{\pi_{D,f}} \mathbb{P}^k$

$(\pi^*\mathcal{O}(1), \pi^* z_i) \quad (\mathcal{O}(1), z_0, \ldots, z_k)$)

$\quad \mathcal{O}(D), f_i$

D = „very ample" $\Longleftrightarrow$ $\pi_{D,f}$ = closed embedding.

Toric examples:

$\Sigma = \mathcal{N}(\Delta)$ ($\Delta \subseteq M_{\mathbb{R}}$ lattice pol.)

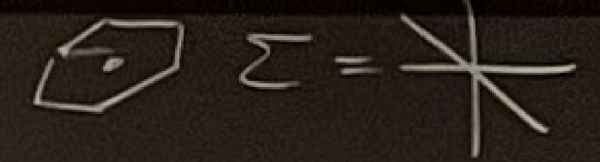
∇ = „Mink. summand" of Δ $\Longleftrightarrow$ $\mathcal{O}(\nabla)$ = inv. sheaf; globally gen ($\Longleftarrow$ bp free)

$\hookrightarrow TV(\Sigma) \longrightarrow \mathbb{P}^{(\nabla \cap M)-1}$

The image is $\mathbb{P}(\nabla)$ $\rightsquigarrow$ $TV(\Sigma) \longrightarrow \mathbb{P}(\nabla) \subseteq \mathbb{P}^{(\nabla \cap M)-1}$

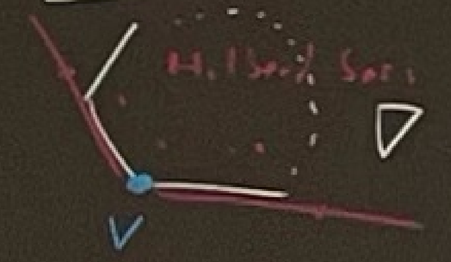
$TV(\Sigma) \to \mathbb{P}(\nabla)$ is an isom $\Longleftrightarrow$ • $\mathcal{N}(\nabla) = \mathcal{N}(\Delta) = \Sigma$

(ex: Δ =



$\nabla = \nabla$

•



$\mathbb{R}_{\geq 0}(\nabla - v) \hat{=} \sigma_v^\vee$

$\forall v$: Hilbert basis of $\sigma_v^\vee \subseteq \nabla$ $\Big\}$ $\nabla = (k \gg 0) \cdot \nabla$ $\rightsquigarrow$ ok.

Def: D = Cartier div.

$\mathcal{O}(D)$ = „ample" $:\Longleftrightarrow$

$\exists k \gg 0$: $k \cdot D$ is v. ample!

Corollary: ample LB on $TV(\Sigma)$ are exactly those $\mathcal{O}(\nabla)$ with $\mathcal{N}(\nabla) = \Sigma$.

Q: Is (sometimes) ω_X = ample ? ($\rightsquigarrow$ „canonical embeddings") „X = „general type".

(ex: $X = E = EC$ $\rightarrow$ is wrong!!)

Q': Is (– " –) ω_X^{-1} = ample ? $\cancel{K_X}$ $-K_X$ $\rightsquigarrow$ „anti canonical embeddings" „X = Fano-varieties"

(toric: $\mathcal{O}(-K_X) = \mathcal{O}(\sum_\rho D_\rho)$)

Ex: $\mathbb{P}^n$ $\rightsquigarrow$ $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$

$\omega^{-1} = \mathcal{O}(n+1)$ = v. ample

What are the toric Fano varieties ?

$\omega_X^{-1} \cong -K_X$

$\nabla \rightsquigarrow \mathcal{O}(\nabla)$ = sa. + ... = ample $\quad \mathcal{N}(\nabla) = \Sigma$

$\sigma \in \Sigma \Leftrightarrow v_\sigma \in \nabla$

$\omega_X^{-1} \hat{=} \{r \mid \langle r, \rho \rangle \geq -1 \ \forall \rho \in \Sigma(1)\}$

