COMMUTATIVE ALGEBRA/ ALGEBRAIC GEOMETRY (BMS-LECTURE WS 2021/22 + SS 2022 + WS 2022/23 FUB)

KLAUS ALTMANN

1. Rings and ideals

20.10.21 (1)

1.1. **Rings and ideals.** Units, zerodivisors, nilpotent elements, prim- and maximal ideals in rings (Example: in $\mathbb{Z}/n\mathbb{Z}$ und $k[X,Y]/(X^2-Y^3) = k[t^2,t^3]$ with k being a field). Operations with Ideals: $+, \cap, \cdot, \sqrt{\ }$; moving ideals along ring homomorphisms.

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1.2. **Algebraic sets.** $k = \overline{k}$ field $\rightsquigarrow k[\mathbf{x}] := k[x_1,\ldots,x_n]$ is the ring of "regular functions" $A(k^n)$; "*closed algebraic subsets*" of k^n are the vanishing loci $V(J) \subseteq k^n$ for subsets or (radical) ideals $J \subseteq k[\mathbf{x}] \rightsquigarrow$ $\boxed{\text{ZARISKI } \textit{topology}}$ on k^n: $\bigcap_i V(J_i) = V(\bigcup_i J_i) = V(\sum_i J_i)$ and $V(J_1) \cup V(J_2) \subseteq V(J_1 \cap J_2) \subseteq V(J_1 J_2) \subseteq V(J_1) \cup V(J_2)$.

Examples: $V(y^2 - x^3)$, $V\Big(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \le 1\Big)$, $\mathrm{SL}(n,k) \subseteq \mathbb{M}(n,k) = k^{n^2}$.

Subset $Z \subseteq k^n \rightsquigarrow$ radical ideal $I(Z) := \{f \in k[\mathbf{x}] \mid f|_Z = 0\} \subseteq k[\mathbf{x}]$. Properties: $I(\subseteq) = \supseteq$ and $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$. Moreover, $Z \subseteq V(I(Z)) =$ "algebraic closure" and $I(V(J)) \supseteq \sqrt{J}$ (even "=" by HNS (7.3)). In particular, for $Z = V(J)$ algebraic: $Z \subseteq V\big(I(V(J))_{\supseteq J}\big) \subseteq V(J) = Z$. Thus, HNS (7.3) $\rightsquigarrow$ order reversing bijection

$$\{\text{closed, algebraic} \subseteq k^n\} \quad \underset{I(\bullet)}{\overset{V(\bullet)}{\rightleftarrows}} \quad \{\text{radical ideals in } k[x_1,\ldots,x_n]\}.$$

Properties: $I(\bigcap_i Z_i) = \sqrt{\sum_i I(Z_i)}$; Z is irreducible $\Leftrightarrow I(Z)$ is a prime ideal.

"Regular functions" on closed algebraic $Z = V(J)$: Reduced "coordinate ring" $A(Z) := k[\mathbf{x}]/I(Z)$ (integral for irreducible $Z =$ "*affine varieties*"); same bijection as above for Z and $A(Z)$; the smallest example is $Z = \{p\}$ with $A(\{p\}) = k[\mathbf{x}]/\mathfrak{m}_p = k$. Open subsets $D(g \in A(Z)) := [g \neq 0] = Z \setminus V(g)$ yields a basis of the open subsets;
3.11.21 (5) $D(g_i)(i \in I)$ cover $Z \Leftrightarrow V(g_i \mid i \in I) = \emptyset \Leftrightarrow (g_i)_{i \in I} = (1)$ in $A(Z)$ by HNS.

1.3. **Functoriality of algebraic sets.** Regular algebraic maps $f : k^m \to k^n$ are, by definition, n-tuples $f = (f_1,\ldots,f_n)$ with $f_i \in k[\mathbf{x}] = k[x_1,\ldots,x_m]$. This is equivalent to k-algebra homomorphisms $f^* : k[\mathbf{y}] := k[y_1,\ldots,y_n] \to k[\mathbf{x}]$ sending $y_i \mapsto f_i(\mathbf{x})$. This map coincides with the pull-back of regular functions, i.e. $f^*(g \in$

$k[\mathbf{y}]) = g \circ f$. If $J \subseteq k[\mathbf{y}]$, then $f^{-1}V(J) = V(f^*(J)k[\mathbf{x}])$, i.e. regular fuctions are continous.

More generalized: If $X \subseteq k^m$ and $Y \subseteq k^n$ are (Zariski-) closed algebraic subsets, then regular maps $f : X \to Y$ are, by definition, given by asking for extendability to commutative diagrams

$$\begin{array}{ccc} k^m & \xrightarrow{F} & k^n \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

where F is regular as before. An equivalent condition for such a diagram is a ring homomorphism $F^* : k[\mathbf{y}] \to k[\mathbf{x}]$ with $F^*(I(Y)) \subseteq I(X)$. In particular, regular maps $f : X \to Y$ are provided by k-algebra homomorphisms $f^* : A(Y) \to A(X)$. This category is equivalent to the opposite of the category of reduced, finitely generated k-algebras.

Thus, (Zariski-) closed algebraic subsets form a category; their isomorphism classes (i.e. neglecting the embedding into an ambient space k^n) are called *affine sets*. This category is equivalent to the opposite of the category of reduced, finitely generated k-algebras.

A special case: If $f \in k[\mathbf{x}]$, then we obtain $g(\mathbf{x}, t) := f(\mathbf{x}) \cdot t - 1 \in k[\mathbf{x}, t]$ and $Z_f := V(g)$ is a closed subset of k^{m+1}, i.e.

$$\begin{array}{ccc} k^{m+1} & \xrightarrow{\text{pr}} & k^m \\ \uparrow & & \uparrow \\ Z_f & \xrightarrow{p} & D(f) \end{array}$$

where p denotes the restriction of the projection map $\text{pr} : (\mathbf{x}, t) \mapsto \mathbf{x}$. It is bijective; the inverse map is $\mathbf{x} \mapsto (\mathbf{x}, 1/f(\mathbf{x}))$. While all maps are continous with respect to the Zariski topology, p does even become a homeomorphism. Moreover, despite it is not a closed subset in k^m, this construction provides $k[\mathbf{x}, t]/(ft-1) = k[\mathbf{x}, 1/f(\mathbf{x})] \subseteq k(\mathbf{x})$ as the associated ring of regular functions.

1.4. **Prime avoiding and two radicals.** A ring is local $\Leftrightarrow$ the non-units form an ideal. "*Nil radical*": $\sqrt{(0)} = \bigcap\{\text{prime ideals}\}$ (*Proof*: $f \notin \sqrt{(0)} \Rightarrow 0 \notin \{f^{\mathbb{N}}\} =: S$, and use Zorn's lemma with ideals disjoint to S). "*Jacobson radical*": $\bigcap\{\text{maximal ideals}\} = \{a \in R \mid 1 + aR \subseteq R^*\}$.

Lemma 1. 1) *Prime ideal* $P \supseteq IJ \Leftrightarrow P \supseteq I \cap J \Leftrightarrow P \supseteq I$ *or* $P \supseteq J$.
2) $J \subseteq \bigcup_{i=1}^k P_i$ *(prime ideales with at most 2 exceptions)* $\Rightarrow \exists\, i \colon J \subseteq P_i$.

Proof. $J \not\subseteq P_i \Rightarrow$ induction yields $x_i \in J \setminus \bigcup_{j \neq i} P_j \rightsquigarrow x_i \in P_i$. For $k = 2$ consider $y := x_1 + x_2 \in J \setminus \bigcup_i P_i$; for $k \geq 3$ consider $y := x_1 + x_2 \cdot \ldots \cdot x_k$ if $P_1 =$ prime. $\square$

1.5. **Chinese Remainders.** The generalization of $\mathbb{Z}/(mn)\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is:

Proposition 2 (Chinese Remainder Theorem)**.** $I_1, \ldots, I_k \subseteq R$ *with* $I_i + I_j = (1)$ *for all* $i \neq j$. *Then,* $\prod_i I_i = \bigcap_i I_i$, *and* $\pi : R/\prod_i I_i \xrightarrow{\sim} \prod_i R/I_i$ *is an isomorphism.*

Proof. $k = 2$: $x_1 + x_2 = 1$ ($x_i \in I_i$) and $y \in I_1 \cap I_2$ yields $y = x_1 y + x_2 y \in I_1 I_2$. Moreover, $\pi(x_1) = (1, 0)$; $\pi(x_2) = (0, 1)$ imply the surjectivity of π.

10.11.21 (7) Induction: Since $x_i + x_k^{(i)} = 1$ (with $x_i \in I_i$, $x_k^{(i)} \in I_k$) yields $\prod_i x_i = \prod_i (1 - x_k^{(i)}) \in \left(\prod_{i=1}^{k-1} I_i\right) \cap (1 + I_k)$, we have $\left(\prod_{i=1}^{k-1} I_i\right) + I_k = (1)$. □

1.6. **The spectrum of a ring.** $\operatorname{Spec} R := \{P \subseteq R \mid \text{primie ideales}\} \supseteq \operatorname{MaxSpec} R$ is a topological space ("Zariski-*Topology*"): $V(J) := \{P \supseteq J\}$ are the closed subsets; $V(I) \cup V(J) = V(I \cap J) = V(IJ)$ and $\bigcap_{i \in I} V(J_i) = V(\sum_{i \in I} J_i)$.

$\operatorname{Spec} R$ is quasicompact: $\bigcap_{i \in I} V(J_i) = \emptyset \Leftrightarrow \sum_{i \in I} J_i \ni 1$. Basis of the open subsets via $D(f) := (\operatorname{Spec} R) \setminus V(f) = \{P \in \operatorname{Spec} R \mid f \notin P\}$; one has $D(f) \cap D(g) = D(fg)$.

Examples: $\mathbb{A}^n := \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$ and $\boxed{\operatorname{Spec} A \supseteq V(J) = \operatorname{Spec} A/J}$.

Subset $Z \subseteq \operatorname{Spec} R \rightsquigarrow$ reduced ideal $I(Z) := \bigcap_{P \in Z} P \subseteq R$. Properties: $I(\subseteq) = \supseteq$ and $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$. Moreover, $Z \subseteq V(I(Z)) =$ "algebraic closure" and $I(V(J)) = \bigcap_{P \supseteq J} P = \sqrt{J}$ (no HNS needed!). In particular, for $Z = V(J)$ algebraic: $Z = V(I(Z))$ as in (1.2).

1.7. **Affine schemes.** $k = \overline{k}$ as in (1.2) $\rightsquigarrow$ another form of HNS (7.3): Every maximal ideal of $k[\mathbf{x}]$ is of the form $\mathfrak{m}_p = (x_1 - p_1, \ldots, x_n - p_n)$ for some $p \in k^n$. Thus, $k^n \xrightarrow{\sim} \operatorname{MaxSpec} k[\mathbf{x}]$, $p \mapsto \mathfrak{m}_p$ is a homeomorphism. Moreover, $\operatorname{MaxSpec} k[\mathbf{x}] \subseteq \mathbb{A}^n_k$ is exactly the set of closed points.

Hence, in $X = \operatorname{Spec} R$, the ring R is considered the ring of regular functions on X: The value of $r \in R$ in $P \in X$ is $\overline{r} \in K(P) := \operatorname{Quot} R/P$ (Example: $K(\mathfrak{m}_p) = k[\mathbf{x}]/\mathfrak{m}_p = k$). In particular, $r \in R$ vanishes on $P \in X \Leftrightarrow r \in P$, and $r \in R$ vanishes on $Z \subseteq X \Leftrightarrow r \in P$ for all $P \in Z \Leftrightarrow r \in I(Z)$.

Regular maps in (1.2): Continuous $f : (Z \subseteq k^n) \to (Z' \subseteq k^{n'})$ such that $f^* : g \mapsto g \circ f$ induces a ring homomorphism $f^* : A(Z') \to A(Z)$ (equivalent: $f = (f_1, \ldots, f_{n'})$ with $k[\mathbf{x}] \twoheadrightarrow A(Z) \ni f_i$). The embedding $Z \hookrightarrow k^n$ corresponds to $k[\mathbf{x}] \twoheadrightarrow A(Z)$.

Ring homomorphisms $\varphi : R \to S \rightsquigarrow$ continuous $\varphi^\# : \operatorname{Spec} S \to \operatorname{Spec} R$; example: $\mathbb{C}[x, y]/(y^2 - x^3) = \mathbb{C}[t^2, t^3] \subseteq \mathbb{C}[t]$. "*Affine scheme*" $\operatorname{Spec} R := (\operatorname{Spec} R, R)$ with morphisms $\operatorname{Hom}_{\mathit{affSch}}(\operatorname{Spec} S, \operatorname{Spec} R) := \operatorname{Hom}_{\mathcal{R}ings}(R, S)$, cf. (19.3), (19.1), and Proposition 56.

2. R-modules, localization/factorization

2.1. **Basics of R-modules.** Operations $\oplus$, $\sum$, $\cap$, Hom, $\otimes$ of R-modules – the latter
17.11.21 (9) is defined via $\operatorname{Hom}_R(M \otimes_R N, P) = \operatorname{Hom}_R(M, N; P) := \{\text{bilinear maps } M \times N \to P\}$. If $M, N \subseteq L$ (e.g. $M, N =$ ideals), then $(M : N) := \{r \in R \mid rN \subseteq M\}$. This includes $(0 : N) = \operatorname{Ann}_R N$. Exact sequences; the 5-lemma.

2.2. **Testing exactness by applying the** Hom **functor.**

Lemma 3. *$M_\bullet = [0 \to M_1 \to M_2 \to M_3]$ is exact $\Leftrightarrow$ $\forall K$: $\mathrm{Hom}_R(K, M_\bullet)$ is exact. Similarily for $[M_1 \to M_2 \to M_3 \to 0]$ und $\mathrm{Hom}_R(\bullet, N)$. In particular, both* Hom *functors are left exact.* 24.11.21 (11)

Proof. Choose $K := R$ for the first claim and $N := \mathrm{coker}(M_2 \to M_3)$ and $N := \mathrm{coker}(M_1 \to M_2)$ for the second. □

For R-modules M, N, P we have $\mathrm{Hom}_R(M \otimes_R N,\, P) = \mathrm{Hom}_R(M,\, \mathrm{Hom}_R(N, P))$, i.e. $(\otimes_R N) \dashv \mathrm{Hom}(N, \bullet)$ ("*adjoint*"). The functor $(\otimes_R N)$ admits a right adjoint $\Rightarrow$ $(\otimes_R N)$ is right exact.

2.3. **Localization.**

$S \subseteq R$ is called *multiplicative closed* $:\Leftrightarrow 1 \in S$ and $S \cdot S \subseteq S$; *Localization* $S^{-1}M := \{m/s \mid m \in M, s \in S\}$ (with $m/s = m'/s' :\Leftrightarrow \exists t \in S: t(ms' - m's) = 0$) is $(S^{-1}R)$-module; $M \to S^{-1}M$ $(m \mapsto m/1)$ is injective $\Leftrightarrow$ S does not contain M-zero divisors.

Examples: $f \in R$, $S := \{f^{\mathbb{N}}\} \rightsquigarrow M_f$. Prime ideal $P \in \mathrm{Spec}\, R$, $S := R \setminus P \rightsquigarrow M_P$; this turns $\boxed{R_P}$ into a $\boxed{\text{local ring}}$ (via 2.5). "Total quotient ring": $S := \{\text{Non-zero divisors of } R\}$.

2.4. **Comparison with factorization.**

(LocFac1) $I \subseteq R$ ideal; $S \subseteq R$ multiplicative closed $\Rightarrow$ $R \to R/I$ is universal with $I \to 0$; $R \to S^{-1}R$ is universal with $S \to \{\text{units}\}$.

(LocFac2) (R/I)-modules are R-modules with $IM = 0$; $(S^{-1}R)$-modules are R-modules with $[S \to \mathrm{Aut}_R(M)] \subseteq [R \to \mathrm{End}_R(M)]$.

(LocFac3) $M \mapsto M/IM = M \otimes_R R/I$ is right exact; $M \mapsto S^{-1}M = M \otimes_R S^{-1}R$ is exact ($R \to S^{-1}R$ is $\boxed{\text{flat}}$).

2.5. **Behavior of ideals via** $[R \to S^{-1}R]$**.**

Let $I \subseteq R$, $J \subseteq S^{-1}R$ be ideals $\Rightarrow$ $I \cdot S^{-1}R = S^{-1}I$ with $S^{-1}I = R \Leftrightarrow I \cap S \neq \emptyset$. Moreover, $S^{-1}(J \cap R) = J$; $I \subseteq (S^{-1}I) \cap R$, but only for *prime* ideals $P \subseteq R \setminus S$ it holds true that $[a/s \in S^{-1}P \Rightarrow a \in P]$, hence $P = (S^{-1}P) \cap R$. This implies 1.12.21 (13)

(LocFac4) $\mathrm{Spec}\, S^{-1}R = \{P \in \mathrm{Spec}\, R \mid P \cap S = \emptyset\}$, in particular, $\mathrm{Spec}\, R_f = D(f) := \mathrm{Spec}\, R \setminus V(f) \subseteq \mathrm{Spec}\, R$ is an open subset. The set $\mathrm{Spec}\, R/I = V(I) \subseteq \mathrm{Spec}\, R$ is closed.

(LocFac4') For $P \in \mathrm{Spec}\, R$ we have: In R/P ideals $\boxed{\text{above } P}$ survive; in R_P ideals $\boxed{\text{below } P}$ survive.

(LocFac5) $S^{-1}(R/I) = S^{-1}R \otimes_R R/I = (S^{-1}R)/(S^{-1}I)$.

2.6. **Local tests.** Many properties of R-modules can be tested locally:

Proposition 4. *An R-linear map $f : M \to N$ is zero/surjective/injective/an isomorphism $\Leftrightarrow$ the same holds true for all $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ with $\mathfrak{m} \in \operatorname{MaxSpec} R$.*

Proof. $a \in M$ with $a/1 = 0$ in all $M_{\mathfrak{m}} \Rightarrow \forall \mathfrak{m}$: $\operatorname{Ann} a \not\subseteq \mathfrak{m} \Rightarrow \operatorname{Ann} a = R$, i.e. $a = 0$. In particular, $[\forall \mathfrak{m}$: $M_{\mathfrak{m}} = 0]$ implies $M = 0$. □

Corollary 5. *Exactness is a local property. M is R-flat $\Leftrightarrow \forall \mathfrak{m}$: $M_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-flat.*

2.7. **The Nakayama lemma.** Let M be a $\boxed{\text{finitely generated}}$ R-module.

