

The. 3/14. Exam Solution 10-19

proper maps $X \rightarrow Y$ ex. closed \Rightarrow $f^{-1}(y)$ finite map, projective in.

$K = \text{field}$, $v: K^* \rightarrow \mathbb{Z}$ valuation ($A = \text{valuation, ordered group, es. } \mathbb{Z}$)

$(\Leftrightarrow \forall a, s \in K^* : v(as) = v(a) + v(s)$
 $v(a+s) \geq \min\{v(a), v(s)\}$)

(thm) $a \in K^* \Rightarrow a \in R$ or $\frac{1}{a} \in R$

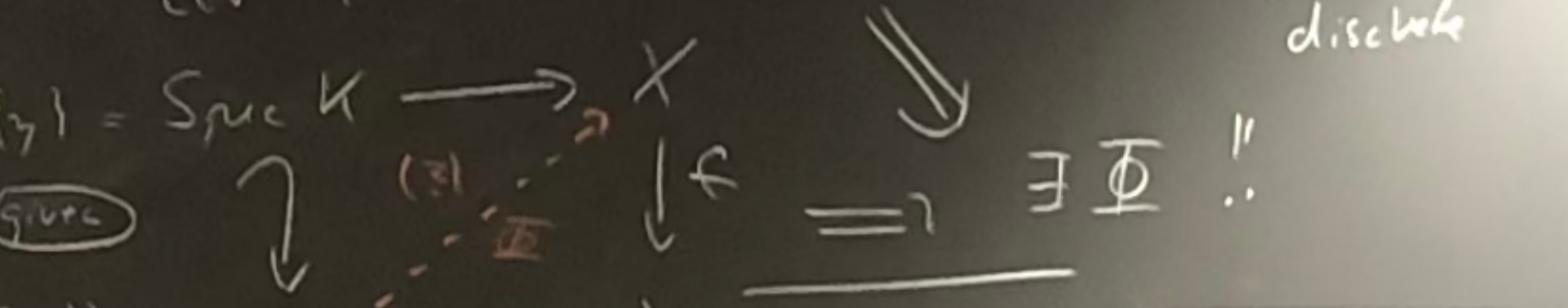
(*) $\Rightarrow R$ is a valuation ring: $A := K^*/R^*$, ord: $a \geq b \Leftrightarrow a/b \in R$

Ex. $X = \text{plane}$, $x \in X$ smooth pt of curve 1 (source pt γ_D of a PD D)

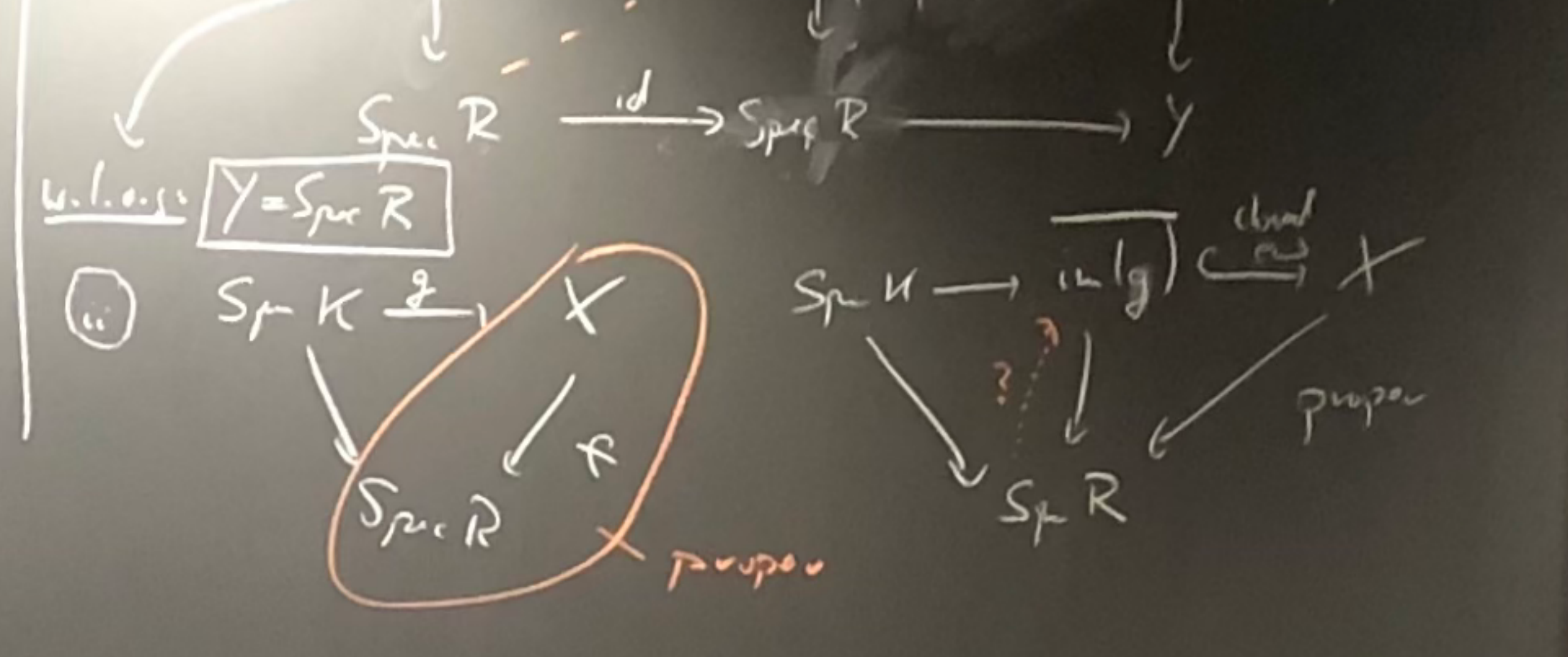
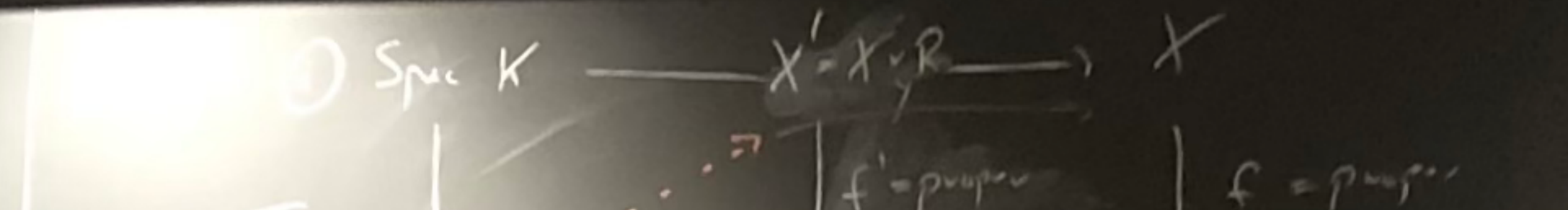
$\hookrightarrow \mathcal{O}_{X,x} = \text{discrete val. ring (DVR)} : m_{X,x} = (t) \quad v(t^k) = k$

