proper morphisms (among schemes) (related: complete varieties)

absolute version of properness
$X|k$ is complete $\Longleftrightarrow$ $X \to \operatorname{Spec} k$ proper.

$f: X \to Y$

Def. $f: X \to Y$ is proper $:\Longleftrightarrow$
- f f. type
- f = separated
- f = universally closed ([stable under base change]-version of „closed")

f_S: $X \times_Y S \to S$

„proper" is local on the target! (0)

Examples
- $X \hookrightarrow Y$ closed embedding.
  e.g. $\operatorname{Spec} A/I \to \operatorname{Spec} A$ ✓
- $X \to Y$ finite map (i.e. affine, $\operatorname{Spec} B \to \operatorname{Spec} A$: B = finite A-module)
  need to check: $\boxed{\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A \quad \text{finite}} \xrightarrow{?}$ surjective (?)

  (going Up/Down theorems)
  $(*)$ if $A \hookrightarrow B$ injective

  $A \xrightarrow{\varphi} B \supseteq J$ $\Longrightarrow$ $\operatorname{Spec} B/J \to \operatorname{Spec} A/\varphi^{-1}(J)$ $= f(\operatorname{Spec} B/J)$ „scheme theoretic closure" / image

  $A/\varphi^{-1}J \hookrightarrow B/J \overset{(*)}{\Longrightarrow} \operatorname{Spec} B/J \to \operatorname{Spec} A/\varphi^{-1}J$ is surjective, i.e. $f(\operatorname{Spec} B/J) = \overline{f(\operatorname{Spec} B/J)}$ □

- $U \hookrightarrow X$ open embedding $\rightsquigarrow$ usually not proper

Expl. $(\mathbb{A}^1 \setminus 0) \hookrightarrow \mathbb{A}^1$ open embedding $\hat{=}$ $k[x] \hookrightarrow k[x]_x$
not closed!!

- $\mathbb{A}^1 \to \operatorname{Spec} k$ closed, but not universally closed!
  $\mathbb{A}^2 \xrightarrow{pr} S = \mathbb{A}^1$
  $V(xy-1) \to \mathbb{A}^1 \setminus 0$
  ($\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$)

Prop: $\mathbb{P}^n_k$ is proper over $\operatorname{Spec} k$
$\Uparrow$
$\mathbb{P}^n_{\mathbb{Z}}$ is proper over $\operatorname{Spec} \mathbb{Z}$

Find properties of „proper":

(1) • $X \xrightarrow{f} Y \xrightarrow{g} Z$, f, g = proper $\Longrightarrow$ $g \circ f$ = proper („composition")

(2) • base change: $X \xrightarrow{\text{proper}} Y$, $\Downarrow$ proper, $X \times_Y S \to S \leftarrow T$

$X \times_Y T = (X \times_Y S) \times_S T$

3) $X \xrightarrow{f} Y \xrightarrow{g} Z$, f proper $\Leftarrow$ g sep., $g \circ f$ proper

Expl. $Z = \operatorname{Spec} k$, $X \xrightarrow{f} Y$ k-variety morphism, $X \to \operatorname{Spec} k \leftarrow Y$

X = complete $\Longrightarrow$ f = proper

4) $X \xrightarrow{f} Y \xrightarrow{g} Z$, f surjective, $g \circ f$ proper $\Rightarrow$ g proper

Theorem: $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper.

Proof. show universal closedness. Have to check: $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A \to \operatorname{Spec} A$ closed (?)
$\mathbb{P}^n_A \xrightarrow{\pi}$

Let $Z \subseteq \mathbb{P}^n_A$ closed sub scheme
$\Longrightarrow Z = V(J)$, $J = (f_1, \dots, f_k) \subseteq A[z_0, \dots, z_n]$ homogeneous ($\deg z_i = 1$)
$\deg f_i = d_i$

Goal: Understand $\pi(V(J)) \subseteq \operatorname{Spec} A$.

$\bigoplus_{i=1}^{k} f_i \cdot A[z]_{d-d_i} \xrightarrow{\beta_d} A[z]_d$ $\quad$ $d \in \mathbb{N}$ – some degree

Let $P \in \operatorname{Spec} A$. $\quad P \notin \pi(V(J)) \Longleftrightarrow \pi^{-1}(P) \cap V(J) = \emptyset$, note: $\pi^{-1}(P) \cap V(f_1, \dots, f_k)$, $f_i \in A[z]$
$\pi^{-1}(P) = \mathbb{P}^n_{\kappa(P)}$, $\operatorname{Proj} A[z]/J$
$= V(\bar f_1, \dots, \bar f_k) \subseteq \mathbb{P}^n_{\kappa(P)}$
$\bar f_i$ = image of $f_i \in A$ in $\kappa(P)$