Proposition 6 (Cayley-Hamilton)**.** *$I \subseteq R$ ideal, $\varphi : M \to IM \Rightarrow \exists p = \sum_j p_j x^{n-j} \in R[x]$: $p_0 = 1$, $p_j \in I^j$ and $p(\varphi) = 0$ in $\operatorname{End}_R(M)$.*

Proof. $m_1, \ldots, m_k \in M$ generators; $\varphi(m_i) = \sum_j a_{ij} m_j \Rightarrow (xI_k - A) \cdot \underline{m} = 0 \in M^k$ (M turns, via φ, into an $R[x]$-module). Multiplication with $\operatorname{adj}(xI_k - A) \rightsquigarrow p(x) := \det(xI_k - A)$ kills all m_i, thus M. □

Corollary 7. 1) *$M = IM \Rightarrow \exists\, p \in 1 + I \subseteq R$: $pM = 0$ ($1 + I \subseteq R^* \Rightarrow M = 0$).*
2) *$f : M \to M$ surjective $\Rightarrow f$ is an isomorphism.*
3) ("Nakayama-Lemma") *$(R, \mathfrak{m})$ local, $m_i \in M$ $\boxed{\text{generate } M/\mathfrak{m}M \Rightarrow \text{generate } M}$.*

Proof. (1) $\varphi := \mathrm{id}_M$; (2) $I := (x) \subseteq R[x] =: R$ with x acting as $f \Rightarrow p(x) = 1 + xq(x)$ kills M since (1), thus $f^{-1} = -q(f)$; (3) $N := \operatorname{span}_R\{m_i\} \Rightarrow$ apply (1) to M/N. □

Application: Minimal sets of generators, minimal resolutions for modules over local rings $(R, \mathfrak{m})$. If $F = R^s$, then $p : F \twoheadrightarrow M$ induces an isomorphism $\overline{p} : F/\mathfrak{m}F \xrightarrow{\sim} M/\mathfrak{m}M \Leftrightarrow \ker p \subseteq \mathfrak{m}F$.

8.12.21 (15)

2.8. **Support of modules.** M=R-module $\rightsquigarrow \operatorname{supp} M := \{P \in \operatorname{Spec} R \mid M_P \neq 0\}$ and, by abuse of notation, $\operatorname{supp} I := \operatorname{supp} R/I$.

- $0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow \operatorname{supp} M = (\operatorname{supp} M') \cup (\operatorname{supp} M'')$;
- M finitely generated $\Rightarrow (S^{-1}N : S^{-1}M) = S^{-1}(N : M) \Rightarrow \operatorname{supp} M = V(\operatorname{Ann} M)$ (via $(0 : M)_P \neq (1) \Leftrightarrow P \supseteq \operatorname{Ann} M$).

2.9. Hom **commutes with flat base change.** $R \to S$ algebra $\rightsquigarrow$ canonical S-linear map $\alpha_M : \operatorname{Hom}_R(M, N) \otimes_R S \to \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S)$.

Proposition 8. *$R \to S$ flat, M finitely presented $\Rightarrow \alpha_M$ is an isomorphism.* (Example: Localisations $R \to S^{-1}R$.)

Proof. $R^a \to R^b \to M \to 0 \Rightarrow$ w.l.o.g.: $M = R^n$. □

3. Notherian rings

3.1. **Chain conditions.** $(\Sigma, \leq)$ poset $\rightsquigarrow$ [strongly ascending chains do always terminate $\Leftrightarrow$ each subset of Σ has maximal elements].
(Examples: open subsets of topological spaces with $\subseteq$, submodules with $\subseteq/\supseteq$).

Definition 9. M is a *noetherian* R-module :$\Leftrightarrow$ each submodule is finitely generated $\Leftrightarrow$ $\Sigma :=$ {submodules} satisfies the *ascending* chain condition (ACC).

Lemma 10. $0 \to M' \to M \xrightarrow{\pi} M'' \to 0$ *exact* $\Rightarrow$ [M *noetherian* $\Leftrightarrow$ M', M'' *noetherian*]. (Special case: $M = M' \oplus M''$, thus finite direct sums.)

Proof. For ($\Leftarrow$) consider intersections with M' and images in M''; afterwards one uses: $N_1 \subseteq N_2 \subseteq M$ with $N_1 \cap M' = N_2 \cap M'$ and $\pi(N_1) = \pi(N_2) \Rightarrow N_1 = N_2$. (This follows from $0 \to N_i \cap M' \to N_i \to \pi(N_i) \to 0$ by using the 5-lemma.) □

R = "*noetherian ring*" :$\Leftrightarrow$ all ideals are finitely generated $\Leftrightarrow$ R is a noetherian R-module. If R is noetherian, then all finitely generated R-modules are noetherian, i.e. "f.g." is bequeathed to the submodules and implies "finitely presented".

3.2. **Hilbert's basis theorem.** The property "noetherian ring" is bequeathed as follows:

Proposition 11. 1) *R noetherian $\Rightarrow$ R/I and $S^{-1}R$ are noetherian.*
2) *R noetherian $\Rightarrow$ finitely generated R-algebras (as $R[x]$) are noetherian.*

Proof. $S^{-1}R$: For $J_i \subseteq S^{-1}R$ use $J_i = S^{-1}(J_i \cap R)$.
"Hilbert's basis theorem": R noetherian; $I \subseteq R[x]$ ideal $\rightsquigarrow$ let $I_0 \subseteq R$ be the ideal of the highest coefficients of polynomials from $I \Rightarrow I_0 = (a^1, \dots, a^k)$. Choose $f_i \in I$ with highest coefficient $a^i \rightsquigarrow I' := (f_1, \dots, f_k) \subseteq R[x]$. Defining $N := \max_i(\deg f_i)$ we conclude $I = I' + (\langle 1, x, \dots, x^{N-1}\rangle \cap I)$, and the second summand is a submodule of a finitely generated R-module. Thus, I is finitely generated. □

In particular, localizations of finitely generated $\mathbb{Z}$- or k-algebras are noetherian.

3.3. **An important filtration.** Let R be a noetherian ring and M a finitely generated R-module (*Example*: $M = k[\mathbf{x}]/$[monomial ideal]).

Proposition 12. *There is a finite* ("nice") *filtration $M = M_0 \supseteq \ldots \supseteq M_m = 0$ with factors $M_{k-1}/M_k \cong R/P_k$ for suitable (possibly equal) prime ideals $P_k \subseteq R$.*

Proof. Induction by #(generators of M) $\rightsquigarrow$ w.l.o.g. $M = R/I$. If I is not a prime ideal $\rightsquigarrow x, y \in R \setminus I$ with $xy \in I$. We obtain $I + (x) \supsetneq I$ and $I : (x) \supseteq I + (y) \supsetneq I$ and $0 \to R/(I : x) \xrightarrow{\cdot x} R/I \to R/[I + (x)] \to 0$. Because of "noetherian", these enlargements of I terminate. □

15.12.21 (17)

3.4. **Associated primes.** Let R and M be as in (3.3).
$\mathrm{Ass}(M) := \{P \in \operatorname{Spec} R \,|\, \exists\, R/P \hookrightarrow M\} = \{\mathrm{Ann}(m) \in \operatorname{Spec} R\}_{m\in M} \subseteq V(\mathrm{Ann}\, M)$.
In particular, using the notation of Proposition 12, $P_m \in \mathrm{Ass}(M) \rightsquigarrow \mathrm{Ass}(M) \neq \emptyset$.

Proposition 13. $0 \to M' \to M \to M'' \to 0$ *exact* $\Rightarrow \mathrm{Ass}(M') \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$. *In particular* (cf. Prop. 12), $\mathrm{Ass}(M) \subseteq \{P_1, \ldots, P_m\}$ *is finite.*

Proof. Let $P \in \mathrm{Ass}(M)\backslash\mathrm{Ass}(M') \Rightarrow R/P \hookrightarrow M \twoheadrightarrow M''$ with kernel $K := M' \cap R/P$. Since each $0 \neq a \in K$ would yield an $R/P \overset{a}{\hookrightarrow} K \subseteq M'$, we obtain $K = 0$. □

3.5. **Minimal primes.** Denote $\mathrm{Min}(M) := \{\text{minimal primes above } \mathrm{Ann}(M)\}$.

Lemma 14. *For each ideal I there exists a finite representation $\sqrt{I} = P_1 \cap \ldots \cap P_k$.*

Proof. If $\sqrt{I}$ is not prime, then choose $x, y \notin \sqrt{I} \ni xy \rightsquigarrow \sqrt{I} = \sqrt{I+(x)} \cap \sqrt{I+(y)}$: Assume $\sqrt{I} = 0$ (R is now reduced) and $a \in \sqrt{(x)} \cap \sqrt{(y)}$. Then $a^m \in (x)$ and $a^n \in (y)$, hence $a^{m+n} \in (xy) = 0$. Now do noetherian induction. □

Lemma 1 implies that unshortenable representations fulfill $\{P_1, \ldots, P_k\} = \mathrm{Min}(R/I)$ and, moreover, that each $P \in V(I) \subseteq \operatorname{Spec} R$ contains an element of $\mathrm{Min}(R/I)$.

Proposition 15. *Let R be a noetherian ring and M a finitely generated R-module.*
1) *For multiplicative closed $S \subseteq R$ we have* $\mathrm{Ass}(S^{-1}M) = \mathrm{Ass}(M) \cap \operatorname{Spec}(S^{-1}R)$.
2) $P \supseteq \mathrm{Ann}\, M$ *minimal prime above* $\mathrm{Ann}\, M \Rightarrow P \in \mathrm{Ass}(M)$.

Proof. (1) Let $F : S^{-1}R/S^{-1}P \hookrightarrow S^{-1}M$ be given by $1 \mapsto m/s \Rightarrow \exists t \in S$: $P \cdot tm = 0$. Then, $f : R/P \to M$, $1 \mapsto tm$ is well-defined, and $S^{-1}f \sim F$ is injective. Eventually, the injectivity of $R/P \hookrightarrow S^{-1}(R/P)$ implies this of f.
2) $P = \mathfrak{m}$ in a local ring $(R, \mathfrak{m}) \Rightarrow \emptyset \neq \mathrm{Ass}(M) \subseteq V(\mathrm{Ann}\, M) = \{\mathfrak{m}\}$. □

$$\boxed{\mathrm{Min}(M) \subseteq \mathrm{Ass}(M) \subseteq \{P_1, \ldots, P_m \text{ of Proposition 12}\} \subseteq \mathrm{supp}(M) = \overline{\mathrm{Min}(M)}.}$$

5.1.22 (19) 3.6. **Zero divisors.** Let R be noetherian and M a finitely generated R-module $\Rightarrow$ $\bigcup \mathrm{Ass}(M) = \{\text{zero divisors of } M\} \cup \{0\}$:
Let $r \in R$ be a zero divisor, i.e. $r \in \mathrm{Ann}(m) \neq (1)$ for some $m \in M$. If $\mathrm{Ann}(m)$ is not prime, then there are $x, y \in R$ with $xy \in \mathrm{Ann}(m)$, but $x, y \notin \mathrm{Ann}(m)$. Thus $\mathrm{Ann}(m) \subsetneq \mathrm{Ann}(xm) \neq (1) \rightsquigarrow$ Noether induction.

4. Modules of finite length and Artin rings

4.1. **Composition series.** R = ring, M = finitely generated R-module $\rightsquigarrow$ "*composition series*" (the factors are simple, i.e. isomorphic to $R/\mathfrak{m}$); $\ell(M) :=$ "*length* of (the shortest composition series of) M" $\leq \infty$.
Examples: 1) $(R, \mathfrak{m})$ local k-algebra with field extension $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ of degree $d \Rightarrow d \cdot \ell(M) = \dim_k M$.
2) $(R, \mathfrak{m})$ local with $\sqrt{0} = \mathfrak{m}$ ($\Leftrightarrow \operatorname{Spec} R = \{\mathfrak{m}\}$) $\Rightarrow \ell(M) < \infty$ (Proposition 12).

Proposition 16. $\boxed{\ell(\bullet) \text{ is additive}}$ *(in particular, strictly monotonic increasing), each filtration of an* R*-module* M *has length* $\leq \ell(M)$ *and (in case of* $\ell(M) < \infty$*) can be refined toward a composition series of* M*. The latter are characterized by* $[\ell(\text{factors}) = 1]$ *or by* $[\text{length} = \ell(M)]$.

Proof. $\ell(\bullet)$ is strictly monotonic increasing: $N \subsetneq M \Rightarrow$ each minimal composition series $\{M_j\}$ of M yields the N-filtration $\{N_j := M_j \cap N\}$ with $N_j/N_{j+1} \subseteq M_j/M_{j+1}$. Thus, for an *arbitrary filtration* $\{M_j\}$ of M on has $\ell(M_j) > \ell(M_{j+1})$, i.e. $\ell(M) \geq$ [length of the filtration]. □

4.2. **Artinian R-modules.** $:\Leftrightarrow$ {submodules} satisfies the $\boxed{\textit{descending}}$ chain condition (DCC); similarily: "*Artinian ring*"; Lemma 10 does still apply.

Examples: (0) $k[\varepsilon]/\varepsilon^2$. (1) $\mathbb{Z}$ is noetherian, but not artinan. (2) $A := \mathbb{Z}_p/\mathbb{Z}$ is an artinian, but not noetherian $\mathbb{Z}$-Modul: $\gcd(a,p) = 1 \Rightarrow a/p^n \sim 1/p^n$ $(ab + p^n c = 1$ implies $1/p^n = b \cdot a/p^n)$; hence $A_n := 1/p^n \cdot \mathbb{Z} \subseteq A$ are the only submodules at all. (3) $\mathbb{Z}_p$ satisfies neither (ACC)/(DCC).

4.3. **Artinian rings.** Despite (2) in (4.2), *rings* R satisfy:

Proposition 17. *R is* $\boxed{\textit{artinian} \Leftrightarrow \ell_R(R) < \infty}$ $\Leftrightarrow$ *R is noetherian with* $\operatorname{MaxSpec} R = \operatorname{Spec} R$*, i.e. every prime ideal is maximal. If so, then* $\operatorname{Spec} R$ *is a finite set.*

Proof. (i) "$\ell_R(R) < \infty$" implies "artinian" and "noetherian" via Proposition 16.
(ii) Let R be noetherian with $\ell_R(R) = \infty$; let $I \subseteq R$ be maximal with "$\ell_R(R/I) = \infty$" $\Rightarrow I$ is prime: $\nearrow$ proof of Proposition 12. On the other hand, since $\ell_R(R/I) = \infty$, the domain R/I is not a field.
(iii) Let R be artinian; let $J \subseteq R$ be the *smallest* ideal being the product of finitely many maximal ideals $\Rightarrow J^2 = J$ and $J = J\mathfrak{m} \subseteq \mathfrak{m}$ $(\forall \mathfrak{m} \in \operatorname{MaxSpec} R) \Rightarrow J = 0$ (Nakayama – if J is finitely generated).
WorkAround (if J is not finitely generated): Let I be the smallest ideal with $IJ \neq 0$ $\Rightarrow IJ = I$ (since $(IJ)J = IJ^2 = IJ \neq 0$), and there is an $f \in I$: $fJ \neq 0 \rightsquigarrow I = (f)$. Thus, I is finitely generated, hence Nakayama applies, hence $I = 0$ (↯).
(iv) $(0) = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k$ provides a filtration of R with its factors being the finite-dimensional (because of "artinian") $R/\mathfrak{m}_i$-vector spaces $\mathfrak{m}_1 \ldots \mathfrak{m}_{i-1}/\mathfrak{m}_1 \ldots \mathfrak{m}_i$.
(v) $P \in \operatorname{Spec} R \Rightarrow P \supseteq \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k \Rightarrow \exists i$: $P \supseteq \mathfrak{m}_i$. □

4.4. **Multiplicities.** M = finitely generated module over a noetherian ring R; let $P \supseteq \operatorname{Ann}(M)$ be minimal above $\operatorname{Ann}(M)$, i.e. $P \in \operatorname{Min}(M)$.

Proposition 18. *In each "nice" filtration of* M (according to Proposition 12) *the factor* $\boxed{R/P \textit{ appears exactly } \ell_{R_P}(M_P)\textit{-times.}}$ *In particular, this multiplicity* $(< \infty)$ *does not depend on the special choice of the filtration.*

Proof. [filtration] $\otimes_R R_P \rightsquigarrow$ factors R/Q with $Q \not\subseteq P$ disappear, and R/P becomes the field $R_P/PR_P = \operatorname{Quot}(R/P)$. □

5. Primary decomposition

12.1.22 (21) 5.1. ***P*-primary ideals.** R = noetherian. $Q \subseteq R$ *primary* $:\Leftrightarrow$ in R/Q the zero divisors are nilpotent.
$Q \subseteq R$ primary $\Rightarrow P := \sqrt{Q}$ is prime ("Q is P-primary") $\Rightarrow \exists\, n : P^n \subseteq Q \subseteq P$. An ideal Q with prime $P := \sqrt{Q}$ is (P-) primary $\Leftrightarrow \forall x, y \in R$: $[xy \in Q,\, x \notin Q \Rightarrow y \in P]$. Thus, intersections of P-primary ideales are P-primary.
Examples: 1) $Q = (x, y^2) \subseteq k[x, y]$ is P-primary with $P = (x, y)$; $P^2 \subseteq Q \subseteq P$.
2) $P := (x, z) \subseteq k[x, y, z]/(xy - z^2) \Rightarrow P^2$ is not primary(!)
$Q \subseteq R$ with $\boxed{\mathfrak{m} := \sqrt{Q} \text{ maximal ideal} \Rightarrow Q \text{ is } \mathfrak{m}\text{-primary}}$ ($\sqrt{Q} = \sqrt{0} \subseteq R/Q$ is then the only prime ideal, hence $\{R/Q - \text{zero divisors}\} = \bigcup \mathrm{Ass}(R/Q) = \sqrt{(0)}$).

5.2. **Existence.** R= noetherian $\rightsquigarrow$ every ideal $I \subseteq R$ is a finite intersection of $\cap$-irreducible ideals.

Lemma 19. *In noetherian rings, all $\cap$-irreducible ideals are primary.*

Proof. $\forall y \in R\ \exists k$: $\mathrm{Ann}(y^k) = \mathrm{Ann}(y^{k+1}) \Rightarrow \mathrm{Ann}(y) \cap (y^k) = (0)$. Hence, if (0) is irreducible, then $\mathrm{Ann}(y) \neq 0$ (i.e. y is a zero divisor) implies $y^k = 0$. □

In particular, all $I \subseteq R$ admit a primary decomposition $I = \bigcap_{i=1}^k Q_i$ which is minimal, i.e. unshortenable with mutually different radicals $P_i = \sqrt{Q_i}$. Example in [Eis, 3.8, S.103-105]: $(x) \cap (x^2, xy, y^2) = (x^2, xy) = (x) \cap (x^2, y)$.

