Thu. 3/14: Exam Schedule 10–14

proper morphisms $X \to Y$: ex: closed ... morph., projective m.

K = field, $v: K^* \to A$ valuation (A = abelian, ordered group, e.g. $\mathbb{Z}$)

($:\Leftrightarrow \forall a, b \in K^*$: $\cdot\ v(ab) = v(a) + v(b)$

$\cdot\ v(a+b) \geq \min\{v(a), v(b)\}$)

"valuation ring" $R := \{a \in K^* \mid v(a) \geq 0\} \cup \{0\}$

(thm. $a \in K^* \overset{(*)}{\Longrightarrow} a \in R$ or $\frac{1}{a} \in R$

$(*) \Longrightarrow R$ is a valuation ring (def.): $A := K^*/R^*$, ord: $a \geq b :\Leftrightarrow a/b \in R$)

Example. X = scheme, $x \in X$ smooth pt of codim 1 (generic pt η_D of a PD D)

$\hookrightarrow \mathcal{O}_{X,x}$ = discrete val. ring (DVR): $m_{X,x} = (t)$ $\quad v(t^k) = k$

e.g.: $x \in C$ = smooth curve

$\frac{\alpha}{q^N} \in R \subseteq \mathcal{O}_x \Rightarrow \alpha \in \mathcal{O}_x$

THUS: $\mathcal{O}_x = K$

$\Rightarrow f^*: R \to \mathcal{O}_{x,x} = K$, $x = 0 \Rightarrow f(x) = \{\eta\}$ ↯

THUS: $\boxed{R \xrightarrow{\sim} \mathcal{O}_{x,r}}$

Theorem (valuative criterion of properness)

Let $f: X \to Y$ proper; (K, R) = val. ring (discrete)

$$\begin{tikzcd} \{\eta\} = \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{\eta, \xi\} = \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists\, \Phi"] & Y \end{tikzcd}$$

(given) $\Longrightarrow \exists\, \Phi$!!

$\operatorname{Spec} R \hookleftarrow \operatorname{Spec} K$

$\mathcal{O}_{\operatorname{Spec} R, \eta} = K \longleftarrow \mathcal{O}_{X,x}$, $\quad R = \mathcal{O}_{\operatorname{Spec} R, \xi}$, f^*

Thus:
- $Y = \operatorname{Spec} R$
- $g: \operatorname{Spec} K \to X$ is dominant!
- X = reduced
- Let $f(x) = \xi$

(1)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X' = X \times_Y R \arrow[r] \arrow[d, "f' = \text{proper}"] & X \arrow[d, "f = \text{proper}"] \\ \operatorname{Spec} R \arrow[r, "id"] \arrow[ur, dashed, "?"] & \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

w.l.o.g. $\boxed{Y = \operatorname{Spec} R}$

(2) $\operatorname{Spec} K \xrightarrow{g} X \xrightarrow{f} \operatorname{Spec} R$ (proper)

$\operatorname{Spec} K \to \overline{\operatorname{im}(g)} \overset{\text{closed}}{\hookrightarrow} X$ (proper) $\to \operatorname{Spec} R$

$R \hookrightarrow \mathcal{O}_{x,x} \hookrightarrow K = \operatorname{Quot}(R) = K(X)$ $\qquad$ Valuation $v: K^* \to A$.

Goal: $\mathcal{O} = R$ or $\mathcal{O} = K$

$\mathcal{O} \neq R \Longrightarrow \exists\, q \in \mathcal{O} \setminus R \leadsto v(q) < 0$

$\forall\, \alpha \in K \leadsto v(\alpha) \in A$

$\Rightarrow v\left(\frac{\alpha}{q^N}\right) = v(\alpha) - N \cdot v(q) \geq 0$ if $N \gg 0$

Theorem $f: X \to Y$ is proper (e.g. projective).

(Recall. $\mathcal{F}$ = quasi-coherent sheaf on $X \Longrightarrow f_* \mathcal{F}$ = quasi-coherent on Y)

Now: $\mathcal{F}/X$ coherent $\Longrightarrow f_* \mathcal{F}$ = coherent.

(Example. $X \to \operatorname{Spec} k$ complete
$\mathcal{F}/X$ = coherent $\Rightarrow \Gamma(X, \mathcal{F}) = \boxed{f.}$ d. k-vs!)

Sheaf cohomology

X = scheme (top. space), $\mathcal{F}$ = sheaf of $\mathcal{A}b \mid \operatorname{Mod}(\mathcal{O}_X) = \mathcal{C}$ $\leadsto \Gamma(X, \cdot): Sh \to \mathcal{C}$ left exact.