$\boxed{\pi: \mathbb{P}^n_A \to \operatorname{Spec} A}$
$\mathbb{P}^n_{\kappa(P)} \to \operatorname{Spec} \kappa(P)$ $\quad$ ($\kappa(P) = Q(A/P)$)

$\Longleftrightarrow V(\bar f_1, \dots, \bar f_k) \subseteq \mathbb{P}^n_{\kappa(P)}$ is empty
(HNS: $V_P(\bar f_1, \dots, \bar f_k) = \emptyset$ in $\mathbb{P}^n_\kappa$ $\Longleftrightarrow V_A(\bar f_1, \dots, \bar f_k) \subseteq \{0\}$ in $\mathbb{A}^{n+1}_\kappa$
$\Longleftrightarrow \sqrt{(\bar f_1, \dots, \bar f_k)} \supseteq (\underline{z}) \Longleftrightarrow \exists N: (\bar f_1, \dots, \bar f_k) \supseteq (\underline{z}^N)$)

$\Longleftrightarrow \exists N: (\bar f_1, \dots, \bar f_k) \supseteq (\underline{z})^N$ inside $\kappa(P)[\underline{z}]$

$\Longleftrightarrow \beta^{\kappa(P)}_d: \bigoplus f_i \cdot \kappa(P)[z]_{d-d_i} \to \kappa(P)[z]_d$ $\quad$ $d \geq N \rightsquigarrow \beta^{\kappa(P)}_d$ = surjective

$X \times Z \hookrightarrow X \xrightarrow{\text{proper}} Y$, open, not proper

$\beta_d \hat{=}$ matrix: $(?) \times \binom{d+n+1}{d}$, $= D$ $\rightsquigarrow$ D-minors of β_d: $m_v(d)$ $(v = 1, \dots)$ $\rightsquigarrow$ $m_v(d) \in \kappa(P)$, $\in A$
det ($D \times D$ - submatrices)

$m(d) := \text{ideal}(m_v(d) \mid v = 1, \dots) \subseteq A$ equals (1) after mapping $A \to \kappa(P) = A_P/P_P$ $\Longleftrightarrow$ ($m(d) \subseteq A$ ideal of minors, $m(d) = (1)$ in A_P)
$\kappa(P)$ $\quad$ ($A_P \supseteq P_P$, $A_P \supseteq A_P^\times$)

$\Longleftrightarrow m(d) \cap (A \setminus P) \neq \emptyset \Longleftrightarrow m(d) \not\subseteq P$
$\Longleftrightarrow P \notin V(m(d)) \subseteq \operatorname{Spec} A$. $\quad$ (for some $d \gg 0$)
$m(d) \subseteq A$

$P \in \pi(V(J)) \Longleftrightarrow \forall d: P \in V(m(d))$

$\pi(V(J)) = \bigcap_d V(m(d))$

$\operatorname{Proj} S \hookrightarrow \mathbb{P}^n_A \to \operatorname{Spec} A$

Corollary X = scheme, $\mathcal{S}$ = graded algebra of $\mathcal{O}_X$-modules $\rightsquigarrow$ $\operatorname{Proj}_X \mathcal{S} \to X$ $\quad$ π = proper

Ex: blowing ups. $\quad$ $S = \dots$ of A-modules $\quad$ $\operatorname{Proj} S \xrightarrow{\pi} \operatorname{Spec} A$ $\quad$ $S = A[z_0, \dots, z_n]/I$

4) $X \xrightarrow[\text{surjective}]{f} Y \xrightarrow[\text{proper}]{g} Z$, $g \circ f$ proper

Thm. $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is proper.

Proof. show universal closedness: Have to check: $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A \longrightarrow \operatorname{Spec} A$ closed ?

$\mathbb{P}^n_A \xrightarrow{\pi} \operatorname{Spec} A$

Let $Z \subseteq \mathbb{P}^n_A$ closed sub scheme

$\Longrightarrow Z = V(J)$, $J = (f_1, \dots, f_k) \subseteq A[z_0, \dots, z_n]$ homog. ideal, $\deg f_i = d_i$ $(\deg z_i = 1)$

Goal Understand $\pi(V(J)) \subseteq \operatorname{Spec} A$.