5.3. **First uniqueness.** Let Q be P-primary; $x \in R \Rightarrow (Q : x) = (1)$ if $x \in Q$, and $(Q : x) = P$-primary otherwise (from $Q \subseteq (Q : x) \subseteq P$ one derives $\sqrt{(Q : x)} = P$).

Theorem 20. *$I = \bigcap_i Q_i$ minimal primary decomposition $\Rightarrow \{P_i := \sqrt{Q_i}\} = \mathrm{Ass}(R/I)$. In particular, we obtain $\sqrt{I} = \bigcap \mathrm{Ass}(R/I) = \bigcap \mathrm{Min}(R/I)$ again.*

Proof. $I = 0$. $x \in R \Rightarrow \sqrt{\mathrm{Ann}\, x} = \bigcap_i \sqrt{(Q_i : x)} = \bigcap_{x \notin Q_i} P_i$. If $\mathrm{Ann}\, x$ is prime, then so is $\sqrt{\mathrm{Ann}\, x}$, hence $\mathrm{Ann}\, x = \sqrt{\mathrm{Ann}\, x} = P_i$ for some i.
Conversely, if $0 \neq x \in I_i := \bigcap_{j \neq i} Q_j$, then $x \notin Q_i$ and $\sqrt{\mathrm{Ann}\, x} = P_i$. If $0 \neq x \in P_i^m I_i$ with $P_i^{m+1} I_i = 0$ (exists because of $P_i^{\gg 0} \subseteq Q_i$), then $P_i x = 0$, hence $P_i \subseteq \mathrm{Ann}\, x \subseteq \sqrt{\mathrm{Ann}\, x} = P_i$. □

In particular, primary ideals Q are alternatively characterized by $\#\,\mathrm{Ass}(R/Q) = 1$.

5.4. **Second uniqueness.** The primary Q_i partners of the associated $P_i \in \mathrm{Ass}(R/I)$ are not all uniquely determined, but:

Theorem 21. *For* $\boxed{\textit{minimal}}$ *$P_i \in \mathrm{Min}(R/I)$, the Q_i are uniquely determined by I.*

Proof. $\otimes_R R_{P_i}$ respects intersections (exact) and kills all Q_j with $P_j \not\subseteq P_i \Rightarrow I R_{P_i} = Q_i R_{P_i}$. On the other hand, for primary ideals, $Q_i R_{P_i} = Q'_i R_{P_i}$ implies $Q_i = Q'_i$. □

5.5. **Monomial ideals.** Generalizing the example in (5.2), let $I \subseteq k[x,y]$ be a monomial ideal $\rightsquigarrow S := \{a \in \mathbb{N}^2 \mid x^a \notin I\}$ "*standard monomials*" with $[S \ni a \geq b \in \mathbb{N}^2$ (i.e. $a - b \in \mathbb{N}^2) \Rightarrow b \in S]$; assume $S \neq \mathbb{N}^2$.

$S(1) := \{a \in S \mid a + (0 \times \mathbb{N}) \subseteq S\} = [0,\alpha] \times \mathbb{N}$ for some (maximal) $\alpha \in \mathbb{Z}_{\geq -1}$
$S(2) := \{a \in S \mid a + (\mathbb{N} \times 0) \subseteq S\} = \mathbb{N} \times [0,\beta]$ for some (maximal) $\beta \in \mathbb{Z}_{\geq -1}$
$S(12) := \overline{S \setminus (S(1) \cup S(2))}$ (closure with respect to "$\leq$") is finite.

$\Rightarrow S(1), S(2), S(12)$ correspond to ideals being (x)-, (y)- and, (x,y)-primary, and $S = S(1) \cup S(2) \cup S(12)$ yields a decomposition. Here, $S(12)$ could be replaced by each larger, "$\leq$"-closed, but still finite set.

6. Integral ring extensions

6.1. **Integral vs. finite.** $A \subseteq B$ rings: $x \in B$ is *integral* over $A :\Leftrightarrow x$ satisfies an equation $x^n + \sum_{v=0}^{n-1} a_v x^v = 0$ with $a_v \in A$; integral closure $=: \overline{A}^{(B)}$.

Examples: R factorial $\Rightarrow R$ is integrally closed in $\operatorname{Quot}(R)$ ("normal"); $w := (\sqrt{5}+1)/2$ satisfies $w^2 - w + 1 = 0$ (over $\mathbb{Z}$).

Proposition 22. *For $A \subseteq B \ni b$ the following facts are equivalent:*

(1) *b is integral over A,*
(2) *$B \supseteq A[b]$ is a finite A-algebra, i.e. finitely generated as an A-module,*
(3) *$\exists$ a finite A-algebra C: $A[b] \subseteq C \subseteq B$,*
(4) *$\exists\, B \supseteq A[b]$-module M: $\operatorname{Ann}_{A[b]} M = 0$, and M is finitely generated over A.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial; (4) $\Rightarrow$ (1) follows from Proposition 6: $\varphi := (\cdot\, b)$; $I = R = A$. □

Consequences: $b_i \in B$ are integral over $A \Leftrightarrow A[b_1, \ldots, b_k]$ is a finitely generated A-module; the A-integral elements of B form a subring; integrality of ring extensions is transitiv.

Integrality (and "integral closure") is a local property, i.e. b is integral over $A \Leftrightarrow$ it is integral over all A_P; A is normal $\Leftrightarrow$ all A_P are normal (even the $A_{\mathfrak{m}}$ suffice): For the first, lift from the A_P to A_{f_i} with $(f_1, \ldots, f_k) = (1)$. The normality statement follows from $A = \bigcap_{\mathfrak{m} \in \operatorname{MaxSpec} A} A_{\mathfrak{m}}$ (for $b \in \operatorname{Quot} A$ consider $\{a \in A \mid ab \in A\}$).

6.2. **Integrality over ideals.** $I \subseteq A$ ideal $\rightsquigarrow$ analogous notion "$b \in B \supseteq A$ is integral over I" via $b^n + \sum_{v=0}^{n-1} a_v b^v = 0$ with $a_v \in I$. We have $\overline{I}^{(B)} = \sqrt{I\,\overline{A}^B}$: If $b \in I\,\overline{A}^{(B)}$, thus $b = \sum_v a_v c_v$ with $a_v \in I$ and $c_v \in \overline{A}^{(B)}$, then $M := A[c_\bullet]$ is a finitely generated A-module. Now, one uses Proposition 6 with $\varphi := (\cdot\, b)$ and I.

Proposition 23. *$A \subseteq B$ domains with normal A. Then, $b \in B$ is integral over $I \subseteq A \Leftrightarrow b$ is algebraic over* $\operatorname{Quot} A$ *with minimal polynomial from $x^n + \sqrt{I}[x]_{<n}$.*

Proof. The coefficients of the minimal polynomial are from $\operatorname{Quot} A$. On the other hand, as symmetric functions in the roots ($\in \overline{\operatorname{Quot} A}$, integral over I) they are also integral over I. □

6.3. **Going up and down.** Let $A \subseteq B$ be an integral extension; denote $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$, $Q \mapsto Q \cap A$.

Proposition 24. (1) *If $A, B =$ domains, then [A is a field $\Leftrightarrow$ B is a field].*

(2) *$Q \in \operatorname{Spec} B$ is maximal $\Leftrightarrow$ $Q \cap A$ is maximal in A.*

(3) *φ is injective on chains of prime ideals of B, i.e. $Q_2 \subseteq Q_1$ together with $\varphi(Q_2) = \varphi(Q_1)$ implies $Q_2 = Q_1$.*

(4) *φ is surjective (on chains) – a successively increasing lifting is possible.*

(5) *A, B integral domains, $A =$ normal $\Rightarrow$ successively decreasing liftings are possible, too.*

Proof. (1) $\Rightarrow$ (2) via factorisation; (2) $\Rightarrow$ (3) via localization by $P := Q_i \cap A$.
(4) If $(A, \mathfrak{m})$ is local, then by (2) every maximal ideal in B is a preimage of $\mathfrak{m}$; localization $\rightsquigarrow$ general case.
(5) Let $P_2 \subseteq P_1 \subseteq A$ and $Q_1 \subseteq B$ with $P_1 = Q_1 \cap A$; we show that P_2 is the restriction of a prime ideal via $A_{P_1} \hookrightarrow B_{Q_1}$. *Problem*: This inclusion is not integral anymore – thus one has to check directly that $P_2 B_{Q_1} \cap A \subseteq P_2$ (and can, afterwards, choose a maximal ideal in $(A \setminus P_2)^{-1} B_{Q_1}$ over P_2): Let $A \ni x = y/s$ with $y \in P_2 B$ and $s \in B \setminus Q_1 \Rightarrow y$ is integral over P_2, i.e. it has over $\operatorname{Quot} A$ a minimal polynomial $y^n + a_1 y^{n-1} + \ldots + a_n = 0$ with $a_v \in P_2$. For s the minimal polynomial becomes $s^n + (a_1/x) s^{n-1} + \ldots + (a_n/x^n) = 0$; integrality $\Rightarrow a_v/x^v \in A$ with $x^v \cdot (a_v/x^v) \in P_2$. Finally, if $x \notin P_2$, then we would obtain $s^n \in P_2 B \subseteq Q_1$. □

19.1.22 (23)

6.4. **Finiteness of the normalization.** Integral closures of domains in fields are, under sufficiently good assumptions, finitely generated modules:

Proposition 25. *Let A be a domain and $L \supseteq \operatorname{Quot} A$ a finite field extension. If*

(i) *A is a finitely generated k-algebra* (with e.g. $L = \operatorname{Quot} A$), *or*

(ii) *A is noetherian, normal, and $L | \operatorname{Quot} A$ is separable,*

then $B := \overline{A}^{(L)}$ is a finitely generated A-module.

Proof. $A =$ finitely generated k-Algebra: See [ZS, ch. V, Th 9, S.267].
Normal/Separable: Let $K := \operatorname{Quot} A$ and $b_1, \ldots, b_m \in B$ a K-basis of $L = \operatorname{Quot} B = B \otimes_A K$ (the equality follows from $s \in L \Rightarrow \exists a \in A$: The minimal polynomial of s turns into an integrality relation of as). With $d := \det \operatorname{Tr}_{L|K}(b_i b_j) \in A \setminus \{0\}$ (separable!), the $\operatorname{Tr}_{L|K}(\bullet, \bullet)$-dual basis is some $b'_1, \ldots, b'_m \in \frac{1}{d} B$. For $b \in L$ it follows that $b = \sum_i \operatorname{Tr}_{L|K}(b b_i) b'_i$, and for $b \in B$, the coefficients stem from A. Hence, $B \subseteq \sum_i A b'_i$. □

7. The Hilbert Nullstellensatz

7.1. **The Weierstrass Preparation Theorem.** (Trivial form for polynomials) $\#k = \infty$, $f \in k[x_1, \ldots, x_n] \Rightarrow$ there is a linear change of coordinates $\psi : x_i \mapsto x_i + a_i x_n$ ($\mathbf{a} \in k^n$; $i = n$: $x_n \mapsto x_n$, but $a_n := 1$) with

$$\psi(f) = (\text{const} \neq 0) \cdot x_n^N + \sum_{i=0}^{N-1} c_i(x_1, \ldots, x_{n-1}) \cdot x_n^i \quad (\text{and } \deg c_i \leq N - i).$$

($N := \deg f \Rightarrow f \mapsto \psi(f)$ produces x_n^N with coefficients $\mathbf{a}^r$ for every monomial $\mathbf{x}^r$ of degree N. The entire coefficient of x_n^N in $\psi(f)$ is then $f_{[\deg = N]}(\mathbf{a})$; hence choose an $\mathbf{a} = (a_1, \ldots, a_{n-1}, 1) \in k^n$ with $f_{[\deg=N]}(\mathbf{a}) \neq 0$.)

Proposition 26 ("Noether-Normalization")**.** *$\#k = \infty$, $k[x_1, \ldots, x_n] \twoheadrightarrow A$ finitely generated k-algebra $\Rightarrow \exists y_1, \ldots, y_d \in \operatorname{span}_k(x_1, \ldots, x_n)$: $k[y_1, \ldots, y_d] \hookrightarrow A$ is integral.*

Proof. $k[x_1, \ldots, x_n] \to A$ finite, not injective $\Rightarrow f \in \ker$ has w.l.o.g. the above shape $\Rightarrow k[x_1, \ldots, x_{n-1}] \to k[x_1, \ldots, x_n]/f$ is finite. □

7.2. **The cool version of the HNS.**

Corollary 27 (HNS1)**.** *Let k be a field and A a finitely generated k-algebra being a field, too. Then, $A|k$ is a* finite field extension*, i.e. $[A : k] < \infty$. In particular, if $k = \overline{k}$, then this implies $A = k$.*

Proof. a) [$\#k = \infty$]: Proposition 26 $\Rightarrow k[y_1, \ldots, y_d] \hookrightarrow A$ is integral; then Proposition 24 implies that $k[y_1, \ldots, y_d]$ is a field $\Rightarrow d = 0$.
b) [Without Proposition 26]: Let $a_1, \ldots, a_n \in A$ be algebra generators. If $n = 0$, then we are done. We proceed by induction on n:
$k \hookrightarrow k[a_1] \hookrightarrow k(a_1) \hookrightarrow A \Rightarrow [A : k(a_1)] < \infty$. Let $f \in k[a_1]$ be a common denomiator of the integrality relations of the remaining $a_2, \ldots, a_n \Rightarrow A$ is integral over $k[a_1]_f \Rightarrow k[a_1]_f$ is a field, i.e. a_1 is not transzendental over k. □

7.3. **The standard version of the HNS.** Let $k = \overline{k}$ be an algebraically closed field.

Proposition 28 (HNS2)**.** *Let $k = \overline{k}$.*

1. *Every maximal ideal of $k[\mathbf{x}]$ is of the form $\mathfrak{m}_p = (x_1 - p_1, \ldots, x_n - p_n)$.*
2. *Let $J \subseteq k[\mathbf{x}]$ be an ideal with $V(J) = \emptyset$ in the sense of (1.2) $\Rightarrow J = (1)$.*
3. *$J \subseteq k[\mathbf{x}]$ ideal $\Rightarrow I(V(J)) = \sqrt{J}$ in the sense of (1.2).*

Proof. Corollary 27 $\Rightarrow$ (1) $\Rightarrow$ (2). (3): $f \in I(V(J)) \Rightarrow V(J, f(\mathbf{x})\, t - 1) = \emptyset \Rightarrow J + (ft - 1) = (1)$. Now, substitute $t \mapsto 1/f$ in the coefficients. □

7.4. **Algebraically not closed fields.** Example for $k \subset \overline{k}$: $J := (x^2 + 1) \subseteq \mathbb{R}[x]$. In (1.7) we have defined $f(P) \in K(P) := \operatorname{Quot}(R/P) = R_P/PR_P$ ("residue field" of P). *Example:* $R = k[\mathbf{x}] \Rightarrow x_i \in R$ yields $x_i(P) \in K(P) =$ "i-th coordinate". If $\mathfrak{m} := (x^2 + 1) \in \operatorname{Spec} \mathbb{R}[x] \Rightarrow K(\mathfrak{m}) = \mathbb{R}[x]/(x^2 + 1) = \mathbb{C}$, and $x(\mathfrak{m}) = \sqrt{-1}$.

8. Projective resolutions

26.1.22 (25) 8.1. **Projective modules.** $:\Leftrightarrow \operatorname{Hom}_R(P, \bullet)$ is exact $\Leftrightarrow$ all $M \twoheadrightarrow P$ split $\Leftrightarrow$ P is the direct $\boxed{\text{summand of a free } R\text{-module}}$ $R^I := R^{\oplus I}$ ($\operatorname{Hom}(P, \bullet)$ is then a summand of $\operatorname{Hom}(R^I, \bullet)$) $\Rightarrow$ P is flat (for the same reason with $P\otimes$ and $R^I\otimes$).

Base change (e.g. localization) preserves "projective" (R^I-summands); for P with finite presentation it holds true: P is projective $\Leftrightarrow \forall \mathfrak{m} \in \operatorname{MaxSpec} R$: $P_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module (for $M \twoheadrightarrow N$ localize $\operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N)$).

Example: $(2, 1 + \sqrt{-5}) \subseteq \mathbb{Z}[\sqrt{-5}]$ is projective, but not free; smooth points of an affine, elliptic curve yield those ideals, too.

$(R, \mathfrak{m})$ local, P = projective with finite presentation $\Rightarrow$ P is $\boxed{\text{(locally) free}}$: Let $R^n \twoheadrightarrow P$ be minimal and $R^n = P \oplus P' \Rightarrow P' \otimes R/\mathfrak{m} = 0$, hence $P' = 0$ (Nakayama).

8.2. **Complexes and Qis'.** $\mathcal{A}$ = abelian category (e.g. $\mathcal{M}od_R = \{R\text{-modules}\}$). complexes $M_\bullet$ (with $d_i : M_i \to M_{i-1}$, left shift $M[1]_i := M_{i-1}$, $M^i := M_{-i}$, hence $d^i : M^i \to M^{i+1}$ and $M[1]^i = M^{i+1}$; $d[1] := -d$); (co-)homlogy $\mathrm{H}_i(M_\bullet) := \mathrm{Z}_i(M_\bullet)/\,\mathrm{B}_i(M_\bullet)$ with $\mathrm{H}_i(M_\bullet) = \mathrm{H}_0(M_\bullet[-i])$; morphisms of complexes $f : M_\bullet \to N_\bullet$; the long exact homology sequence (is functorial); $\boxed{\text{"Qis":=“quasiisomorphisms”}}$ (*not* stable under the application of functors).

Example: $f : 0_\bullet \to M_\bullet := [0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0]$ exact $\Rightarrow$ f is Qis and $\mathrm{H}_\bullet(\mathrm{id}_M) = 0$. However, $N := \mathbb{Z}/2\mathbb{Z}$ yields $f \otimes \mathrm{id}_N \neq$ Qis and $\mathrm{H}_\bullet(\mathrm{id}_M \otimes \mathrm{id}_N) \neq 0$.

Double complexes $M_{\bullet\bullet}$ have differentials $d' : M_{i\bullet} \to M_{(i-1)\bullet}$ and $d'' : M_{\bullet j} \to M_{\bullet(j-1)}$ with $d'd'' + d''d' = 0$; the associated "total complex" is $\operatorname{Tot}_\bullet(M_{\bullet\bullet})$ with $\operatorname{Tot}_n := \oplus_{i+j=n} M_{ij}$ and $d := d' + d''$.