(Alg. 1: $(\otimes_R N): \operatorname{Mod} \to \operatorname{Mod}$ right exact

$\forall\, M_\bullet \in \operatorname{Mod}$ $\exists\, P_\bullet \xrightarrow{q.i.} M_\bullet$

$M_\bullet$ = complex, bounded. $\to \cdots \to 0 \to 0$

F = right exact $\Rightarrow$ $\mathbb{L}F(M) := F(P_\bullet)$

$L_i F(M_\bullet) := H_i(-\,''\,-)$)

Special case: $M_\bullet = M$ $(= M_0)$

$\boxed{\cdots \to P_2 \to P_1 \to P_0 \cancel{\to 0} \to M \to 0}$ exact

$\updownarrow$ q.i.

$0 \to M \to 0$

Now: (Γ = left exact) $\quad A \in \mathcal{A}b/\operatorname{Mod}_R \overset{(!)}{\leadsto} \exists$ injective module I: $A \hookrightarrow I$

(Ex: I = inj. ab. grp $\Leftrightarrow \forall\, k \in \mathbb{Z}$ $(k \neq 0)$, $\forall\, a \in I$ $\exists\, b \in I$: $a = k \cdot b$ $\quad$ („$b = a/k$“)

Remark: I = inj. R-module $\Rightarrow \forall f \in R$: $I \longrightarrow I_f$ is surjective. (e.g. $(\mathbb{Z}/6\mathbb{Z}) \to (\mathbb{Z}/6\mathbb{Z})_2$)

(Proof for I = torsion free). I = injective $\Leftrightarrow \operatorname{Hom}_R(\cdot, I)$ = exact $\Leftrightarrow \forall\, M \hookrightarrow N$, $M \to I$, $N \dashrightarrow I$

$I \hookrightarrow I_f \leadsto (\cdot f)$ is an iso of I)

$I \xrightarrow{id} I$

Corollary: I = inj. R-module

$\leadsto \tilde{I}$ = „flasque“ (quasi-coh.) sheaf on $\operatorname{Spec} R$

X = scheme, $\mathcal{F}$ = $\mathcal{O}_X$-module.

- $\boxed{\mathcal{F} = \text{inj.}} \Leftrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{F})$ = exact.
- $\boxed{\mathcal{F} = \text{flasque}} \Leftrightarrow \forall\, V \subseteq U$: $\mathcal{F}(U) \xrightarrow{res} \mathcal{F}(V)$ surjective

Prop: „injective $\Rightarrow$ flasque“

Proof: $U \overset{j}{\subseteq} X$ $\quad$ (?) $\Gamma(U, \mathcal{F}) \longleftarrow \Gamma(X, \mathcal{F})$

$\operatorname{Hom}(j_! \mathcal{O}_U, \mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j^* \mathcal{F}) \xleftarrow{(?)} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$

$f^* \dashv f_*$

$j_! \dashv j^* \dashv j_*$

$j_! \mathcal{O}_U \hookrightarrow \mathcal{O}_X$

$(j_! \mathcal{G})(V \subseteq X) = \begin{cases} \mathcal{G}(V) & \text{if } V \subseteq U \\ 0 & \text{if not} \end{cases}$ $\quad (\mathcal{G}|U)$

$\mathcal F \rightsquigarrow \exists$ inj. resol. $\mathcal F \to I^\bullet \rightsquigarrow \Gamma(X, I^\bullet) =: R\Gamma(X,\mathcal F)$

$R^p\Gamma(X,\mathcal F) = H^p(\Gamma(X, I^\bullet))$ $I_0 \to I_1 \to \dots$ (inj. sheaves)   complex of ab. grps ✓

$\| $

$H^p(X,\mathcal F)$.

instead: it is allowed to take $\mathcal F \xrightarrow[\text{(resol.)}]{} A^\bullet$; A^i = „acyclic for Γ" $\Rightarrow H^\nu(X, A^i) = 0 \ \forall \nu \geq 1$.

Prop: $\mathcal F$ = flasque $\Rightarrow H^\nu(X,\mathcal F) = 0 \ \forall \nu\ge 1$.

Proof: $0 \to \mathcal F \hookrightarrow I \to \mathcal G_{\text{flasque}} \to 0$ (inj. $\Rightarrow$ flasque) $=$

• $\forall U$: $\boxed{0\to \mathcal F(U) \to I(U) \to \mathcal G(U) \to 0}$ exact. $H^1(U,\mathcal F) \to H^1(U, I) = 0$

$\dots \to I(X) \to \mathcal G(X) \to 0$

$H^\nu(U,\mathcal G) \xrightarrow{\sim} H^{\nu+1}(U,\mathcal F)$ $(\nu\ge1)$

$\|$

0

$X = \operatorname{Spec} A \Rightarrow [\, I = $ inj. A-mod $\Rightarrow \tilde I$ = flasque $]$

$\Rightarrow \forall \mathcal F$ = qu. coh.: we are allowed to use resol. by (radical) qu. coh. sheaves $\tilde I$!

$\Rightarrow H^{\nu\ge1}(\operatorname{Spec} A, \mathcal F = \text{qu. coh.}) = 0$!!