es: $x \in C = \text{smooth curve} \quad \frac{1}{q^N} \in R \subseteq \mathcal{O}_x \Rightarrow a \in \mathcal{O}_x \Rightarrow f^*: R \rightarrow \mathcal{O}_x = K, x=0 \Rightarrow f(x) = \gamma + i$
 THUS: $\mathcal{O}_x = K$ THUS: $R \cong \mathcal{O}_{X,x}$

Theorem (valuative criterion of properness)
 Let $f: X \rightarrow Y$ proper, $(K, R) = \text{val. ring}$ discrete



Thus: $Y = \text{Spec } R$
 $g: \text{Spec } K \rightarrow X$ is dominant!
 $X = \text{red. closed}$
 Let $f(x) = s$



$R \subseteq \mathcal{O}_{X,x} \subseteq K = \text{Quot}(R) = K(x)$ valuation $v: K^* \rightarrow \mathbb{Z}$
 Goal: $\mathcal{O} = R$ or $\mathcal{O} = K$
 $\mathcal{O} \neq R \Rightarrow \exists q \in \mathcal{O} - R \sim v(q) < 0 \Rightarrow \forall d \in K, v(d) \in \mathbb{Z} \Rightarrow v(\frac{d}{q^N}) = v(d) - N \cdot v(q) \geq 0$ if $N \gg 0$

Theorem: $f: X \rightarrow Y$ is proper (ex. projective).

(Recall: $F = \text{quasi-coherent sheaf on } X \Rightarrow f_* F = \text{quasi-coherent on } Y$)

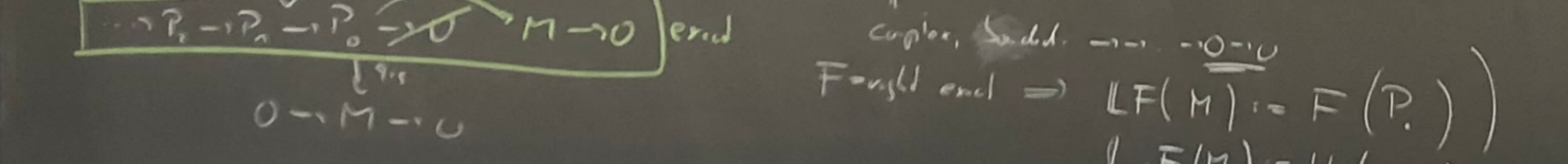
Now: $F|_X$ coherent $\Rightarrow f_* F = \text{coherent}$. (Ex. $X \rightarrow \text{Spec } R$ complete $F|_X = \text{coherent} \Rightarrow \Gamma(X, F) = \hat{F}$ d. s.-vs.)