8.3. **Mapping cones.** $f : M_\bullet \to N_\bullet \rightsquigarrow$ *mapping cone* $\operatorname{Cone}(f)_\bullet := \operatorname{Tot}(M_\bullet \to N_\bullet)$ where the complexes $M_\bullet$ and $N_\bullet$ sit in row 1 and 0, respectively, with $d' := f$ and $d'' := (-d_M)/d_N$. Down to earth, this means that $\operatorname{Cone}(f)_\bullet := N_\bullet \oplus M_\bullet[1]$ with differential $d_{\mathrm{Cone}} := \begin{pmatrix} d_N & f \\ 0 & -d_M \end{pmatrix}$; in particular, $0 \to N_\bullet \to \operatorname{Cone}(f)_\bullet \to M_\bullet[1] \to 0$ is an exact sequence of complexes (where each layer separately splits); the connecting homomorphism equals $\mathrm{H}_\bullet(f)$. The complex $\operatorname{Cone}(f)$ is exact $\Leftrightarrow$ $f : M_\bullet \to N_\bullet$ is Qis.

Note: $M_\bullet[1] \hookrightarrow \operatorname{Cone}(f)$ and $\operatorname{Cone}(f) \twoheadrightarrow N_\bullet$ are *not* maps of complexes, i.e. they are not compatible with the respective differentials. In particular, the above sequence does not split as a sequence of complexes.

8.4. **Homotopies.** A *homotopy* $H : f \sim 0$ is a $H : M_\bullet \to N_\bullet[-1]$ (not compatible with d) with $Hd + dH = f$. Homotopies $H : 0 \sim 0$ are degree one morphisms of complexes.

$K^{(+/-/b)}(\mathcal{A}) :=$ homotopy category of bounded (from below, above, or both) $\mathcal{A}$-complexes with

$$\mathrm{Hom}_K(M,N) := \{\text{maps of complexes}\}/\text{homotopy} = \mathrm{H}_0\,\mathrm{Hom}_\bullet(M,N)$$

$\rightsquigarrow$ *homotopy equivalences* ($f: M_\bullet \to N_\bullet$ and $g: N_\bullet \to M_\bullet$ with $gf \sim \mathrm{id}_M$ and $fg \sim \mathrm{id}_N$) become isomorphisms in $K(\mathcal{A}) \rightsquigarrow$ Qis's:

Proposition 29. 1) $f \sim 0 \Rightarrow \mathrm{H}_\bullet(f) = 0$. *Thus,* $\mathrm{H}_0 : K(\mathcal{A}) \to \mathcal{A}$ *makes sense.*
2) *Let* $P_\bullet \in K^-(\mathrm{proj}\,\mathcal{A}) \subseteq K^-(\mathcal{A})$ *"projective" (i.e. all* P_i *are projective) and* $C_\bullet \in K(\mathcal{A})$ *exact* $\Rightarrow$ *every* $f : P_\bullet \to C_\bullet$ *is* 0*-homotopic.* □ 2.2.22 (27)

8.5. **The** Hom **complex.** Let $M_\bullet, N_\bullet \in K^b(\mathcal{A})$; then we define the double complex $\mathrm{Hom}_{\bullet\bullet}(M_\bullet, N_\bullet)$ via $\mathrm{Hom}_{ij} := \mathrm{Hom}(M_{-i}, N_j)$. The ordinary Hom complex is obtained as $\mathrm{Hom}_\bullet(M_\bullet, N_\bullet) := \mathrm{Tot}\,\mathrm{Hom}_{\bullet\bullet}(M_\bullet, N_\bullet)$, i.e. $\mathrm{Hom}_n(M_\bullet, N_\bullet) = \oplus_j \mathrm{Hom}(M_{j-n}, N_j)$ with $d(\varphi) = d_N\,\varphi - \varphi\,d_M$. In particular, $\mathrm{Z}_n(\mathrm{Hom}_\bullet)$ is the set of degree n homomorphisms of complexes.

For $f : M_\bullet \to N_\bullet$ and $A_\bullet \in K^b(\mathcal{A})$ the functor $\mathrm{Hom}_\bullet(A_\bullet, -)$ and the Cone construction commute; in particular, we obtain an exact sequence of complexes

$$0 \to \mathrm{Hom}_\bullet(A_\bullet, N_\bullet) \to \mathrm{Hom}_\bullet(A_\bullet, \mathrm{Cone}\,(f)_\bullet) \to \mathrm{Hom}_\bullet(A_\bullet, M_\bullet[1]) \to 0.$$

(Note that one has to be more careful with unbounded complexes; direct sums might be to replaced by direct products...)

8.6. **Projective resolutions become canonical.** Assume that the abelian category $\mathcal{A}$ has *enough projectives*, i.e. every object attracts a surjection from a projective one. Then, in $K^-(\mathcal{A})$ there exist unique and functorial projective resolutions (similar with injective resolutions in $K^+(\mathcal{A})$):

Proposition 30. 1) *Let* $P_\bullet \in K^-(\mathrm{proj}\,\mathcal{A})$ *be "projective" and* $A_\bullet \stackrel{q}{\to} B_\bullet$ *be a Qis in* $K(\mathcal{A}) \Rightarrow q$ *induces an isomorphism* $\mathrm{Hom}_{K(\mathcal{A})}(P_\bullet, A_\bullet) \stackrel{\sim}{\to} \mathrm{Hom}_{K(\mathcal{A})}(P_\bullet, B_\bullet)$.
2) *Each* $M_\bullet \in K^-(\mathcal{A})$ *admits a unique projective resolution* $P_\bullet \stackrel{\mathrm{qis}}{\to} M_\bullet$. *This construction yields a* *functor* $K^-(\mathcal{A}) \to K^-(\mathrm{proj}\,\mathcal{A})$ *transforming Qis' into isomorphisms*.

Proof. 1) Since q is a Qis, the complex $\mathrm{Cone}\,(q)$ is exact, i.e. for all $n \in \mathbb{Z}$ we have $\mathrm{H}_n(\mathrm{Hom}_\bullet(P_\bullet, \mathrm{Cone}\,(q)) = \mathrm{Hom}_{K(\mathcal{A})}(P_\bullet, \mathrm{Cone}\,(q)[n]) = 0$ by Proposition 29(2). Using the exact sequence of (8.5), this means that $\mathrm{Hom}_\bullet(P_\bullet, A_\bullet) \to \mathrm{Hom}_\bullet(P_\bullet, B_\bullet)$ is a qis.

2) Let $f_\bullet : P_\bullet \stackrel{\mathrm{qis}}{\to} M_\bullet$ for $< i$, and $f_i : P_i \to M_i$ inducing a surjective $\ker(P_i \to P_{i-1}) \twoheadrightarrow \ker(M_i \to M_{i-1})$. Then, one lifts $P'_{i+1} \twoheadrightarrow f_i^{-1}\big(\mathrm{im}(M_{i+1} \to M_i)\big) \cap Z_i(P_\bullet) \to \to \mathrm{im}(M_{i+1} \to M_i)$ toward M_{i+1}. Hence, $P_{i+1} := P'_{i+1} \oplus P''_{i+1}$ with surjective $P''_{i+1} \to \to \ker(M_{i+1} \to M_i)$.

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\text{qis}} & M_\bullet \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ P'_\bullet & \xrightarrow{\text{qis}} & M'_\bullet \end{array}$$

(i) For a given f and for given resolutions $P_\bullet \to M_\bullet$ and $P'_\bullet \to M'_\bullet$, there exists a unique F in $K^-(\operatorname{proj} \mathcal{A})$.

(ii) If $f = \mathrm{id}$ (or $f = \mathrm{qis}$) then F is a qis, too. Its inverse within $K^-(\mathcal{A})$ can be obtained via $\mathrm{Hom}_K(P', P) \xrightarrow{\sim} \mathrm{Hom}_K(P', P') \ni \mathrm{id}$.

Why is $G \mapsto \mathrm{id}_{P'}$ inverse to F? By definition, we know that $F \circ G = \mathrm{id}_{P'}$. In particular, G is a qis. This yields

$$\mathrm{Hom}_K(P,P) \underset{\sim}{\xrightarrow{F}} \mathrm{Hom}_K(P,P') \underset{\sim}{\xrightarrow{G}} \mathrm{Hom}_K(P,P) \underset{\sim}{\xrightarrow{F}} \mathrm{Hom}_K(P,P').$$
$$\underbrace{\qquad\qquad\qquad}_{\Phi} \qquad \underbrace{\qquad\qquad\qquad}_{\mathrm{id}_{P'}}$$

Thus, since we already know that the horizontal maps in the previous line are isomorphisms, $\mathrm{Hom}_K(G) = \mathrm{Hom}_K(F)^{-1}$, hence, for the map $\Phi : \mathrm{Hom}_K(P,P) \to \mathrm{Hom}_K(P,P)$ we get,

$$\mathrm{Hom}_K(GF) = \mathrm{Hom}_K(G) \circ \mathrm{Hom}_K(F) = \mathrm{id}\,.$$

These two incarnations of Φ, however, send id_P to $GF = \mathrm{id}_P$, respectively. □

9. Tor(sion) and Ext(ensions)

Every object $M \in \mathcal{A}$ gives rise to a complex supported on the 0-th spot only. Then, for a complex $P_\bullet = [\ldots P_2 \to P_1 \to P_0]$, a quasiisomorphism $P_\bullet \to M$ is equivalent to an exact sequence $\ldots P_2 \to P_1 \to P_0 \to M \to 0$.

9.1. **Derived functors.** Let $\mathcal{A}$ a be an abelian category with *enough projectives* and let $F : \mathcal{A} \to \mathcal{B}$ be an (additive) right exact functor, e.g. $F = (\otimes_R N) : \mathcal{M}od_R \to \mathcal{M}od_R$. Then, the *derived functors* $\mathrm{L}_i F : \mathcal{A} \to \mathcal{B}$ ($i \geq 0$) are charcterized by (i) $\boxed{\mathrm{L}_0 F = F}$, (ii) $\boxed{\mathrm{L}_{\geq 1} F(\text{projective}) = 0}$, and (iii) $[0 \to M' \to M \to M'' \to 0] \mapsto$ [natural transformation $\mathrm{L}_i F(M'') \to \mathrm{L}_{i-1} F(M')$ with $\boxed{\text{long exact homology sequence}}$]. In particular, $\mathrm{L}_{\geq 1}(\text{exact } F) = 0$.

Construction: $P_\bullet \to M$ projective resolution $\rightsquigarrow \mathrm{L}_i F(M) := \mathrm{H}_i\big(F(P_\bullet)\big)$.

(Proof of (iii): Projective resolutions $P'_\bullet \stackrel{\text{qis}}{\to} M'$ and $P_\bullet \stackrel{\text{qis}}{\to} M \rightsquigarrow f : P'_\bullet \to P_\bullet \rightsquigarrow \mathrm{Cone}\,(f) \stackrel{\text{qis}}{\to} \mathrm{Cone}\,(M' \to M) \stackrel{\text{qis}}{\to} M''$; now take the long exact homology sequence for $F\big(0 \to P_\bullet \to \mathrm{Cone}\,(f) \to P'_\bullet[1] \to 0\big)$.

The overall picture: $M_\bullet \in K^-(\mathcal{A})$ with projective resolution $K^-(\operatorname{proj} \mathcal{A}) \ni P_\bullet \xrightarrow{\text{qis}} M_\bullet \Rightarrow \mathbb{L}F(M_\bullet) := F(P_\bullet) \in K^-(\mathcal{B})$. There is a natural transformation $\mathbb{L}F \to F$, and $\mathbb{L}_i F M_\bullet := \mathrm{H}_i(\mathbb{L} F M_\bullet)$. If $f : M_\bullet \stackrel{\text{qis}}{\to} N_\bullet$ is a qis, then, in contrast to $F(f)$, the map $\mathbb{L}F(f)$ preserves this property. However, if F is exact, then $F(P_\bullet) \to F(M_\bullet)$ stays a qis, hence $\mathbb{L}F \to F$ is a qis, too.

9.2. Tor **and** Ext **as derived functors.** $\operatorname{Tor}^R(\bullet, N) := (\otimes^{\mathbb{L}}_R N)$, $\operatorname{Ext}_R(\bullet, N) := \mathbb{R}\operatorname{Hom}_R(\bullet, N)$; example: $R = \mathbb{Z}$; compatibility of Tor_i^R with flat base change $R \to S$
9.2.22 (29)
($P_\bullet \to M$ yields projective S-resolution $P_\bullet \otimes_R S \to M \otimes_R S$) – and similarly for Ext^i_R, if the P_i are of finite presentation. Moreover, one can choose the argument to resolve ($\rightsquigarrow$ usage of $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ for Ext):

Proposition 31. *Let $P_\bullet \xrightarrow{\text{qis}} M$, $Q_\bullet \xrightarrow{\text{qis}} N$, and $N \xrightarrow{\text{qis}} I^\bullet$ be projective and injective resolutions, respectively. Then,* $\operatorname{Tor}_i^R(M, N) = \operatorname{H}_i(P_\bullet \otimes_R N) = \operatorname{H}_i(M \otimes_R Q_\bullet)$ *and* $\operatorname{Ext}^i_R(M, N) = \operatorname{H}^i \operatorname{Hom}(P_\bullet, N) = \operatorname{H}^i \operatorname{Hom}(M, I^\bullet)$.

Proof. The first equalities are the definitions; for the second check the properties (i)-(iii) from (9.1). □

9.3. **Yoneda's Extensions.** $\operatorname{Ex}^1_R(M, N) := \{0 \to N \to \bullet \to M \to 0\}/\text{isom} \rightsquigarrow$ provides a bifunctor on $\mathcal{A}^{\text{opp}} \times \mathcal{A}$ ($m : M' \to M$ induces $0 \to N \to \bullet \times_M M' \to M' \to 0$; similarily for $n : N \to N'$) with R-algebra structure (addition via doubling the sequence and additional application of $M \to M \oplus M$ and $N \oplus N \to N$).

Proposition 32. $\operatorname{Ext}^1_R(M, N) \xrightarrow{\sim} \operatorname{Ex}^1_R(M, N)$ *as R-modules.*

Proof. $M \twoheadleftarrow P_0$ projective $\rightsquigarrow$ $(*)$ $0 \to K \to P_0 \to M \to 0$. With $\operatorname{Hom}(P_0, N) \to \operatorname{Hom}(K, N) \to \operatorname{Ext}^1_R(M, N) \to 0$ let $\operatorname{Hom}(K, N) \ni p \mapsto p_*(*)$. □

10. Flatness and Syzygies

10.1. [M projective $\Leftrightarrow \operatorname{Ext}^1_R(M, \bullet) = 0$] and [$N$ flat $\Leftrightarrow \operatorname{Tor}^R_1(\bullet, N) = 0$].

Proposition 33. *Let N be an R-module of finite presentation. Then, N is projective $\Leftrightarrow$ N is flat $\Leftrightarrow$ $\forall \mathfrak{m} \in \operatorname{MaxSpec} R$: $\operatorname{Tor}^R_1(R/\mathfrak{m}, N) = 0$.*

Proof. Projectivity can be checked locally, Tor^R_i commutes with localization $\rightsquigarrow$ w.l.o.g.. $(R, \mathfrak{m})$ is a local ring. Copy (8.1): $R^n \twoheadrightarrow N$ minimal $\rightsquigarrow$ Nakayama. □

If $0 \to K \to P_{n-1} \to \ldots \to P_0 \to N \to 0$ is exact (with projective P_i) $\Rightarrow$ $\operatorname{L}_i F(K) = \operatorname{L}_{i+n} F(N)$ for $i \geq 1$. In particular, it follows for finitely generated N over noetherian rings R: If $\operatorname{Tor}^R_{n+1}(R/\mathfrak{m}, N) = 0$ (for all $\mathfrak{m}$), then K is projective, i.e. $\operatorname{pd}(N) \leq n$.

Corollary 34 (Hilbert syzygy theorem)**.** *Every finitely generated $\mathbb{C}[x_1, \ldots, x_n]$-module has a projective* $\boxed{\textit{resolution of length } n}$*, i.e. its projective dimension is $\leq n$.*

Proof. The *Koszul complex* of (10.2) (e.g. $\mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]^2 \to \mathbb{C}[\mathbf{x}]$ for $n = 2$) provides a free resolution of length n of $\mathbb{C}[\mathbf{x}]/\mathfrak{m} \cong \mathbb{C}$. □

10.2. **The Koszul complex.** Over $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_n]$, we construct a free resolution of $\mathbb{C} = \mathbb{C}[\mathbf{x}]/(\mathbf{x})$: For $p \in \mathbb{N}$ let

$$K^p := \Lambda^p\, \mathbb{C}[\mathbf{x}]^n = \oplus_{1\le i_1<\ldots<i_p\le n}\, \mathbb{C}[\mathbf{x}] \cdot e_{i_1} \wedge \ldots \wedge e_{i_p} = \oplus_{\underline{i}}\, \mathbb{C}[\mathbf{x}] \cdot e(\underline{i})$$

and $d : K^p \to K^{p+1}$ be the wedge product $\wedge(\sum_{\nu=1}^n x_\nu e_\nu)$. The complex is $\mathbb{Z}^n$-graded by $\deg(\mathbf{x}^r \in \mathbb{C}[\mathbf{x}]) := r$ and $\deg e_i := -e_i$, i.e. $\deg(e(\underline{i})) = -\sum_{v=1}^p e_{i_v}$. Then, if $r_1, \ldots, r_\ell \ge 0$ and $r_{\ell+1} = \ldots = r_n = -1$, the degree r part of $K^\bullet$ equals $\mathbf{x}^r \cdot \boxed{\Lambda^{\bullet-n+\ell}\, \mathbb{C}^\ell} \otimes_{\mathbb{C}} \Lambda^{n-\ell}\, \mathbb{C}^{n-\ell}$ with $\mathbb{C}^\ell$-basis $f_\nu := x_\nu e_\nu$ and differential $d : \Lambda^p \mathbb{C}^\ell \to \Lambda^{p+1}\mathbb{C}^\ell$ equal to $\wedge(\sum_{\nu=1}^\ell f_\nu)$, for the first factor, and where the second factor $\Lambda^{n-\ell}\, \mathbb{C}^{n-\ell} = \mathbb{C} \cdot e(\ell+1, \ldots, n)$ does not matter at all. Ende Alg I

If $\ell \ge 1$, then $h : \Lambda^{p+1}\mathbb{C}^\ell \to \Lambda^p \mathbb{C}^\ell$ with $h(e_1 \wedge e_{i_1} \wedge \ldots \wedge e_{i_p}) := e_{i_1} \wedge \ldots \wedge e_{i_p}$ (and 0 otherwise) provides a homotopy id $\sim^h 0$. If $\ell = 0$ then $K^\bullet(r)$ is concentrated in $K^n(r) = \mathbb{C}$ and provides an isomorphism from this to $\mathbb{C} = \mathbb{C}[\mathbf{x}]/(\mathbf{x})$.