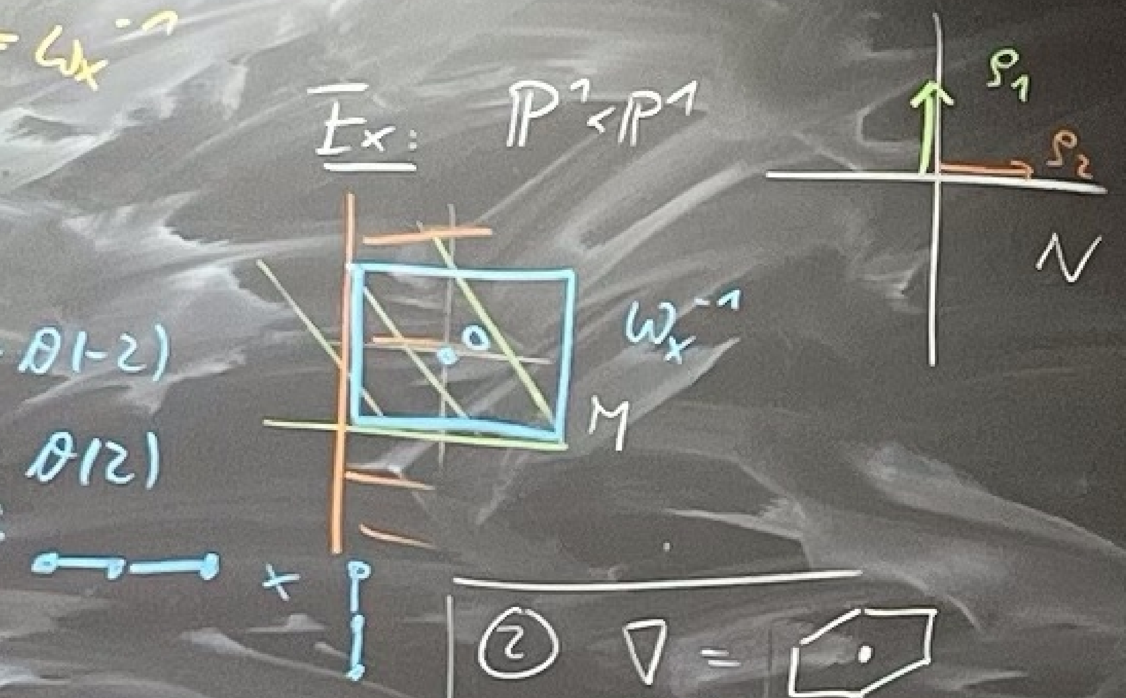
Goal: $\mathcal{O}(\nabla) = \omega_X^{-1}$

$\Rightarrow \nabla = \{r \in M_{\mathbb{R}} \mid \langle r, \rho \rangle \geq -1 \ \forall \rho \in \Sigma(1)\}$

has to have the property: $\mathcal{N}(\nabla) = \Sigma$

· vertices of ∇ = $\Sigma(1)$

Ex: $\mathbb{P}^1 \times \mathbb{P}^1$



$\omega_{\mathbb{P}^1} = \mathcal{O}(-2)$

$\omega_{\mathbb{P}^1}^{-1} = \mathcal{O}(2)$

② $\nabla =$

$\nabla^\vee \cong \nabla$

Def: $P \subseteq M_{\mathbb{R}}$ is a polytope with $0 \in P$

$P^\vee := \{a \in N_{\mathbb{R}} \mid \langle p, a \rangle \geq -1 \ \forall p \in P\}$

$(P \rightsquigarrow \sigma_P := \mathrm{cone}(P \times \{1\})$ — $\sigma_{P^\vee} = \sigma_P^\vee$

$P^\vee \rightsquigarrow \sigma_{P^\vee} = \mathrm{cone}(P^\vee \times \{1\}))$

Result: $X = TV(\Sigma)$ is Fano $\Leftrightarrow$ Q = conv$\{\Sigma(1)\}$

then: Fano $\Leftrightarrow$ Σ = face fan of Q

not Fano

Get Fanos: Recipe · take a polytope Q: $0 \in \mathrm{int}\, Q$, lattice pol! · vertices = primitive lattice pts

$\Downarrow$

Σ := face fan of Q. //

∇: Σ = face fan$(Q) = \mathcal{N}(Q^\vee)$, $\quad \nabla = Q^\vee \subseteq M_{\mathbb{R}}$

Problem: ∇ might be non-lattice!!

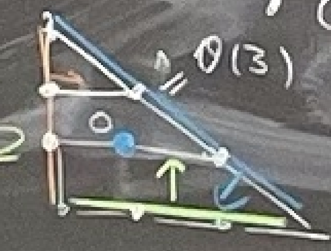
Observation: ∇ = lattice pol. $\Leftrightarrow$ Q-facets have dist. 1 from $0 \in N$

$\Leftrightarrow$ ω_X^{-1} = invertible $\Leftrightarrow$ X = Gorenstein

i.e.: Fano + Gorenstein $\Leftrightarrow$ lattice pol. Q with: · $0 \in \mathrm{int}\, Q$ · Q and $Q^\vee$ are lattice pol.! } "reflexive" polytopes

$\rightsquigarrow \mathbb{P}(\nabla), \mathbb{P}(\nabla^\vee)$: both are Fano varieties. they are hosts CY-varieties.

ex: ① $\mathbb{P}^2$ $\quad \mathcal{O}(3)$ $\quad \omega_{\mathbb{P}^2} = \mathcal{O}(-3)$