Sheaf cohomology

$X = \text{scheme}$, $F = \text{sheaf of } AB / \text{Mod}(\mathcal{O}_X) \sim \Gamma(X, \cdot) : \text{SL} \rightarrow \mathcal{E}$ left exact.

(A.1) $(\mathcal{O}_R M) : \text{Mod} \rightarrow \text{Mod}$ right exact

Special case: $M_i = M (= M_0)$

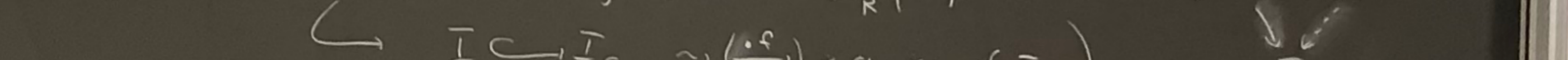


Now: ($\Gamma = \text{left exact}$) $A \in AB / \text{Mod}_R \stackrel{!!}{\sim} \exists$ injective module $I, A \subseteq I$

(Ex: $I = \text{inj. d. sup} \Leftrightarrow \forall k \in \mathbb{Z}, \forall a \in I \exists s \in I : a = k \cdot b$ ("s" = "u")

Remark: $I = \text{inj. } R\text{-module} \Rightarrow \forall f \in R : I \rightarrow I_f$ is surjective. (es. $(\mathbb{Z}/6\mathbb{Z}) \rightarrow (\mathbb{Z}/6\mathbb{Z})_2$)

(Proof for $I = \text{torsion free}$). $I = \text{injective} \Leftrightarrow \text{Hom}_R(\cdot, I) = \text{exact} \Leftrightarrow \forall M \subseteq N$



Corollary: $I = \text{inj. } R\text{-module} \Rightarrow \hat{I} = \text{"flasque" (quasi-coherent) sheaf on Spec } R$

$X = \text{scheme}, F = \mathcal{O}_X\text{-module}$

$\hat{F} = \text{inj.} \Leftrightarrow \text{Hom}_{\mathcal{O}_X}(\cdot, \hat{F}) = \text{exact}$

$\hat{F} = \text{flasque} \Leftrightarrow \forall V \subseteq U : \hat{F}(U) \xrightarrow{\text{res}} \hat{F}(V)$

Prop: "injective \Rightarrow flasque"

Proof: $U \subseteq X$ $\Gamma(U, \hat{F}) \leftarrow \Gamma(X, \hat{F})$
 $\text{Hom}_{\mathcal{O}_U}(D_U, \hat{F}) \leftarrow \text{Hom}_{\mathcal{O}_X}(D_X, \hat{F})$
 $(j_* \hat{F})(V \subseteq X) = \begin{cases} \hat{F}(U) & \text{if } V \subseteq U \\ 0 & \text{if not} \end{cases}$ (S1U)

$\mathcal{F} \sim \mathcal{F}$ is a sheaf. $\mathcal{F} \rightarrow \mathcal{I} \rightarrow \dots \rightarrow \mathcal{I}^i \rightarrow \mathcal{I}^{i+1} \rightarrow \dots$
 $\mathcal{R}P(X, \mathcal{F}) = H^p(\mathcal{R}P(X, \mathcal{I}^i))$
 $H^p(X, \mathcal{F})$
 instead: it is allowed to take $\mathcal{F} \rightarrow \mathcal{A}^i$ (resol.) where $\mathcal{A}^i =$ "acyclic for Γ "
 $= H^p(X, \mathcal{A}^i) = 0 \quad \forall v \geq 1$
 Proof: $\mathcal{F} = \mathcal{F}_{\text{loc}} \Rightarrow H^v(X, \mathcal{F}) = 0 \quad \forall v \geq 1$
 Proof: $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{G} \rightarrow 0$
 $(i_j = \text{flips}) = \mathcal{F}_{\text{loc}}$
 $H^p(U, \mathcal{F}) \rightarrow H^p(U, \mathcal{I})$
 $0 \rightarrow \mathcal{F}|_U \rightarrow \mathcal{I}|_U \rightarrow \mathcal{G}|_U \rightarrow 0$ exact. 0
 $H^v(U, \mathcal{G}) \rightarrow H^{v+1}(U, \mathcal{F})$ ($v \geq 1$)