20.4.22 (31)

10.3. **No finite generation.** Flatness encodes "continuity" of families $\operatorname{Spec} S \to \operatorname{Spec} R$. (*Example*: Flat projection $R = \mathbb{C}[t] \hookrightarrow \mathbb{C}[x,t]/(x^2 - t) = S$ of the parabola and the non-flat projection $\mathbb{C}[t] \to \mathbb{C}[x,t]/(tx - t)$; comparison of the fibers in ± 1, 0 (and in the generic point η) in both cases – also over $\mathbb{R}$.) Higher dimension of the fibers $\rightsquigarrow$ the occuring modules (e.g. S over R) are no longer finitely generated!

Proposition 35. *Let N be an R-module. Then, N is flat $\Leftrightarrow$ $\operatorname{Tor}_1^R(R/I, N) = 0$ for all finitely generated ideals $I \subseteq R$*

Proof. ($\Leftarrow$) $M' \subseteq M \rightsquigarrow$ it suffices to test injectivity of $M' \otimes_R N \to M \otimes_R N$ only for finitely generated M', M: $x = \sum_i m_i \otimes n_i \in M' \otimes_R N \Rightarrow$ for $x \mapsto 0$ only finitely many bilinear relations in $M \otimes_R N$ are used. Thus, using filtrations, everything can be reduced to $I \subseteq R$, and again the finitely generated ideals suffice. □

Applications: 1) A $k[\varepsilon]/\varepsilon^2$-module N is flat $\Leftrightarrow$ $N/\varepsilon N \xrightarrow{\cdot\varepsilon} \varepsilon N$ is (also) injective. Identifying $k[\varepsilon]/\varepsilon^2$-modules N with pairs (V, φ) consisting of a k-vector space V and $\varphi \in \operatorname{End}_k(V)$ with $\varphi^2 = 0$, i.e. with $\operatorname{im}\varphi \subseteq \ker\varphi$, then (V, φ) is flat iff $\operatorname{im}\varphi = \ker\varphi$.
2) R = domain $\rightsquigarrow$ [flat $\Rightarrow$ torsion free]; R = principal ideal domain $\rightsquigarrow$ "$\Leftrightarrow$".
(Counter) *examples:* $\mathbb{Z}/2\mathbb{Z}$ is a flat (even projective) $\mathbb{Z}/6\mathbb{Z}$-module. The ideal $(x,y) \subset k[x,y]$ is torsion free, but not flat.

11. Graded rings and modules

11.1. **Graded rings and modules.** $\mathbb{Z}$ or more general abelian grading groups A; example $S = k[\mathbf{x}]$; homogeneous ideals and graded submodules; shifts $M(d)$ or $S(d)$; homogeneous resolutions.
Example: $(xz - y^2, wy - z^2, xw - yz)$, using $w(xz - y^2) + y(wy - z^2) + z(xw - yz) = 0$, with respect to the usual $\mathbb{Z}$-grading or to $\deg(x, y, z, w) := (1, i)$ with $i = 1, 2, 3, 4$.

27.4.22 (33) 11.2. **Homogenization.** $w \in \mathbb{R}^n_{\geq 0} \rightsquigarrow \deg_w x_i := w_i$ defines a grading on $k[\mathbf{x}]$; *homogenization*: $f \in k[\mathbf{x}] \rightsquigarrow k[t,\mathbf{x}] \ni f^h(t,\mathbf{x}) := t^{\deg_w f} f(t^{-w}\mathbf{x}) = \mathrm{in}_w f + t\cdot\text{remainder}$; with $\deg t := 1$ the f^h becomes homogeneous of degree $\deg_w f$; *dehomogenization* $f^h(1,\mathbf{x}) = f(\mathbf{x})$. For $a + \deg f = \deg g$ one has $t^a f^h + g^h = t^\bullet (f+g)^h$. This follows from $F(1,\mathbf{x})^h \cdot t^\bullet = F(t,\mathbf{x})$ for homogeneous $F(t,\mathbf{x})$.

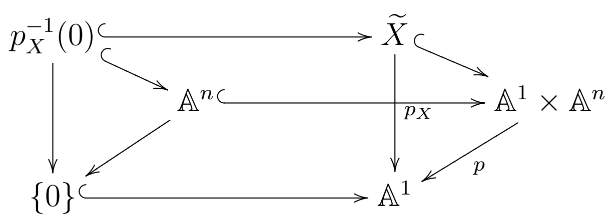
If $I \subseteq k[\mathbf{x}]$ is an ideal and $\leq_w$ is a term order breaking ties for $\deg_w \Rightarrow \mathrm{in}_w I$ is generated by $\mathrm{in}_w\{\leq_w\text{-GB of } I\}$; $I^h := (f^h \mid f \in I)$ is a homogeneous ideal; substituting $t \mapsto 1$ yields $I^h \mapsto I$.

Example: $w = \underline{1}$ and $I = (y - x^2, z - x^2)$ (GB for $y, z > x^2$ but not for $x^2 > y, z$; the latter requires $y - z$) yields $I^h = (yt - x^2, zt - x^2, y - z)$.

Lemma 36. *Let* $I = (f_1, \ldots, f_k)$*. Then* $I^h = ((f_1^h, \ldots, f_k^h) : t^\infty) = (I^h : t^\infty)$*. If* $\{f_1, \ldots, f_k\}$ =[$\leq_w$-Gröbner basis]*, then* $(f_1^h, \ldots, f_k^h)$ *is already* t*-saturated.*

Proof. $g(t,\mathbf{x})$ homogeneous with $t^\ell g \in I^h \Rightarrow g(1,\mathbf{x}) \in I \Rightarrow g(t,\mathbf{x}) = t^\bullet g(1,\mathbf{x})^h \in I^h$. Alternatively, $g(1,\mathbf{x}) = \sum_i \lambda_i(\mathbf{x}) f_i(\mathbf{x}) \Rightarrow \exists\, k, k_i \geq 0:\ t^k g(1,\mathbf{x})^h = \sum_i t^{k_i} \lambda_i^h f_i^h \Rightarrow g \in ((f_1^h, \ldots, f_k^h) : t^\infty)$. If $\{f_i\} = $ GB, then $\mathrm{in}_{\leq_w}(\lambda_i f_i) \leq \mathrm{in}_{\leq_w} g(1,\mathbf{x}) \Rightarrow \deg_w \lambda_i + \deg_w f_i \leq \deg_w g(1,\mathbf{x}) \Rightarrow k = 0$ is possible, i.e. $g \in (f_1^h, \ldots, f_k^h)$. $\square$

11.3. **Gröbner degenerations understood as flat families.** $w \in \mathbb{R}^n_{\geq 0} \rightsquigarrow X := \operatorname{Spec} k[\mathbf{x}]/I \subseteq \mathbb{A}^n$, $\widetilde{X} := \operatorname{Spec} k[t,\mathbf{x}]/I^h \subseteq \mathbb{A}^1 \times \mathbb{A}^n \xrightarrow{p} \mathbb{A}^1$



$\boxed{p_X \text{ is flat}}$ since $k[t,\mathbf{x}]/I^h$ is a flat $k[t]$-module $\Leftrightarrow$ t-torsion free $\Leftrightarrow I^h = (I^h : t^\infty)$ and $p_X^{-1}(\mathbb{A}^1 \setminus 0) \cong X \times (\mathbb{A}^1 \setminus 0)$ via the $k[t^{\pm 1}]$-linear $\begin{array}{ccc} k[t^{\pm1},\mathbf{x}]/I & \xrightarrow{\sim} & k[t^{\pm1},\mathbf{x}]/I^h \\ \mathbf{x}, f & \mapsto & t^{-w}\mathbf{x},\ t^{-\deg f} f^h; \end{array}$
$p_X^{-1}(0) = \operatorname{Spec} k[t,\mathbf{x}]/((t) + I^h) = \operatorname{Spec} k[\mathbf{x}]/\mathrm{in}_w(I)$.

11.4. **Limits.** The punctured $p_X^{-1}(\mathbb{A}^1 \setminus 0) \to (\mathbb{A}^1 \setminus 0)$ is a trivial family. Moreover, by Problem 57(c), $\overline{p_X^{-1}(\mathbb{A}^1 \setminus 0)} = \overline{V(I^h) \setminus V(t)} = V(I^h : t^\infty) = V(I^h)$; hence $X_0 := p_X^{-1}(0) = \boxed{\text{“}\lim_{t\to 0}\text{”}}\, p_X^{-1}(t)$.

$I = (f_1, \ldots, f_k)$ Gröbner basis $\Rightarrow X = V(f_i)$, $\widetilde{X} = V(f_i^h)$ and $X_0 = V(\mathrm{in}_w f_i)$. For non-GB we just have $X_0 \subseteq V(\mathrm{in}_w f_i)$.

Example: $I = (x - z, y - z)$, $w = (0,0,1) \Rightarrow V(tx - z, ty - z) = k \cdot (1,1,t)$ over $\mathbb{A}^1 \setminus 0$, but has the 0-fiber $V(z)$ which is bigger than the wanted $V(z, x - y)$.

Different term orders yield different degenerations: See [Eis, S.342-347]. This motivates the usage of non-reduced, 0-dimensional schemes.

11.5. **Artin-Rees.** A noetherian; $I \subseteq A$ ideal $\rightsquigarrow \widetilde{A} := \oplus_{\nu \geq 0} I^\nu$ is a finitely generated A-algebra $\Rightarrow$ noetherian, too. M = finitely generated A-module with "*I-filtration*", i.e. $\{M_\nu\}_{\nu \geq 0}$ with $IM_\nu \subseteq M_{\nu+1} \subseteq M_\nu$ (Example: $M_\nu = I^\nu M$) $\rightsquigarrow \widetilde{M} := \oplus_{\nu \geq 0} M_\nu$ is a graded $\widetilde{A}$-module.

Proposition 37. *$\widetilde{M}$ is noetherian $\Leftrightarrow M_{\nu+1} = IM_\nu$ for $\nu \gg 0$ ("I-stable").*

Proof. ($\Rightarrow$) $M^k := (\oplus_{\nu \leq k} M_\nu) \oplus (\oplus_{\nu \geq 1} I^\nu M_k)$ is an ascending chain in $\widetilde{M}$. □

Corollary 38. *(1) $M' \subseteq M \Rightarrow I(I^\nu M \cap M') = I^{\nu+1} M \cap M'$ for $\nu \gg 0$, i.e. $\exists c$: $I^k M' \supseteq I^k(I^c M \cap M') = I^{k+c} M \cap M' \supseteq I^{k+c} M'$ for $k \geq 0$ ("Artin-Rees").*
(2) $1 + I \subseteq A^*$ (e.g. $I = \mathfrak{m}$ in a local ring) $\Rightarrow \bigcap_{k \geq 0} I^k M = 0$.

Proof. (1) $\widetilde{M'} = \oplus_\nu (I^\nu M \cap M') \subseteq \oplus_\nu I^\nu M = \widetilde{M}$ is a noetherian $\widetilde{A}$-module.
(2) follows from (1) with $M' := \bigcap_k I^k M$ and Nakayama. □

11.6. **The local criterion of flatness.** A homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is called *local* $:\Leftrightarrow \varphi(\mathfrak{m}) \subseteq \mathfrak{n} \Leftrightarrow \varphi^\#(\mathfrak{n}) = \mathfrak{m}$. Counter *example:* $\mathbb{C}[x]_{(x)} \hookrightarrow \mathbb{C}(x)$.

Proposition 39 (Local criterion of flatness)**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphismus of noetherian rings, let N be a finitely generated S-module. Then N is flat over $R \Leftrightarrow \operatorname{Tor}_1^R(R/\mathfrak{m}, N) = 0$.*

Proof. Let $I \subseteq R$ be an ideal and $I \otimes_R N \ni u \mapsto 0 \in IN \subseteq N$; we show that $u = 0$: $I \otimes_R N$ is a finite S-Modul, and $\mathfrak{m}^a(I \otimes_R N) \subseteq \mathfrak{n}^a(I \otimes_R N) \Rightarrow \bigcap_a \mathfrak{m}^a(I \otimes_R N) = 0$; Artin-Rees $\rightsquigarrow \mathfrak{m}^{a' \gg a} \cap I \subseteq \mathfrak{m}^a I \Rightarrow$ it suffices to show that u is contained, for all $a' \in \mathbb{N}$, in the image of $(\mathfrak{m}^{a'} \cap I) \otimes_R N \to I \otimes_R N$ i.e. that u vanishes in $I/(\mathfrak{m}^{a'} \cap I) \otimes_R N$.

$$\begin{array}{ccccc} u & & I \otimes_R N & \longrightarrow & I/(\mathfrak{m}^{a'} \cap I) \otimes_R N \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & N = R \otimes_R N & \longrightarrow & R/\mathfrak{m}^{a'} \otimes_R N \end{array}$$

On the other hand, the right hand column is injective since $\operatorname{Tor}_1^R(M, N) = 0$ for all R-modules M of finite length – this follows via induction from the hypothesis. □

12. Hilbert polynomials

20.7.22 (53)

12.1. **Poincaré series.** S_0 noetherian ring; λ: {finitely generated S_0-modules} $\to \mathbb{N}$ additive. $S = \oplus_{\nu \geq 0} S_\nu$ finitely generated, graded S_0-algebra: $a_1, \ldots, a_n$ homogeneous generators with $\deg a_i = d_i \geq 1$. If M = finitely generated, graded S-module $\Rightarrow$ "*Poincaré series*" $P(M, t) := \sum_{\nu \geq 0} \lambda(M_\nu) \cdot t^\nu \in \mathbb{N}[[t]]$ (cut off the negative part).

Theorem 40 (Hilbert-Serre)**.** $\prod_{i=1}^n (1 - t^{d_i}) \cdot P(M, t) \in \mathbb{Z}[t]$.

Proof. $n = 0 \Rightarrow P(M,t) \in \mathbb{N}[t]$. In general: $K, L :=$ kernel/cokernel of $M \xrightarrow{\cdot a_n} M$ $\Rightarrow \lambda(K_\nu) - \lambda(M_\nu) + \lambda(M_{\nu+d_n}) - \lambda(L_{\nu+d_n}) = 0$, hence

$$t^{d_n} P(K,t) - t^{d_n} P(M,t) + P(M,t) - P(L,t) = \sum_{v=0}^{d_n-1} (\lambda(M_v) - \lambda(L_v))t^v =: g \in \mathbb{N}[t]$$

$\Rightarrow (1 - t^{d_n})P(M,t) = P(L,t) - t^{d_n} P(K,t) + g(t)$. And, since a_n annhilates the modules K, L, they are modules over $S_0[a_1, \ldots, a_{n-1}] \subseteq S$. □

12.2. **Pole orders.** $d(M) :=$ [pole order of $P(M,t)$ in $t = 1$] $\leq n$. On the other hand, $d(M) \leq 0$ indicates that M does λ-live only in finitely many degrees: $P(M,t) \cdot \prod_i (\sum_{v=0}^{d_i-1} t^v) \in (1-t)^{-d(M)}\mathbb{Z}[t] \subseteq \mathbb{Z}[t]$ enforces that $P(M,t) \in \mathbb{N}[t]$. From $d(M) < 0$ it even follows that $P(M,t) = 0$.

Example: $P(k[x_1, \ldots, x_n], t) = \sum_\nu \binom{\nu+n-1}{n-1} t^\nu = 1/(1-t)^n$ (this easily follows via the $\mathbb{Z}^n$ grading and $\sum_{r \in \mathbb{N}^n} t^r = \prod_i \sum_{k\geq 0} t_i^k) \Rightarrow d(k[x_1, \ldots, x_n]) = n$.

Proposition 41. *If $a \in S$ is a non-zero divisor of M with $\deg a \geq 1$, then $d(M/aM) = d(M) - 1$.*

Proof. $M \xrightarrow{\cdot a} M$ has $K = 0$, hence $(1 - t^{\deg a})P(M,t) = P(M/aM,t) + g(t)$ with $g \in \mathbb{N}[t]$. In the case $d(M/aM) = 0$ it first follows that $d(M) \leq 1$ and $P(M/aM,t) + g(t) \in \mathbb{N}[t]$. However, with $d(M) = 0$ one would additionally obtain that $P(M/aM,1) + g(1) = 0$. □

12.3. **Numerical polynomials.** The coefficients $\lambda(M_\nu)$ of $P(M,t)$ themselves behave like polynomials in ν ("*Hilbert polynomial*"); $f \in \mathbb{R}[t]$ is called a *numerical polynomial* $:\Leftrightarrow f(g) \in \mathbb{Z}$ for sufficiently large $g \in \mathbb{Z} \Leftrightarrow f = \sum_{i=0}^{\deg f} c_i \binom{t}{i}$ with (uniquely determined) $c_i \in \mathbb{Z}$.
(($\Rightarrow$) via induction by $\deg f$: $g(t) := f(t+1) - f(t) = \sum_{i=0}^{\deg f - 1} c_{i+1}\binom{t}{i}$.)

Proposition 42. *Let S be generated in degree 1 over S_0 $(d_i = 1) \Rightarrow$ for $\nu \gg 0$, one has $[\nu \mapsto \lambda(M_\nu)] = H_M(\nu) \in \mathbb{Q}[\nu]$ with $\deg H_M = d(M) - 1$.*

Proof. $P(M,t) = f(t)/(1-t)^{d(M)} = f(t) \cdot \sum_{k\geq 0} \binom{d+k-1}{d-1} t^k \Rightarrow$ with $f(t) = \sum_{k=0}^N a_k t^k$ we have $\lambda(M_\nu) = \sum_{k=0}^N a_k \binom{d+\nu-k-1}{d-1}$ for $\nu \geq N$. Since $\sum_k a_k = f(1) \neq 0$, the coefficients of v^{d-1} do not cancel each other. □

Example: $H_{k[x_0,\ldots,x_n]}(v) = \binom{v+n}{n} = 1/n!\, v^n + \ldots$ and, for a homogeneous $f \in k[\mathbf{x}]_d$, $H_{k[\mathbf{x}]/f}(v) = \binom{v+n}{n} - \binom{v+n-d}{n} = d/(n-1)!\, v^{n-1} + \ldots$. In particular, for $S = k[\mathbf{x}]/I$, the *degree* $\deg(S) := \deg(H_S)! \cdot$ [leading coefficient of H_S] generalizes the degree of a polynomial.

13. Dimension of local rings

13.1. **$\mathfrak{m}$-primary ideals.** $(A, \mathfrak{m})$ noetherian local ring, $\mathfrak{m}^r \subseteq Q \subseteq \mathfrak{m}$ ($\mathfrak{m}$-primary) ideal $\Rightarrow S := \mathrm{Gr}_Q(A) := \oplus_{\nu\geq 0} Q^\nu/Q^{\nu+1}$ with $S_0 = A/Q$ (artinian) and $\lambda := \ell =$ length.