$\bigoplus_{i=1}^{k} f_i \cdot A[\underline{z}]_{d-d_i} \xrightarrow{\beta_d} A[\underline{z}]_d$ $\quad d \in \mathbb{N}$ = some degree

Let $P \in \operatorname{Spec} A$ $\quad P \notin \pi(V(J)) \Longleftrightarrow \pi^{-1}(P) \cap V(J) = \emptyset$; **note** $\pi^{-1}(P) \cap V(f_1, \dots, f_k) = V(\bar f_1, \dots, \bar f_k) \subseteq \mathbb{P}^n_{K(P)}$, $f_i \in A[\underline{z}]$

$\pi^{-1}(P) = \mathbb{P}^n_{K(P)}$, $V(J) = \operatorname{Proj} A[\underline{z}]/J$

$\bar f_i$ = image of $f_i \in A$ in $K(P)$

$\boxed{\pi : \mathbb{P}^n_A \longrightarrow \operatorname{Spec} A}$

$\mathbb{P}^n_{K(P)} \longrightarrow \operatorname{Spec} K(P)$ $\quad (K(P) = Q(A/P))$

$\Longleftrightarrow V(\bar f_1, \dots, \bar f_k) \subseteq \mathbb{P}^n_{K(P)}$ is empty

(**HNS**: $V_{\mathbb{P}}(\bar f_1, \dots, \bar f_k) = \emptyset$ in $\mathbb{P}^n_K \Longleftrightarrow V_A(\bar f_1, \dots, \bar f_k) \subseteq \{0\}$ in $\mathbb{A}^{n+1}_K$

$\Longleftrightarrow \sqrt{(\bar f_1, \dots, \bar f_k)} = (\underline{z}) \Longleftrightarrow \exists N : (\bar f_1, \dots, \bar f_k) \supseteq (\underline{z}^N)$.)

$\Longleftrightarrow \exists N \quad (\bar f_1, \dots, \bar f_k) \supseteq (\underline{z})^N$ inside $K(P)[\underline{z}]$

$\Longleftrightarrow \beta_d^{K(P)} : \bigoplus f_i \cdot K(P)[\underline{z}]_{d-d_i} \longrightarrow K(P)[\underline{z}]_d$. $\quad d \geq N \rightsquigarrow \beta_d^{K(P)}$ = surjective

$\boxed{X \times Z \hookrightarrow X \xrightarrow{\text{proper}} Y \text{ open, not proper}}$

$\beta_d \,\hat{=}\,$ matrix $(?) \times \binom{d+n+1}{d}$, $\binom{d+n+1}{d} = D$ $\rightsquigarrow$ D-minors of β_d: $m_v(d)$ $(v = 1, \dots)$ $\rightsquigarrow$ $m_v(d) \in K(P)$

D-minors $=$ det ($D \times D$ - submatrices), $m_v(d) \in A$

$m(d) := $ ideal $(m_v(d) \mid v = 1 \dots) \subseteq A$ equals (1) after mapping $A \to K(P) = A_P/P_P$ $\Longleftrightarrow$ ($m(d) \subseteq A$ ideal of minors; $m(d) = (1)$ in A_P)

$(A_P \overset{?}{\supseteq} P_P$, $A_P \supseteq A_P^{\times})$

$\Longleftrightarrow m(d) \cap (A \setminus P) \neq \emptyset \Longleftrightarrow m(d) \not\subseteq P$

$\Longleftrightarrow P \notin V(m(d)) \subseteq \operatorname{Spec} A$. (for some $d \gg 0$)

$P \in \pi(V(J)) \Longleftrightarrow \forall d : P \in V(m(d))$ //

$\pi(V(J)) = \bigcap_d V(m(d))$

$\operatorname{Proj} S \hookrightarrow \mathbb{P}^n_A \to \operatorname{Spec} A$

Corollary X = scheme, $\mathcal{S}$ = graded algebra of $\mathcal{O}_X$-modules $\rightsquigarrow$ $\operatorname{Proj}_X \mathcal{S} \longrightarrow X$

S = — " — of A-modules $\quad \operatorname{Proj} S \xrightarrow{\pi} \operatorname{Spec} A$ // π = proper, $S = A[z_0, \dots, z_n]/I$

Ex blowing ups.

$\mathcal{S} = \bigoplus_{d=0}^{\infty} \mathcal{S}_d$, $\mathcal{S}_d = \mathcal{O}_X$-modules, $\mathcal{S}_d \times \mathcal{S}_e \longrightarrow \mathcal{S}_{d+e}$ $\quad$ graded $\mathcal{O}_X$-alg. $\Longrightarrow \operatorname{Proj}_X \mathcal{S} \longrightarrow X$

$S = \bigoplus_{d=0}^{\infty} S_d$ $\quad$ A-mod. $\quad$ A-alg $= S_0$ $\Longrightarrow \operatorname{Proj} S \longrightarrow \operatorname{Spec} A$ proper.

Σ = fan inside $N_{\mathbb{R}} \cong \mathbb{R}^d$. $\quad X = TV(\Sigma)$.