$X = \text{Spec } A \Rightarrow [\mathcal{I} = \mathcal{I}_j; A\text{-mod} \Rightarrow \tilde{\mathcal{I}} = \text{flashes}]$
 $\Rightarrow \forall \mathcal{F} = \text{qu. coh.}$ we are allowed to use resol. by (radical) qu. coh. sheaves $\tilde{\mathcal{I}}$!
 $\Rightarrow H^{v \geq 1}(\text{Spec } A, \mathcal{F} = \text{qu. coh.}) = 0$!!
Ex (X, \mathcal{O}_X) , $\mathcal{O}_X^* = \text{sh. of ab. grps.}$ $H^1(X, \mathcal{O}_X^*) = \text{Pic } X$
 $H^1(\text{Spec } A, \mathcal{O}^*) \neq 0$ "general".
Cech cohomology, $\mathcal{F} = \text{sheaf}$, $\mathcal{U} = \{U_i\} = \text{open cov. of } X$ ($i \in I = \text{well-ordered}$)
 $0 \rightarrow \mathcal{F}(X) \rightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightarrow \prod_{i < j} \mathcal{F}(U_i \cap U_j) \rightarrow \dots \rightarrow \prod_{i_0 < \dots < i_n} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_n}) \rightarrow \dots$
 $\prod_{i_0 < \dots < i_n} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_n}) = C^k(X, \mathcal{F})$
 $\prod_{i_0 < \dots < i_n} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_n}) = \sum_{v=0}^n (-1)^v \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_n})$
Clar. exact: constant id ~ 0 as a complex of sheaves
 fix one index $j \in I$
technical trick: • Satisfy the Cech complex
 • consider stalks in $x \in X \Rightarrow \exists U_j \ni x \Rightarrow \text{take } \mathcal{F}(U_j)$

$\mathcal{F}(X) \rightarrow C^0(X, \mathcal{F}) \rightarrow C^1(X, \mathcal{F}) \rightarrow \dots$
 sheaf. $\mathcal{F} \rightarrow \mathcal{E}^0(\mathcal{F}) \rightarrow \mathcal{E}^1(\mathcal{F}) \rightarrow \dots$ exact of $\mathcal{E}^v(\mathcal{F}) = \mathcal{E}^v(\mathcal{U}, \mathcal{F})$
 $C^0(X, \mathcal{F}) = \mathcal{P}(X, \mathcal{E}^0(\mathcal{U}, \mathcal{F}))$
Def: $\check{H}^p(X, \mathcal{U}, \mathcal{F}) = H^p(C^0(X, \mathcal{F}))$ - Lopez $\check{H}^p = H^p$ ex: $\mathcal{U} = \{X\}$
 $\check{H}^0(X, \mathcal{F}) = \mathcal{P}(X, \mathcal{F})$
 $\check{H}^1(X, \mathcal{F}) = 0$ (2-1)
Theorem: • $\mathcal{V} \leq \mathcal{U} \Rightarrow \check{H}^p(\mathcal{V}) \leftarrow \check{H}^p(\mathcal{U})$
 • $H^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$
 • $\mathcal{F} = \text{qu. coh.} \Rightarrow H^p(X, \mathcal{F}) = \check{H}^p(\mathcal{U}_{\text{aff}}, \mathcal{F})$ Ex: $\mathbb{P}^1, U_0, U_1, \mathcal{O}(1)$
Ex: $H^1(\mathcal{O}^*) \leftarrow H^1(\mathcal{O}^*)$ $0 \rightarrow \mathcal{O}^*(X) \rightarrow \prod \mathcal{O}^*(U_i) \rightarrow \prod \mathcal{O}^*(U_{ij})$
 $t_i \mapsto S_i$ cocycle $\rightarrow 0$
 $(t_i) \mapsto S_i = \epsilon_i / \epsilon_i$

toric situation (e.g. $X = \mathbb{P}^1$)
 $X = \text{TV}(\Sigma)$ $\mathcal{L} \in \text{Pic } X$: nef sheaves $\mathcal{O}(\Delta)$
 $\mathcal{O}(\Delta^+) \otimes \mathcal{O}(\Delta^-)^{-1} = \mathcal{O}(\Delta^+ - \Delta^-)$
 $H^k(\text{TV}(\Sigma), \mathcal{O}(\Delta^+ - \Delta^-)) (m \in M) = \tilde{H}^{k-1}(\Delta^- \setminus (\Delta^+ - m))$
 M -graded
 $\mathbb{P}^1 \leftarrow \mathcal{O} \rightarrow \mathcal{O}(1) = \mathcal{O}(\circ \rightarrow \circ) = \mathcal{O}(\circ \rightarrow \circ \rightarrow \circ) \leftarrow \tilde{H}^{k-1}(\circ \setminus \circ \rightarrow \circ)$
 $\Rightarrow H^0 = H^{-1} = \mathbb{C}$ for 2 d.f. $m = 0, 1$
 $= 0$ otherwise
 $H^1 = 0$
 $\tilde{H}^{-1} = 0 \Rightarrow H^0 = \mathbb{C}$
 $\tilde{H}^0 = 0 \Rightarrow H^1(\mathcal{O}(-2)) = \mathbb{C}$