Proposition 43. *M finitely generated A-module $\Rightarrow \nu \mapsto g(\nu) := \ell(M/Q^\nu M) < \infty$ equals a polynomial $\chi_Q^M \in \sum_{i=0}^n \mathbb{Z}\binom{\nu}{i}$ of degree $d(\mathrm{Gr}_Q(M)) \le n := \#\{Q\text{-generators}\}$ for $\nu \gg 0$.*

Proof. $\mathrm{Gr}_Q(M) := \oplus_{\nu\ge 0} Q^\nu M/Q^{\nu+1}M$ is a finitely and in deg $= 1$ generated $\mathrm{Gr}_Q(A)$-module; Proposition 42 $\rightsquigarrow g(\nu+1) - g(\nu) = \ell(\mathrm{Gr}_Q^\nu(M))$ is a polynomial of degree $< n$. □

$\boxed{d(A) := \deg \chi_Q^A} = d(\mathrm{Gr}_Q(A)) - 1 + 1$ does not depend on Q: $\mathfrak{m}^r \subseteq Q \subseteq \mathfrak{m} \Rightarrow \chi_{\mathfrak{m}}^A(\nu) \le \chi_Q^A(\nu) \le \chi_{\mathfrak{m}}^A(r\nu)$ for $\nu \gg 0$. Hence $\boxed{d(A) \le \delta(A) := \min_Q \#\{Q\text{-generators}\}}$.

Example: $A := k[x_1,\dots,x_n]_{(\mathbf{x})} \hookrightarrow k[|\mathbf{x}|] =: \hat{A}$ have both $\mathrm{Gr}_{(\mathbf{x})}(A) = \mathrm{Gr}_{(\mathbf{x})}(\hat{A}) = k[\mathbf{x}]$. Hence, $\chi_{(\mathbf{x})}(k) = \binom{k-1+n}{n}$; indeed, $H_{\mathrm{Gr}}(k) = \chi(k+1) - \chi(k) = \binom{k+n}{n} - \binom{k-1+n}{n} = \binom{k+n-1}{n-1}$. Thus, $d(\mathbb{A}^n, 0) = n$ and $\mathrm{mult}(\mathbb{A}^n, 0) = 1$ with $\mathrm{mult}(A) := d(A)! \cdot [\text{leading coefficient of } \chi_{\mathfrak{m}}^A]$.

13.2. **Hypersurfaces.** Let $a \in (A, \mathfrak{m})$ be a non-zero divisor for M. Comparable to Proposition 41 we obtain:

Proposition 44. $\deg \chi_{\mathfrak{m}}^{M/aM} \le \deg \chi_{\mathfrak{m}}^M - 1$*; in particular $d(A/aA) \le d(A) - 1$ for $M := A$.*

Proof. $M/\mathfrak{m}^\nu M \twoheadrightarrow M/(a + \mathfrak{m}^\nu M)$ yields $\chi_{\mathfrak{m}}^M(\nu) - \chi_{\mathfrak{m}}^{M/aM}(\nu) = \ell(aM/(aM \cap \mathfrak{m}^\nu M))$, and $\mathfrak{m}^\nu(aM) \subseteq aM \cap \mathfrak{m}^\nu M \overset{\mathrm{Cor38}}{=} \mathfrak{m}^{\nu-\nu_0}(aM \cap \mathfrak{m}^{\nu_0} M) \subseteq \mathfrak{m}^{\nu-\nu_0}(aM)$ implies $\chi_{\mathfrak{m}}(\nu - \nu_0) \le \ell(\dots) \le \chi_{\mathfrak{m}}(\nu)$. Hence $\chi_{\mathfrak{m}}^M(\nu) - \chi_{\mathfrak{m}}^{M/aM}(\nu)$ is a polynomial of the same degree and with the same leading coefficient as $\chi_{\mathfrak{m}}^M$. □

Example: $f \in \mathfrak{m}^d \setminus \mathfrak{m}^{d+1}$ in A, then $\mathrm{in}(f) := \bar{f} \in \mathfrak{m}^d/\mathfrak{m}^{d+1} = \mathrm{Gr}_{\mathfrak{m}}^d(A)$; in particular, there is a natural surjection $\Phi : \mathrm{Gr}_{\mathfrak{m}}(A)/\mathrm{in}(f) \twoheadrightarrow \mathrm{Gr}_{\mathfrak{m}}(A/f)$. If $\mathrm{in}(f)$ is a non-zero divisor in $\mathrm{Gr}_{\mathfrak{m}}(A)$, then Φ is an isomorphism.
Hence, for $k[x_1,\dots,x_n]_{(\mathbf{x})}/(f) \subseteq k[|\mathbf{x}|]/f$ (with $f = f_d + f_{d+1} + \dots$ in the latter) we obtain $\chi(k) = \chi^{(\mathbb{A}^n,0)}(k) - \chi^{(\mathbb{A}^n,0)}(k-d) = \binom{k-1+n}{n} - \binom{k-1-d+n}{n}$. In particular, $d(k[\mathbf{x}]_{(\mathbf{x})}/(f) = n - 1$ and $\mathrm{mult}(k[\mathbf{x}]_{(\mathbf{x})}/(f) = d$.

13.3. **Towers of primes.** "*Height*" of prime ideals $\rightsquigarrow$ "*Krull dimension*" $\dim A := \dim(\operatorname{Spec} A) := \max\{\operatorname{ht} P \mid P \in \operatorname{Spec} A\}$; $P \subseteq A$ is a *minimal* prime ideal $\Leftrightarrow \operatorname{ht} P = 0$.

Proposition 45. *$(A, \mathfrak{m})$ noetherian local ring $\Rightarrow$* $\boxed{\operatorname{ht}\mathfrak{m} =: \dim A \le d(A)}$. *In particular, the height of prime ideals in noetherian local rings is always finite.*

Proof. $d(A) = 0 \Rightarrow \chi_{\mathfrak{m}}^A(\nu) = \ell(A/\mathfrak{m}^\nu)$ constant $\Rightarrow \mathfrak{m}^\nu = 0$ (Nakayama) for $\nu \gg 0$.
Induction by $d(A)$: $P_0 \subset \dots \subset P_r$ chain of prime ideals, $a \in P_1 \setminus P_0 \Rightarrow \overline{A} := A/P_0 + (a)$ does still contain the chain $\overline{P}_1 \subset \dots \subset \overline{P}_r$, and $d(\overline{A}) < d(A/P_0) \le d(A)$. □

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Theorem 46. *$(A,\mathfrak{m})$ noetherian local ring* $\Rightarrow$ $\boxed{\dim(A)=d(A)=\delta(A)}$. *For non-zero divisors $a\in A$ one has* $\dim A/a = \dim A - 1$.

Proof. For $v\le \dim A$ construct inductively $(a_1,\ldots,a_\nu)$ with $[P\supseteq (a_1,\ldots,a_\nu)\Rightarrow \operatorname{ht} P\ge \nu]$: If $P_1,\ldots,P_N$ are the minimal primes over $(a_1,\ldots,a_{\nu-1})$ with $\operatorname{ht} P_i = \nu-1<\dim A \Rightarrow P_i\subset \mathfrak{m}\Rightarrow \exists a_\nu\in\mathfrak{m}\setminus\bigcup_i P_i$.
For $\nu=\dim A$ it follows that $Q:=(a_1,\ldots,a_{\dim A})$ is $\mathfrak{m}$-primary $\rightsquigarrow \delta(A)\le\dim A$.
"$\ge$" (holding without the non-zero divisor assumption): $(\overline{a_1},\ldots,\overline{a_d})=\overline{\mathfrak{m}}$-primary in $A/aA\Rightarrow (a,a_1,\ldots,a_d)=\mathfrak{m}$-primary in A. □

14. Regular local rings

14.1. **Tangent cones.** $\dim(A,\mathfrak{m})=d\rightsquigarrow Q=(a_1,\ldots,a_d)$ $\mathfrak{m}$-primary ("*parameter system*") $\rightsquigarrow \Phi:(A/Q)[x_1,\ldots,x_d]\twoheadrightarrow \operatorname{Gr}_Q A$. It holds true: $\Phi(f)=0\Rightarrow f\mapsto 0\in (A/\mathfrak{m})[x_1,\ldots,x_d]$. (Otherwise, by Problem **??**, the (homogeneous) f is a non-zero divisor, hence

$$d=d(\operatorname{Gr}_Q A)\le d((A/Q)[x_1,\ldots,x_d]/f)<d((A/Q)[x_1,\ldots,x_d])=d.)$$

If $Q=\mathfrak{m}$ is possible, then Φ becomes an isomorphism!

Definition 47. $(A,\mathfrak{m})$ is "*regular*" $:\Leftrightarrow \operatorname{Gr}_{\mathfrak{m}}(A)$ is a polynomial ring $\Leftrightarrow$ $\boxed{\dim\mathfrak{m}/\mathfrak{m}^2}=\dim A \overset{\text{Nakayama}}{\Longleftrightarrow}$ $\mathfrak{m}$ is generated by $(\dim A)$ many elements.
(If $\operatorname{Gr}_{\mathfrak{m}}(A)$ is a polynomial ring, then $\#(\text{variables})=d(\operatorname{Gr}_{\mathfrak{m}}(A))=\dim(A)$.) Regular rings are automatically integral domains (is a consequence of Problem 65).

14.2. **Projective dimension of the residue field.** Regularity of rings can be tested homologically:

Proposition 48. *$(A,\mathfrak{m})$ is regular* $\Leftrightarrow \operatorname{Tor}^A_{\gg 0}(A/\mathfrak{m},A/\mathfrak{m})=0\Leftrightarrow$ *every finitely generated A-module admits a* $\boxed{\text{finite free resolution}}$.

Proof. The equivalence of the two right conditions follows from (10.1).
$(\Rightarrow)$ $\mathfrak{m}=(a_1,\ldots,a_d)\Rightarrow$ the *Koszul complex* is a free A-resolution of $k=A/\mathfrak{m}$ – this follows via $\operatorname{Gr}_{\mathfrak{m}}(A)$ from the corresponding result for polynomial rings in (10.2): $M'\to M\to M''$ with exact $\operatorname{Gr}_{\mathfrak{m}}(M')\to\operatorname{Gr}_{\mathfrak{m}}(M)\to\operatorname{Gr}_{\mathfrak{m}}(M'')$ (homogeneous maps of degree 1) $\Rightarrow \ker\cap\mathfrak{m}^iM\subseteq \operatorname{im}+\big(\ker\cap\mathfrak{m}^{i+1}M\big)\Rightarrow \exists i_0:\ \forall i\ge i_0:\ \ker\subseteq \operatorname{im}+(\ker\cap\mathfrak{m}^iM)=\operatorname{im}+\mathfrak{m}^{i-i_0}(\ker\cap\mathfrak{m}^{i_0}M)\subseteq \operatorname{im}+\mathfrak{m}^{i-i_0}\ker$.
$(\Leftarrow)$ $\mathfrak{m}\setminus\mathfrak{m}^2$ contains non-zero divisors a: Otherwise, by (3.6) and "prime avoidance" (Lemma 1), $\mathfrak{m}\in\operatorname{Ass}(A)$, i.e. $0\to A/\mathfrak{m}\overset{\cdot s}{\to} A\to A/s\to 0\Rightarrow \operatorname{Tor}^A_i(A/s,k)\overset{\sim}{\to}\operatorname{Tor}^A_{i-1}(k,k)$. Along the lines of (10.1), it follows that $0=\operatorname{Tor}^A_{\operatorname{pd}(k)+1}(A/s,k)=\operatorname{Tor}^A_{\operatorname{pd}(k)}(k,k)$ which cannot be true.
$\dim A/a=\dim A-1$; $\dim_k\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2=\dim_k\mathfrak{m}/(a+\mathfrak{m}^2)=\dim_k\mathfrak{m}/\mathfrak{m}^2-1\rightsquigarrow$ induction: Let $F_\bullet\overset{\text{qis}}{\to}\mathfrak{m}$ be a finite, free A-resolution; since $\operatorname{H}_{\ge 1}(F_\bullet\otimes A/a)=\operatorname{Tor}^A_{\ge 1}(\mathfrak{m},A/a)=$

$\mathrm{H}_{\geq 1}(\mathfrak{m} \stackrel{\cdot a}{\to} \mathfrak{m}) = 0$, the morphism $F_\bullet \otimes A/a \stackrel{\text{qis}}{\to} \mathfrak{m}/a\mathfrak{m}$ becomes a free A/a-resolution. The exact sequence $0 \to A/\mathfrak{m} \stackrel{\cdot a}{\to} \mathfrak{m}/a\mathfrak{m} \to \mathfrak{m}/a \to 0$ splits ($A/\mathfrak{m} \hookrightarrow \mathfrak{m}/a\mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ has a section), hence $\mathrm{Tor}^{A/a}_{\gg 0}(k,k) = 0$. □

Corollary 49. *Localizations of regular rings in prime ideals are regular.*

Proof. $\mathrm{Tor}_i^{A_P}(A_P/PA_P, A_P/PA_P) = \mathrm{Tor}_i^A(A/P, A/P) \otimes_A A_P = 0$ for $i \gg 0$. □

15. Global dimension

15.1. **Height vs. codimension.** Let $a_i \in A$ and $P \supseteq (a_1, \ldots, a_r)$ be a minimal prime ideal $\Rightarrow \mathrm{ht}\, P \leq r$ (in A_P the ideal P is the only prime above $(a_1, \ldots, a_r)$; thus, the latter is P-primary).

Proposition 50. 1) *$a \in A$ non-zero divisor $\Rightarrow$ minimal prime ideals P above (a) have height* $\mathrm{ht}\, P = 1$ ("Krull principal ideal theorem").
2) *$A =$ integral domain $\rightsquigarrow$ [factorial $\Leftrightarrow$ prime ideals of height 1 are principal].*

Proof. 1) $\mathrm{ht}\, P = 0 \rightsquigarrow \dim A_P/a \leq \dim A_P - 1 = -1$.
2) Use "factorial" $\Leftrightarrow$ irreducible $f \in A$ yield prime ideals (f): ($\Leftarrow$) $f \in A \Rightarrow$ a minimal $P \ni f$ has $\mathrm{ht} = 1$; ($\Rightarrow$) $\mathrm{ht}\, P = 1 \Rightarrow$ choose an irreducible $f \in P$. □

In particular, "factorial" implies "regular in codimension (height) one". The reversed implication fails: $\mathbb{C}[x,y,z]/(y^2 - xz)$.

15.2. **Krull dimension.** $\dim A := \max_{P \in \mathrm{Spec}\, A} \mathrm{ht}\, P = \dim A/\sqrt{0}$; if $P_1, \ldots, P_r$ are the minimal primes, then $\dim A = \max_i \dim A/P_i$. Proposition 24 implies [$A \subseteq B$ integral $\Rightarrow \dim A = \dim B$].

Example: $A = k[x_1, \ldots, x_n] \rightsquigarrow$ w.l.o.g. $k = \overline{k}$ ($\overline{k}[\mathbf{x}]$ is integral over $k[\mathbf{x}]$) $\stackrel{\text{HNS}}{\Rightarrow}$ $(\mathbf{x})$ is a "typical" maximal ideal $\Rightarrow \dim k[\mathbf{x}] = \dim k[\mathbf{x}]_{(\mathbf{x})} = n$. A chain of primes: $(x_1, \ldots, x_i)$.

15.3. **Transzendental degree.** Let A be a finitely generated k-algebra without zero divisors $\rightsquigarrow X := \mathrm{Spec}\, A$ is irreducible with $K[X] := A$ and "*function field*" $K(X) := \mathrm{Quot}\, A$.

Proposition 51. 1) $\boxed{\dim A = \text{tr-deg}_k \mathrm{Quot}\, A}$.
2) *$P \subseteq A$ prime ideal $\Rightarrow$* $\dim A = \dim A/P + \mathrm{ht}\, P = \dim A/P + \dim A_P$. *In particular,* $\dim A = \dim A_{\mathfrak{m}}$ *for maximal ideals* $\mathfrak{m}$.

Proof. (1) Proposition 26 $\Rightarrow \exists\, k[y_1, \ldots, y_r] \hookrightarrow A$ finite, hence $\text{tr-deg}_k \mathrm{Quot}\, A = \text{tr-deg}_k k(\mathbf{y}) = r = \dim k[\mathbf{y}] = \dim A$.
(2) w.l.o.g. $\mathrm{ht}\, P = 1$ and $A = k[\mathbf{y}]$: Proposition 24(5) $\Rightarrow \mathrm{ht}\, P = \mathrm{ht}(P \cap k[\mathbf{y}])$ and $\dim A/P = \dim k[\mathbf{y}]/(P \cap k[\mathbf{y}])$. Factoriallity of $k[\mathbf{y}] \rightsquigarrow P = (f)$ with an irreducible $f \in k[\mathbf{y}]$; by (7.1) $k[y_1, \ldots, y_r]/f$ is finite over (w.l.o.g.) $k[y_1, \ldots, y_{r-1}]$, hence it is $(r-1)$-dimensional. □

Applications: $\dim A_f = \dim A,\ \dim(X \times Y) = \dim X + \dim Y.$

16. Projective varieties

4.5.22 (35)

16.1. **Recalling affine varieties and spectra.** Equivalences of categories ($k = \overline{k}$):

$$\{\text{closed affine subsets } Z \subseteq \mathbb{A}^n_k\} \leftrightarrow \{\text{radical ideals } I \subseteq k[x_1,\ldots,x_n]\}$$
$$\leftrightarrow \{k[x_1,\ldots,x_n] \twoheadrightarrow A = \text{ reduced}\}$$

or, forgetting the embedding, $\{$affine algebr k-varieties $\} \leftrightarrow \{$f.g. red k-algebras$\}$. Without k, this generalizes to the scheme setup, i.e. to the equivalence of categories

$$\{\text{affine schemes } (\operatorname{Spec} A, A)\} \leftrightarrow \{\text{commutative rings } A\}^{\text{opp}}.$$

A becomes the ring of regular functions on $\operatorname{Spec} A$, we allow nilpotent elements in A, and we do not need a field k at this point.