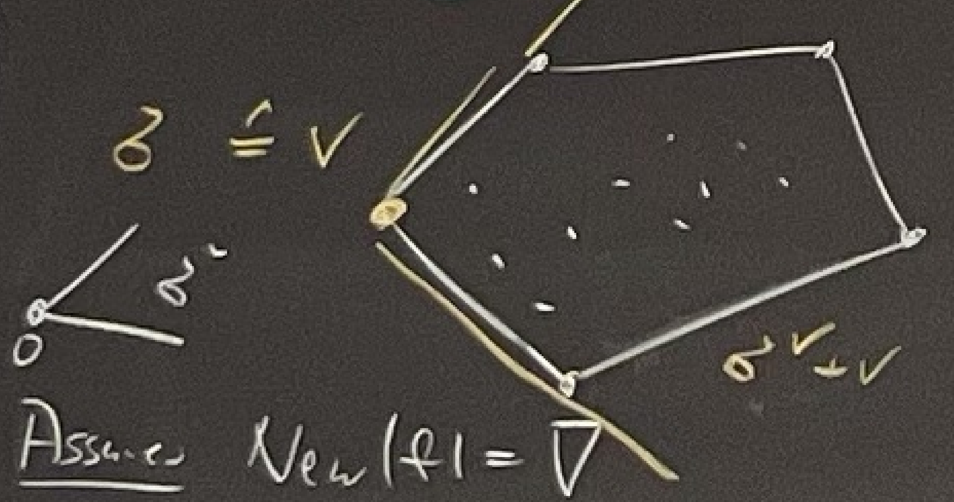
$M \supseteq$ $\quad \langle \cdot, (0,1) \rangle \geq -1 \quad \langle \cdot, (1,0) \rangle \geq -1 \quad \langle \cdot, (-1,-1) \rangle \geq -1$

$\nabla^\vee = \mathrm{conv}\{(0,1),(1,0),(-1,-1)\}$

$\mathbb{P}(\nabla^\vee) \not\cong \mathbb{P}^2$

Let $\nabla \subseteq M_{\mathbb{R}}$ be a lattice pol. $\rightsquigarrow \Sigma = \mathcal{N}(\nabla)$; $\quad TV(\Sigma) \xrightarrow{\sim} \mathbb{P}(\nabla)$

$\sigma \hat{=} v$



$\sigma^\vee \hat{=} v$

Assume Newt$(f) = \nabla$

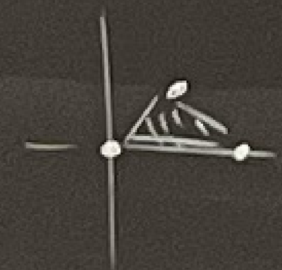
Take $f \in k[M] = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

f = sum of monomials $= \sum_{r \in M} \lambda_r \cdot x^r$, $\quad x^r = \chi^r$

Supp $f := \{r \mid \lambda_r \neq 0\}$

Newt(f) = conv(supp f)

$f(x,y) = 1 + x^2 + xy$



$V(f) \subseteq T = \mathrm{Spec}\, k[M]$

$Y_f := \overline{V(f)} \hookrightarrow TV(\Sigma) = X$

Y = hypersurface in $X \rightsquigarrow \mathcal{J} \subseteq \mathcal{O}_X$

$k[\sigma^\vee \cap M] \ni f \cdot \chi^{-v_\sigma}$ gives the same ideal in T as f does.

$TV(\sigma)$

$\mathcal{J}|_{TV(\sigma)} = (f \cdot \chi^{-v_\sigma}) = f \cdot (\chi^{-v_\sigma}) \Rightarrow \mathcal{J} \cong \mathcal{O}(-\nabla)$

$\mathcal{L}$: $\mathcal{L}|_{TV(\sigma)} = \langle \chi^{-v_\sigma} \rangle = \mathcal{O}(\nabla)$

$\omega_Y = \iota^* \omega_X \otimes (\mathcal{J}/\mathcal{J}^2)^\vee$ $\quad$ adj. formula

$Y \xhookrightarrow{\iota} X$ $\quad$ no "old" cod. -1

$\mathcal{J}|_Y$ $\quad (\mathcal{J}|_Y = \mathcal{J} \otimes \mathcal{O}_Y = \mathcal{J} \otimes \mathcal{O}_X/\mathcal{J} = \mathcal{J}/\mathcal{J}^2)$

$\omega_Y = \mathcal{O}(+\nabla)|_Y \otimes \mathcal{O}(-\nabla) = \mathcal{O}_Y.$

$\boxed{\text{Let } \nabla = \text{reflexive}}$

$0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \to 0$

$\det(\mathcal{J}/\mathcal{J}^2) \otimes \omega_Y = \omega_X \otimes \mathcal{O}_Y$

$\Rightarrow$ $Y = CY$

Serre $\quad \mathcal{J} = \dots \Rightarrow \dots$ l.free