Ex: $(X, \mathcal O_X)$; $\mathcal O_X^*$ = sh. of ab. grps. $H^1(X, \mathcal O_X^*) = \operatorname{Pic} X$

$H^1(\operatorname{Spec} A, \mathcal O^*) \ne 0$ in general.

Čech cohomology: $\mathcal F$ = sheaf ; $\mathcal U = \{U_i\}$ = open cov. of X ($i\in I$ = well-ordered)

$0\to \mathcal F(X) \to \prod_{i\in I}\mathcal F(U_i) \to \prod_{i_0<i_1}\mathcal F(U_{i_0}\cap U_{i_1}) \to \dots \to \prod_{i_0<\dots<i_k} \mathcal F(U_{i_0}\cap\dots\cap U_{i_k}) = C^k(X,\mathcal F)$

Claim: exact, coschol. [illegible] as a complex of sheaves $\prod_{i_0<\dots<i_{k-1}} \mathcal F(\dots) \to$ $d(s_{\dots})_{i_0\dots i_k} := \sum_{\nu=0}^{k} (-1)^\nu s_{i_0 \dots \hat{i_\nu}\dots i_k}|$

fix one index j ≙ U_j

technical trick: • Sheafify the Čech complex

• consider stalks in $x\in X \Rightarrow \exists U_j \ni x \rightsquigarrow$ take $d(s_j)$. □

$\mathcal F(X) \to C^0(X,\mathcal F) \to C^1(X,\mathcal F) \to \dots$

sheafified: $\mathcal F \to \mathcal C^0(\mathcal F) \to \mathcal C^1(\mathcal F)\to\dots$ exact of $\mathcal C^\nu(\mathcal F) = \mathcal C^\nu(\mathcal U, \mathcal F)$

$C^\bullet(X,\mathcal F) = \Gamma(X, \mathcal C^\bullet(\mathcal U, \mathcal F))$

Def: $\check H^p(X, \mathcal U, \mathcal F) := H^p(C^\bullet(X,\mathcal F))$ — Hope: $\check H^p = H^p$ ex: $\mathcal U = \{X\}$ $\Downarrow$ $\check H^0(X,\mathcal F) = \Gamma(X,\mathcal F)$, $\check H^i(X,\mathcal F) = 0$ $(i\ge1)$

Theorem: • $\mathcal V < \mathcal U \Rightarrow \check H^p(\mathcal V) \leftarrow \check H^p(\mathcal U)$

• $H^p(X,\mathcal F) = \lim_{\mathcal U} \check H^p(\mathcal U, \mathcal F)$.

• $\mathcal F$ = qu. coh. $\Rightarrow H^p(X,\mathcal F) = \check H^p(\mathcal U_{\text{aff}}, \mathcal F)$ Ex: $\mathbb P^1$, U_0, U_∞, $\mathcal O(1)$, $\mathcal O(-2)$

Ex: $H^1(\mathcal O^*) \leftrightarrow \check H^1(\mathcal O^*)$ $0\to \mathcal O^*(X) \to \prod \mathcal O^*(U_i) \to \mathcal O^*(U_{ij})$

$t_i \longmapsto s_{ij} \xmapsto{\text{cocycle}} 0$

$(t_i) \longmapsto s_{ij} := t_j/t_i$

toric situation (e.g. $X = \mathbb P^1$)   lattice pol. in M ($\mathcal N(\Delta) \geq \Sigma$)

$X = TV(\Sigma)$ $\mathcal L \in \operatorname{Pic} X$: nef sheaves $\mathcal O(\Delta)$

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$\mathcal O(\Delta^+)\otimes\mathcal O(\Delta^-)^{-1} = \mathcal O(\Delta^+ - \Delta^-)$

$H^k(TV(\Sigma), \mathcal O(\Delta^+ - \Delta^-))_{(m\in M)} = \tilde H^{k-1}(\Delta^- \setminus (\Delta^+ - m))$

M-graded

$\tilde H^{-1}(X) = \begin{cases} \mathbb C & \text{if } X=\emptyset \\ 0 & \text{if } X\ne\emptyset \end{cases}$

$\mathbb P^1$ $\mathcal O(1) = \mathcal O(\dots) = \mathcal O(\dots - \dots)$ (m) $\tilde H^{-1}(\dots \setminus \dots)$

$\Rightarrow H^0 - \tilde H^{-1} = \mathbb C$ for 2 diff. $m = 0, 1$

$= 0$ otherwise

$H^0(\mathcal O(1))$ = 2-dim.

$\mathcal O(-2) = \mathcal O(\dots - \dots)$

$\dots \setminus \bullet\,(-m)$

$H^1 = 0$

$\tilde H^{-1} = 0 \Rightarrow H^0 = 0$

$\tilde H^0 = 1 \Rightarrow H^1(\mathcal O(-2)) = \mathbb C$.