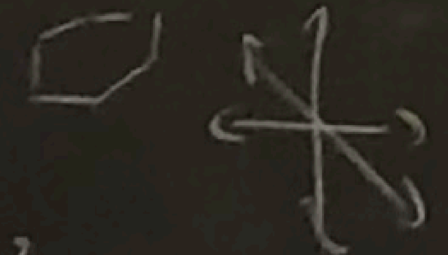
$TV(\Sigma)$ = projective $\Longleftrightarrow \Sigma = \mathcal{N}(\Delta)$; Δ = lattice polytope.

$TV(\Sigma) \hookrightarrow \mathbb{P}^N$ closed

$\Downarrow$ $TV(\Sigma) = \mathbb{P}(\Delta) \hookrightarrow \mathbb{P}^N$

($\mathcal{L} \mid TV(\Sigma)$ is ample $\Longrightarrow \mathcal{L}$ = Sp fun

$\Updownarrow$ $\mathcal{O}(\Delta)$ s.t. $\Sigma \in \mathcal{N}(\Delta)$)

eg: 

If $\Sigma \subsetneq \mathcal{N}(\Delta)$

$\longrightarrow TV(\Sigma) \longrightarrow TV(\mathcal{N}(\Delta)) \hookrightarrow \mathbb{P}^N$

Q When is $TV(\Sigma)$ complete?

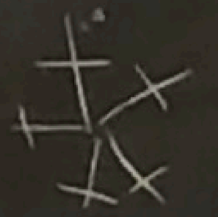
Prop complete $\Longleftrightarrow |\Sigma| = N_{\mathbb{R}}$.

not complete

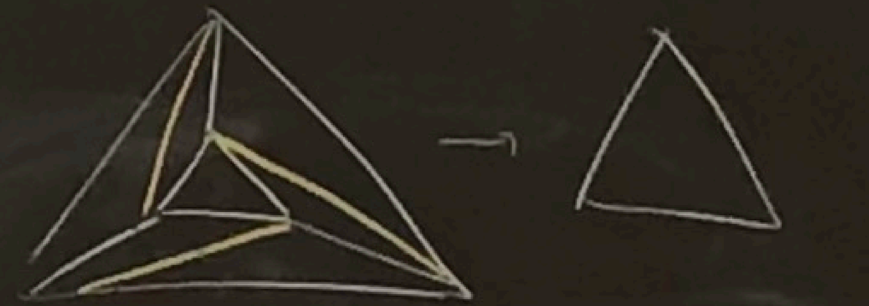
Proof: $(\Longrightarrow)$ Let $|\Sigma| \subsetneq N_{\mathbb{R}}$ $\rightsquigarrow$ Take $\bar\Sigma \supseteq \Sigma$: $|\bar\Sigma| = N_{\mathbb{R}}$, Σ = subfan.

$\Longrightarrow TV(\Sigma) \underset{\text{open}}{\subseteq} TV(\bar\Sigma) \longrightarrow \operatorname{Spec} k$ //

$(\Longleftarrow)$ Let $|\Sigma| = N_{\mathbb{R}}$. : $\Sigma \overset{?}{=} \mathcal{N}(\Delta) \Longrightarrow TV(\Sigma)$ = proj. $\Longrightarrow$ complete.



Counter ex



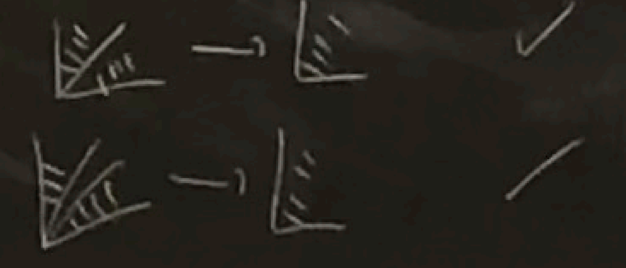
Instead:

$\Sigma \geq \Sigma'$ (subdiv.)

$TV(\Sigma') \longrightarrow TV(\Sigma) \longrightarrow \operatorname{Spec} k$. proper ?

Need: $TV(\Sigma')$ = proper over $\operatorname{Spec} k$ $\quad$ Σ' = easy fan = cutting by hyperplanes!

Ex.



Remark $TV(\Sigma') \longrightarrow TV(\Sigma)$ is proper $\Longleftrightarrow \forall \sigma \in \Sigma$: $g^{-1}(\sigma)$ = covered by Σ'-cones.

$\cdot\ N' \xrightarrow{g} N$

$\cdot\ \forall \sigma' \in \Sigma'\ \exists \sigma \in \Sigma : g(\sigma') \subseteq \sigma$