Examples: 1) Functor of affine toric varieties $\mathbb{TV}(N,\sigma)$ via $(\sigma \subseteq N_{\mathbb{R}}) \mapsto (\sigma^\vee \subseteq M_{\mathbb{R}})$ and $\mathbb{TV}(N,\sigma) := \operatorname{Spec} k[\sigma^\vee \cap M]$;
2) surjections $A \twoheadrightarrow B$ corresponds to closed embeddings $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$;
3) localizations $A \to A_g$ yield $\operatorname{Spec} A_g = D(g) := (\operatorname{Spec} A) \setminus V(g) \subseteq \operatorname{Spec} A$.
4) Faces $\tau \leq \sigma \subseteq N_{\mathbb{R}}$ lead to open embeddings $\mathbb{TV}(N,\tau) \hookrightarrow \mathbb{TV}(N,\sigma)$.

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16.1.1. *The toric $\mathbb{P}^1$-construction.* The easiest concrete instance of (4) is the following: Let $\Sigma := \{\sigma^+, \sigma^-, 0\}$ consisting of the 1-dimensional cones $\sigma^\pm := \mathbb{R}_{\geq/\leq 0}$ and their intersection 0. Then the associated semigroups are $\mathbb{N}$, $-\mathbb{N}$, and $\mathbb{Z}$.

$$\mathbb{N} \hookrightarrow \mathbb{Z} \hookleftarrow -\mathbb{N} \qquad \mathbb{C}[\mathbb{N}] \hookrightarrow \mathbb{C}[\mathbb{Z}] \hookleftarrow \mathbb{C}[-\mathbb{N}] \qquad \mathbb{C}[x] \hookrightarrow \mathbb{C}[x, x^{-1}] \hookleftarrow \mathbb{C}[y]$$

where the y from the bottom right corner maps to x^{-1}. Geometrically, this means that we glue two copies of $\mathbb{A}^1 = \mathbb{C}^1$ with coordinates x and y, respectively, along their open subsets $\mathbb{C}^*$. However, the identification of the two "tori" is done via $y = x^{-1}$.

16.2. **The projective space.** The affine varieties $\mathbb{C}^n$ and, e.g., the quadric $V(x^2 +$ 11.5.22 (37)
$y^2 - 1) \subseteq \mathbb{C}^2$ and the "elliptic curve" $E := V(y^2 - x^3 + x) \subseteq \mathbb{C}^2$ are not compact when considered in the classical topology (the quasicompactness of the Zariski topology is misleading here).

$k = \overline{k}$ field $\rightsquigarrow \mathbb{P}^n_k := \mathbb{P}(k^{n+1})$ with $\boxed{\mathbb{P}_k(V) := (V^\vee \setminus \{0\})/k^*}$; the complex $\mathbb{P}^n_{\mathbb{C}} = S^{2n-1}/\mathbb{C}_1$ is compact in the classical topology; "*projective algebraic subsets*":= vanishing loci $V(J) = V_{\mathbb{P}}(J) \subseteq \mathbb{P}^n_k$ for *homogeneous* ideales $J \subseteq k[\mathbf{z}]$ with $\mathbf{z} =$

$(z_0,\ldots,z_n) \rightsquigarrow$ similarily to (1.2): ZARISKI *topology* on $\mathbb{P}^n_k$; $g\in k[\mathbf{z}]$ homogeneous $\rightsquigarrow$ $D_+(g) := \mathbb{P}^n_k\setminus V(g)$ yield a basis of the open subsets. The special charts $D_+(z_i)\cong k^n$ will be identifyed with the affine schemes $\operatorname{Spec} k[\mathbf{z}]_{(z_i)}$ where

$$k[\mathbf{z}]_{(z_i)} = k[\tfrac{z_0}{z_i},\ldots,\tfrac{z_n}{z_i}] \subset k[\mathbf{z},\tfrac{1}{z_i}] = k[\mathbf{z}]_{z_i}$$

denotes the *homogeneous localization* consisting of the degree 0 elements of the latter, ordinary localization. $\mathbb{P}^n = \bigcup_{i=0}^n D_+(z_i)$ is an open, affine covering. And $\mathbb{P}^n_k = D_+(z_0)\sqcup\mathbb{P}^{n-1}_k$ with $D_+(z_0) = \operatorname{Spec} k[\mathbf{x}]$ where $\mathbf{x}=(x_1,\ldots,x_n)$ and $x_i = z_i/z_0$.

16.3. **Projective subsets.** If we start with an ideal $I\subseteq k[\mathbf{x}]$ corresponding to the affine $\operatorname{Spec} k[\mathbf{x}]/I = V(I)\subseteq\mathbb{A}^n_k \rightsquigarrow$ homogenization $I^h\subseteq k[\mathbf{z}]$ ($\subseteq k[t,\mathbf{x}]$ in (11.2) with $w=\underline{1}$), i.e. after substituting $x_i\mapsto z_i/z_0$ one multiplies with the minimal z_0-power killing all denominators in the polynomials from $I \rightsquigarrow \boxed{V_{\mathbb{P}}(I^h) = \overline{V_{\mathbb{A}}(I)}}$ inside $\mathbb{P}^n$. *Example:* $\overline{E} = V_{\mathbb{P}}(y^2z - x^3 + xz^2)\subseteq\mathbb{P}^2$ carries a group structure; usually the neutral element is chosen as $(0:1:0)$ which is $\overline{E}\setminus E$, cf. (16.2).

The opposite construction: If $J\subseteq k[\mathbf{z}]$ is a homogeneous ideal, then $J^i := J_{(z_i)}\subseteq k[\frac{\mathbf{z}}{z_i}] = k[\mathbf{x}^{(i)}]$ is obtained from substituting $z_\nu\mapsto x^{(i)}_\nu = z_\nu/z_i$ (thus $z_i\mapsto 1$) in the arguments of the polynomials from J. Then, the local structure of $V_{\mathbb{P}}(J)\subseteq\mathbb{P}^n$ in the chart $D_+(z_i)$ is obtained by identifying $V_{\mathbb{P}}(J)\cap D_+(z_i) = \operatorname{Spec} k[\mathbf{x}^{(i)}]/J^i$.

The maximal ideal $(\mathbf{z}) =$ is called the *irrelevant ideal.* $V(J)$ and $V(J:\mathbf{z}^\infty)$ have the same local structure, e.g. $V(\mathbf{z})$ and $V(1)$, or $V(z_0^2 - z_0z_1, z_0z_1 - z_1^2)$ and $V(z_0 - z_1)$, and the ideal $(J:\mathbf{z}^\infty)$ is maximal with this property.

Example: $\operatorname{Grass}(d,V)\subseteq\mathbb{P}(\Lambda^dV^\vee)$ is given by the Plücker relations.

For $Z = V(J)\subseteq\mathbb{P}^n$ we call $\boxed{S(Z) := k[\mathbf{z}]/(J:\mathbf{z}^\infty)}$ the *homogeneous coordinate ring*; it is $\mathbb{Z}$-graded; the affine coordinate ring of the i-th chart $Z\cap D_+(z_i)$ is $S_{(z_i)}$.

Remark. Taking $I(Z)\subseteq k[\mathbf{z}]$ instead of $(J:\mathbf{z}^\infty)$ is to coarse and big if one is interested to preserve a possible non-reduced local structure.

Problem 52. For an ideal $I\subseteq k[\mathbf{x}]$ with $\mathbf{x}=(x_1,\ldots,x_n)$ denote by $I^h := (f^h\mid f\in I)\subseteq k[\mathbf{z}]$ with $\mathbf{z}=(z_0,\ldots,z_n)$ and $x_i = z_i/z_0$ its homogenization. On the contrary, for a homogeneous ideal $J\subseteq k[\mathbf{z}]$ we denote by $J^0\subseteq k[\mathbf{x}]$ its dehomogenization obtained by $z_0\mapsto 1$ and $z_i\mapsto x_i$ for $i\geq 1$. It equals the homogenous localization $J_{(z_0)}$. Eventually, we denote by $V_{\mathbb{P}}(J)\subseteq\mathbb{P}^n$ and $V_{\mathbb{A}}(I)\subseteq\mathbb{A}^n = D_+(z_0)\subset\mathbb{P}^n$ the respective vanishing loci.

a) Recall that $V_{\mathbb{A}}(J^0) = V_{\mathbb{P}}(J)\cap D_+(z_0)$ inside $\mathbb{A}^n = D_+(z_0)$. Assume that $k=\overline{k}$, and use the Hilbert Nullstellensatz to show that then $V_{\mathbb{P}}(I^h) = \overline{V_{\mathbb{A}}(I)}$ inside $\mathbb{P}^n_k$.

b) Show by presenting a suitable example that the equality of (a) fails for $k=\mathbb{R}$.

c) In Subsection (11.2) we had considered $\mathbb{A}' := \mathbb{A}^{n+1}$ instead of $\mathbb{P} := \mathbb{P}^n$. In particular, we denote $V_{\mathbb{A}'}(J)\subseteq\mathbb{A}'$ for the affine subsets induced by homogeneous ideals $J\subseteq k[\mathbf{z}]$. Comparing both situations via $\pi:\mathbb{A}'\setminus 0\to\mathbb{P}$ we have now open subsets $D(z_0)\subset\mathbb{A}'$ and $D_+(z_0)\subset\mathbb{P}$ with $D(z_0) = \pi^{-1}\big(D_+(z_0)\big)$, see Problem 71.

We have seen in Subsection (16.6) that $V_{\mathbb{A}'}(J) \cap D(z_0) = \pi^{-1}\big(V_{\mathbb{A}}(J^0)\big)$ with $V_{\mathbb{A}}(J^0) \subseteq \mathbb{A} = D_+(z_0) \subset \mathbb{P}$. Or, with other symbols, and $V_{\mathbb{A}'}(I^h) \cap D(z_0) = \pi^{-1}\big(V_{\mathbb{A}}(I)\big)$. Using this, we have got in Subsection (11.2) that $V_{\mathbb{A}'}(I^h) = \overline{V_{\mathbb{A}'}(I^h) \cap D(z_0)}$ inside $\mathbb{A}'$. Now, use this to derive $V_{\mathbb{P}}(I^h) = \overline{V_{\mathbb{P}}(I^h) \cap D_+(z_0)}$ inside $\mathbb{P}$.

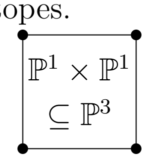
16.4. **Special constructions.** The homogeneous coordinate ring is not an invariant of the projective variety, but it depends on its projective embedding, cf. $\nu_{1,2}$:

1) The *Veronese embedding* $\nu_{n,d} : \mathbb{P}^n \hookrightarrow \mathbb{P}(k[\mathbf{z}]_d) = \mathbb{P}^{\binom{d+n}{n}-1}$ is (locally) an isomorphism onto the image. However, $S(\nu_{n,d}(\mathbb{P}^n)) = \oplus_{d|k} k[\mathbf{z}] \subsetneq k[\mathbf{z}] = S(\mathbb{P}^n)$.
Example: For $\nu_{1,2} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, $(z_0 : z_1) \mapsto (z_0^2 : z_0 z_1 : z_1^2)$ the image is $V(w_0 w_2 - w_1^2)$, and the inverse map consists of the two local pieces $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ $(w_0 : w_1 : w_2) \mapsto (w_0 : w_1)$ (not defined in $(0:0:1)$) and $\mapsto (w_1 : w_2)$ (not defined in $(1:0:0)$).
While $\nu_{1,2} : \mathbb{P}^1 \xrightarrow{\sim} V_{\mathbb{P}}(w_0 w_2 - w_1^2)$, the map between the homogeneous coordinate rings is $\nu_{1,2}^* : k[w_0, w_1, w_2]/(w_0 w_2 - w_1^2) \xrightarrow{\sim} k[z_0^2, z_0 z_1, z_1^2] \subset k[z_0, z_1]$, i.e. the quadric yields only the even degrees inside $k[z_0, z_1]$. All non-degenerate quadrics ("conics") in $\mathbb{P}^2_{\mathbb{C}}$ are, via a linear change of coordinates, equal to $V(w_0 w_2 - w_1^2)$. In particular, they are isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

2) The *Segre embedding* $\mathbb{P}^a \times \mathbb{P}^b \hookrightarrow \mathbb{P}^{(a+1)(b+1)-1}$ or, coordinate free, $\mathbb{P}(V) \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \otimes W)$ gives $\mathbb{P}^a \times \mathbb{P}^b$ the structure of a projective variety. *Example:* $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, $(y_0 : y_1), (z_0 : z_1) \mapsto (y_0 z_0 : y_0 z_1 : y_1 z_0 : y_1 z_1)$ has the image $V(w_{00} w_{11} - w_{10} w_{01})$. In particular, non-degenerate quadrics in $\mathbb{P}^3$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \neq \mathbb{P}^2$). Hence, they contain always two infinite families of lines.

On the contrary, a general cubic surface $S \subseteq \mathbb{P}^3$ contains exactly 27 lines, cf. (17.6).

3) *Projective toric varieties:* $M := \mathbb{Z}^n$, $\Delta \subseteq M_{\mathbb{Q}}$ lattice polytope (convex hull of a 18.5.22 (39)
finite subset of M) $\rightsquigarrow$ $\boxed{\mathbb{P}(\Delta) \subseteq \mathbb{P}_k^{\#(\Delta \cap M)-1}}$ with equations $\prod_v z_v^{\lambda_v} = \prod_v z_v^{\mu_v}$ resulting from the affine dependencies $\sum_v \lambda_v(v,1) = \sum_v \mu_v(v,1)$ where $v \in \Delta \cap M$, $\lambda_v, \mu_v \in \mathbb{N}$. The $(M \oplus \mathbb{Z})$-graded kernel of $k[z_v \mid v \in \Delta \cap M] \to k[M \oplus \mathbb{Z}]$, $z_v \mapsto x^{(v,1)} = x^v t$ is generated from the above equations, hence $S(\mathbb{P}(\Delta)) = k[\mathbb{N} \cdot (\Delta \cap M, 1)] =: k[\Delta]$.
Examples: 3.0) $\Delta^n := \{\mathbf{x} \in \mathbb{Q}^n_{\geq 0} \mid x_1 + \ldots + x_n \leq 1\} = \{\mathbf{x} \in \mathbb{Q}^{n+1}_{\geq 0} \mid x_0 + \ldots + x_n = 1\}$ $\Rightarrow \mathbb{P}(\Delta^n) = \mathbb{P}^n_k$ (there are no affine dependencies at all, hence no equations).
3.1) Veronese′: $d \in \mathbb{Z}_{\geq 1} \rightsquigarrow \mathbb{P}^n_k \cong \mathbb{P}(d\Delta^n) \subseteq \mathbb{P}_k^{\binom{n+d}{d}-1}$, $\underline{z} \mapsto (\underline{z}^r \mid |r| = d)$, but $S(\mathbb{P}(d\Delta^n)) = \oplus_{v \geq 0} S(\mathbb{P}^n_k)_{dv} \subsetneq S(\mathbb{P}^n_k)$. Or, $\mathbb{P}(\Delta) \to \mathbb{P}(d\Delta)$ for normal polytopes.
3.2) Segre′: $\mathbb{P}(\Delta_1) \times \mathbb{P}(\Delta_2) = \mathbb{P}(\Delta_1 \times \Delta_2)$; the relations $(e_i, e_j) + (e_k, e_l) = (e_i, e_l) + (e_k, e_j)$ yield the equations $\operatorname{rank}(z_{ij})_{0 \leq i,j \leq m,n} \leq 1$. There is a natural map $\mathbb{P}(\Delta_1 + \Delta_2) \to \mathbb{P}(\Delta_1 \times \Delta_2)$.

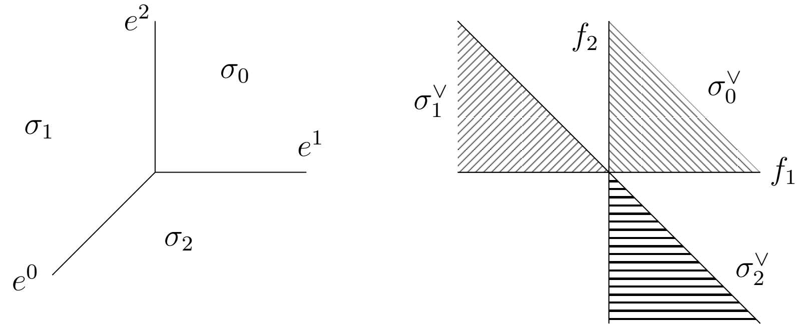


16.5. **Toric varieties.** We recall that affine toric varieties are associated to polyhedral cones and glue this construction afterwards.

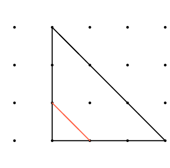
16.5.1. *Affine toric varieties.* $N := \mathbb{Z}^n$, $M := \operatorname{Hom}(N,\mathbb{Z})$, $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q} \rightsquigarrow$ perfect pairing $\langle \bullet, \bullet \rangle : N_{\mathbb{Q}} \otimes M_{\mathbb{Q}} \to \mathbb{Q}$. Polyhedral cones $\sigma \subseteq N_{\mathbb{Q}}$ with apex $\rightsquigarrow \sigma^{\vee} := \{r \in M_{\mathbb{Q}} \mid \langle \sigma, r\rangle \geq 0\}$; *polyhedrial duality* $\sigma^{\vee\vee} = \sigma$ and $(\sigma_1 \cap \sigma_2)^{\vee} = \sigma_1^{\vee} + \sigma_2^{\vee}$.
$\boxed{\text{Functor } \sigma \mapsto \mathbb{TV}(\sigma)} := \mathbb{TV}(\sigma, N) := \operatorname{Spec} k[\sigma^{\vee} \cap M] \subseteq \mathbb{A}^H_k$ as in (17.5); if $\tau \leq \sigma$ is a face, then every $r \in \operatorname{int}(\sigma^{\vee} \cap \tau^{\perp}) \cap M$ yields $\tau = \sigma \cap r^{\perp} \Rightarrow \tau^{\vee} = \sigma^{\vee} - \mathbb{Q}_{\geq 0} \cdot r$, hence $\mathbb{TV}(\tau) = D(\mathbf{x}^r) \subseteq \mathbb{TV}(\sigma)$, cf. (16.5). *Examples:* $\mathbb{TV}(\mathbb{Q}^n_{\geq 0}) = \mathbb{A}^n_k$; $\mathbb{TV}(\sigma_1) \times \mathbb{TV}(\sigma_2) = \mathbb{TV}(\sigma_1 \times \sigma_2)$; $\mathbb{TV}(\mathbb{Q}_{\geq 0}(1,0) + \mathbb{Q}_{\geq 0}(1,2)) = V(z^2 - xy) \subseteq \mathbb{A}^3_k$.

16.5.2. *General toric varieties.* With the notation of (16.5.1): If Σ is a $\boxed{\text{fan}}$ of cones in $N_{\mathbb{Q}}$, then we glue $\boxed{\mathbb{TV}(\Sigma, N) := \varinjlim_{\sigma \in \Sigma} \mathbb{TV}(\sigma)}$; this construction is functorial with respect to $f : (N, \Sigma) \to (N', \Sigma')$ meaning a $\mathbb{Z}$-linear map $f : N \to N'$ such that $\forall \sigma \in \Sigma \; \exists \sigma' \in \Sigma'$: $f(\sigma) \subseteq \sigma'$.

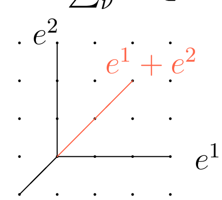
The toric description of $\mathbb{P}^n$*:* $N := \mathbb{Z}^{n+1}/\sum_i e^i$, hence $M := \operatorname{Hom}(N, \mathbb{Z}) = [\sum e^i = 0] \subseteq \mathbb{Z}^{n+1}$ with basis $f_i := e_i - e_0$ $(i = 1, \ldots, n)$. The cones $\sigma_i := \langle e^0, \ldots, \hat{e}^i, \ldots, e^n\rangle \rightsquigarrow \sigma_i^{\vee} = \langle e_{\bullet} - e_i\rangle$ provide $\mathbb{TV}(\sigma_i) = U_i$, and $\tau := \sigma_i \cap \sigma_j$ determines open embeddings $U_i \supseteq D(z_j/z_i) = U_{ij} = D(z_i/z_j) \subseteq U_j$. With $\Sigma := \{\sigma_i \text{ and faces}\}$ we obtain $\mathbb{TV}(\Sigma) = \mathbb{P}^n$.



The toric description of the blow up: $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ is the gluing of $k[x_1, \ldots, x_n] \to k[x_1/x_i, \ldots, x_n/x_i, x_i]$, hence $k[\mathbb{N}^n] \to k[\langle e_{\bullet} - e_i, e_i\rangle \cap \mathbb{Z}^n]$. Thus, the i-th chart corresponds to the inclusion $\sigma_i := \langle e^1, \ldots, \hat{e^i}, \ldots, e^n, \sum_{\nu} e^{\nu}\rangle \subseteq \mathbb{Q}^n_{\geq 0} =: \sigma$, i.e. $\mathbb{Q}^n_{\geq 0}$ will be subdivided by inserting the inner ray $e := \sum_{\nu} e^{\nu} \in \mathbb{Z}^n = N$.



polyhedron $\Delta \subseteq M_{\mathbb{Q}}$



subdivided cone $\mathbb{Q}^2_{\geq 0} \subseteq N_{\mathbb{Q}}$

The map $h : \widetilde{\mathbb{A}^n} \to \mathbb{P}^{n-1}_k$ can be obtained from the projection $N \twoheadrightarrow N/\mathbb{Z}e$ with $e := e^1 + \ldots e^n$. (17.2) $\rightsquigarrow \mathcal{O}_{\mathbb{P}^{n-1}}(-1) =$ sheaf of sections of h; $\Gamma(\mathbb{P}^{n-1}, \mathcal{O}(-1)) = 0$ is illustrated by the non-existence of global *toric* sections of h: There are no hyperplanes meeting all cones of the $\widetilde{\mathbb{A}^n}$-fan at once.

If $\Delta \subseteq M_{\mathbb{Q}}$ is a lattice polyhedron, then we had defined in (16.4)(3) and (17.5) the toric variety $\mathbb{P}(\Delta)$. Let $\Sigma := \mathcal{N}(\Delta) :=$ (inner) *normal fan* of $\Delta \rightsquigarrow n : \mathbb{TV}(\Sigma) \to \mathbb{P}(\Delta)$ is built from gluing the maps $n_w^* : k[x^{v-w} \mid v \in \Delta \cap M] \to k[\mathbb{Q}_{\geq 0} \cdot (\Delta - w) \cap M]$ for (e.g. vertices) $w \in \Delta \cap M$. This becomes an isomorphism ("Δ is *ample*") for $\Delta := (\gg 0) \cdot \Delta$.

16.6. **The affine cone and the Hilbert polynomial.** The local structure of $\pi : \mathbb{A}_k^{n+1} \backslash 0 \to \mathbb{P}_k^n$, $(z_0, \dots, z_n) \mapsto (z_0 : \dots : z_n)$ is $D_+(z_i) \times (\mathbb{A}_k^1 \backslash 0) = D(z_i) \to D_+(z_i)$; on the level of k-algebras, this corresponds to $k[\mathbf{z}]_{(z_i)} \otimes k[z_i^{\pm 1}] = k[\mathbf{z}]_{z_i} \supseteq k[\mathbf{z}]_{(z_i)}$.

$\emptyset \neq Z \subseteq \mathbb{P}^n \rightsquigarrow C(Z) := \overline{\pi^{-1}(Z)} = \pi^{-1}(Z) \cup \{0\}$ is called the *affine cone* over Z; $\dim C(Z) = \dim Z + 1$. In $A(\mathbb{A}^{n+1}) = k[\mathbf{z}] = S(\mathbb{P}^n)$ we have $I_{\mathbb{A}}(C(Z)) = I_{\mathbb{P}}(Z)$. Similarily, if $J \subsetneq k[\mathbf{z}]$ is a homogeneous ideal, then $C(V_{\mathbb{P}}(J)) = V_{\mathbb{A}}(J : \mathbf{z}^\infty)$, leading to $A(C(Z)) = S(Z)$.

Homogeneous/projective HNS: Let $k = \overline{k}$ and $Z = V_{\mathbb{P}}(J) \subseteq \mathbb{P}_k^n$ for a given homogeneous ideal $J \subseteq k[\mathbf{z}]$. Then, if $f \in I_{\mathbb{P}}(Z)$ is homogeneous with $\deg f > 0 \Rightarrow f = 0$ on $\pi^{-1}(Z)$ and $f(0) = 0$, i.e. $f \in I_{\mathbb{A}}(Z) \Rightarrow \exists N\colon\ f^N \in J$. In particular, $V_{\mathbb{P}}(J) = \emptyset$ does only imply that $(\mathbf{z})^N \subseteq J$.

Now, we discuss properties of $Z \subseteq \mathbb{P}^n$ via the local properties of $C(Z)$ in $0 \in \mathbb{A}^{n+1}$: Let $S = \oplus_{d \geq 0} S_d$ be a finitely generated, graded ($S_0 = k$)-algebra with irrelevant ideal $S_+ := \oplus_{d \geq 1} S_d \Rightarrow$ for $S_{\text{loc}} := (S \setminus S_+)^{-1} S$ it holds true that $\operatorname{Gr}_{S_+}(S_{\text{loc}}) = \oplus_{d \geq 0} S_+^d / S_+^{d+1}$. If S is generated in degree 1 (e.g. $S = S(Z)$) $\Rightarrow \operatorname{Gr}_{S_+}(S_{\text{loc}}) = S \Rightarrow H_S(t) = \chi_{S_+}^{S_{\text{loc}}}(t+1) - \chi_{S_+}^{S_{\text{loc}}}(t) \Rightarrow \deg H_S = \dim S_{\text{loc}} - 1$. In particular, $\deg H_{S(Z)} = \dim Z$, and the (normalized with $(\dim Z)!$) leading coefficient of $H_{S(Z)}$ is $\boxed{\deg Z := \operatorname{mult}(C(Z), 0)}$, cf. (12.3) and (13.1).

Example: $\deg V_{\mathbb{P}}(F) (\subseteq \mathbb{P}^n) = \deg F$; $\deg \mathbb{P}(\Delta) = \operatorname{vol}(\Delta)$ where vol is normalized to vol(standard simplex) $= 1$ (quadrics $\nu_2(\mathbb{P}^1)$ and $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$; $\deg \nu_2(\mathbb{P}^2) = 4$).

16.7. **Linear projections.** The map $\pi : \mathbb{A}_k^{n+1} \setminus \{0\} \to \mathbb{P}_k^n$ from (16.6) has the following generalization: Let $L, L' \subseteq \mathbb{P}_k^n$ be disjoint linear subspaces with $\dim L + \dim L' = n - 1 \rightsquigarrow \pi_L : \mathbb{P}_k^n \setminus L \to L'$, $p \mapsto \operatorname{span}(p, L) \cap L'$. Using coordinates, $L = (* : \underline{0})$, $L' = (\underline{0} : *) \Rightarrow \pi_L(\underline{x} : \underline{y}) = (\underline{0} : \underline{y})$. This was already used in (16.4)(1).

16.8. **Global regular functions.** $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \bigcap_i k[\mathbf{z}]_{(z_i)} = \bigcap_i k[z_0/z_i, \dots, z_n/z_i] = k$ ("factorial" $\Rightarrow$ the intersection of just two rings is already k) $\rightsquigarrow \mathbb{P}^{n \geq 1}$ is not affine!

Proposition 53. *Let $Z \subseteq \mathbb{P}_k^n$ be a projective variety* (irreducible) $\Rightarrow \Gamma(Z, \mathcal{O}_Z) = k$.

Proof. $f \in \Gamma(Z, \mathcal{O}_Z) = \bigcap_i S(Z)_{(z_i)} \subseteq \operatorname{Quot} S(Z) \Rightarrow \exists N :\ (\mathbf{z})^N f \subseteq (\mathbf{z})^N \Rightarrow (\mathbf{z})^N f^{q \in \mathbb{N}} \subseteq (\mathbf{z})^N \Rightarrow S(Z)[f] \subseteq z_0^{-N} S(Z)$, i.e. f is integral over $S(Z)$. The coefficients of the integrality relation are, w.l.o.g., homogeneous of degree 0, hence $\in k$. □

On the other hand, $z_0, \dots, z_n$ are global on $\mathbb{P}^n_k$, but they are no functions. Instead, they are global sections of the dual "$\mathcal{O}(1)$" of the locally trivial tautological fibration "$\mathcal{O}(-1)$" on $\mathbb{P}^n$. In general, we define for $d \in \mathbb{Z}$, $\mathcal{O}(-d) := \{(\ell, c) \mid \ell \in \mathbb{P}^n,\ c \in \ell^{\otimes d}\}$ where ℓ is understood as a line, i.e. as a 1-dimensional subspace $\ell \subseteq k^{n+1}$, and for $d < 0$ we define $\ell^{\otimes d} := \mathrm{Hom}_k(\ell^{\otimes(-d)}, k)$. 25.5.22 (39)

16.9. **The definition of** $\mathrm{Proj}\, S$**.** Let $S = \oplus_{d\geq 0} S_d$ be a ($\mathbb{N}$-)graded ring (e.g. $S = S(Z)$ for $Z \subseteq \mathbb{P}^n_k$, i.e. S_1 = finitely generated ($A := S_0$)-module, the A-algebra S is generated from S_1) $\rightsquigarrow$ the topological space $\boxed{\mathrm{Proj}\, S}$ $:= \{P \in \mathrm{Spec}\, S \mid S_{\geq 1} \not\subseteq P = \text{homogeneous}\} \to \mathrm{Spec}\, A$ (recovering Z). Zariski-closed: $V_{\mathbb{P}}(J) \subseteq \mathrm{Proj}\, S$ for homogeneous ideals $J \subseteq S$; open basis $D_+(f) := \mathrm{Proj}\, S \setminus V(f) = \mathrm{Spec}\, S_{(f)}$ for homogeneous $f \in S_{\geq 1}$. The "(affine) *cone*" is $\mathrm{Spec}\, S \setminus V_{\mathbb{A}}(S_{\geq 1}) \to \mathrm{Proj}\, S$, $P \mapsto (P \cap \bigcup_d S_d)$ [Example $x_0(x_1 - c_1) - x_1(x_0 - c_0)$]; locally $D(f) \to D_+(f)$.

Remark. While this construction is similar to $\mathrm{Spec}(A)$ – what is the analogue to the affine scheme $(\mathrm{Spec}\, A, A)$? The problems are: (i) $S = S(Z)$ depends on the embedding, i.e. different rings S and T might encode the same variety; (ii) S does not provide functions on $\mathrm{Proj}\, S$ – what kind of objects are elements of S at all? (iii) global functions on $\mathrm{Proj}\, S$ are constants.

17. Blowing up

17.1. **Blowing up** $0 \in \mathbb{A}^n_k$**.** (cf. picture [Hart, S.29])

$$\begin{array}{ccc} \widetilde{\mathbb{A}^n_k} := V(x_i y_j - x_j y_i) \subseteq \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k \xrightarrow{\ \pi\ } \mathbb{A}^n_k, & & [(x_1, .., x_n), (y_1 : ... : y_n)] \overset{\pi}{\mapsto} (x_1, .., x_n) \\ \downarrow h & & \downarrow h \\ \mathbb{P}^{n-1}_k & & (y_1 : ... : y_n) \end{array}$$

Outside of 0, the map $\pi : \pi^{-1}(\mathbb{A}^n \setminus 0) \xrightarrow{\sim} \mathbb{A}^n \setminus 0$ is an isomorphism; "*exceptional divisor*" $\boxed{E := \pi^{-1}(0) = \mathbb{P}^{n-1}}$; if ℓ is a line through $0 \in \mathbb{A}^n \Rightarrow \pi^{\#}(\ell) := \overline{\pi^{-1}(\ell \setminus \{0\})} = \ell \times \{\ell\}$, i.e. $\pi^{\#}(\ell) \cap E = \{\ell\} \subseteq \mathbb{P}^{n-1}$. We consider $\widetilde{\mathbb{A}^n} = \{(c, \ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid c \in \ell\}$ with $h(c, \ell) = \ell$ the "*universal line*" over $\mathbb{P}^{n-1}$ (generalizes to the tautological bundle = universal subspace over $\mathrm{Grass}(k, V)$).

17.2. **Local description of the blowing up.** On the i-th chart $\mathbb{A}^n_k \times D_+(y_i)$, the space $\widetilde{\mathbb{A}^n_k}$ is given by the equations $\mathbf{x} = x_i \frac{\mathbf{y}}{y_i}$; for the affine coordinate rings this means

$$\begin{array}{c} k[x_i, \mathbf{x}/x_i] = k[x_i, \mathbf{y}/y_i] = k[\mathbf{x}, \mathbf{y}/y_i]/(\mathbf{x} - x_i \tfrac{\mathbf{y}}{y_i}) \xleftarrow{\ \pi^*\ } k[\mathbf{x}] \\ \uparrow h^* \\ k[\mathbf{x}/x_i] = k[\mathbf{y}/y_i] \end{array}$$

and $k[x_i^{\pm 1}, \mathbf{x}/x_i] \xleftarrow{\sim} k[\mathbf{x}]_{x_i}$ for the restriction to $D(x_i) \times D_+(y_i) \to D(x_i)$. While the charts of the blowing up $\widetilde{\mathbb{A}^n_k}$ are obtained from $\mathbb{A}^n_k$ by allowing certain denominators, i.e. while this might remind of a localization procedure, π is $\boxed{\text{not flat}}$.

17.3. **Strict transforms.** $X \subseteq \mathbb{A}^n_k \rightsquigarrow \pi^\#(X) := \overline{\pi^{-1}(X \setminus 0)} \subseteq \widetilde{\mathbb{A}^n_k}$; the "total transform" splits into $\boxed{\pi^{-1}(X) = \pi^\#(X) \cup E}$. The ideal I_E of the exceptional divisor $E = \pi^{-1}(0)$ is locally principal, namely $I_E = (x_i)$ on $h^{-1}(D_+(y_i))$; if $X = V_{\mathbb{A}}(J)$, then the ideal of both the total and strict transform $\pi^{-1}(X)$ and $\pi^\#(X)$ in $h^{-1}(D_+(y_i))$ is $J := J\,k[x_i, \mathbf{x}/x_i]$ and $(J : x_i^\infty)$, respectively.

Example: $X = V(y^2 - x^3) \rightsquigarrow \pi^{-1}(X) \cap h^{-1}(D_+(x)) = V(t^2x^2 - x^3)$ with $t = y/x$, but $\pi^\#(X) = V(t^2 - x)$ is even contained in the $[y \neq 0]$ chart. The morphism $\pi^\#(X) \to X$ becomes $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^3)$ with $x \mapsto t^2$ and $y \mapsto t^3$.

17.4. **Blowing up via** Proj**.** With $I := (\mathbf{x}) \subseteq k[x_1, \ldots, x_n]$, one obtains $\Rightarrow \widetilde{\mathbb{A}^n} = \operatorname{Proj} \oplus_{d\geq 0} I^d \stackrel{\pi}{\to} \operatorname{Spec} k[\mathbf{x}] = \mathbb{A}^n$, namely $\boxed{S := \oplus_{d\geq 0} I^d t^d}$ is a finitely generated, graded $(S_0 = k[\mathbf{x}])$-algebra with $D_+(x_i t) \,\widehat{=}\, S_{(x_i t)} = k[\mathbf{x}][\mathbf{x}/x_i] = k[x_i, \mathbf{x}/x_i]$. Moreover, the closed embedding $\widetilde{\mathbb{A}^n} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ is realized via the surjection $k[\mathbf{x}][\mathbf{y}] \twoheadrightarrow S$, $y_i \mapsto x_i t$. The exceptional divisor E is recovered via $\pi^{-1}(0) = \operatorname{Proj} S/IS = \operatorname{Proj} \oplus_{d\geq 0} I^d/I^{d+1}$. See also (22.5).

17.5. **Toric description of the blowing up.** $\Delta \subseteq M_{\mathbb{Q}}$ lattice polytope $\rightsquigarrow$ the affine charts $D_+(z_v) = \operatorname{Spec} k[\Delta]_{(z_v)}$ are numerated by the $v \in \Delta \cap M$ or just the vertices v of Δ. The affine coordinate rings are the semigroup rings $k[\Delta]_{(z_v)} = k[\mathbb{N} \cdot ((\Delta - v) \cap M)] \subseteq k[\mathbb{Q}_{\geq 0} \cdot (\Delta - v) \cap M]$.

Similarly, $k[x_i, \mathbf{x}/x_i] = k[\mathbb{Q}_{\geq 0} \cdot (\nabla - e^i) \cap \mathbb{Z}^n]$ where $\nabla = \operatorname{conv}\{e^1, \ldots, e^n\} + \mathbb{Q}^n_{\geq 0}$. For those non-compact polyhedra $\Delta = \Delta^c + \operatorname{tail}(\Delta)$, we have a similar construction as in (16.4)(3): $v \in \Delta^c \cap M$ gives rise to a homogeneous coordinate z_v; $w \in H \subseteq \operatorname{tail}(\Delta) \cap M$ (H generates $\operatorname{tail}(\Delta) \cap M$) enumerates ordinary coordinates x_w. Now, $\boxed{\mathbb{P}(\Delta) \subseteq \mathbb{P}_k^{(\Delta^c \cap M)-1} \times \mathbb{A}^H_k}$ is defined by the binomial equations corresponding to the linear dependencies among $(v,1)$ and $(w,0)$ inside $M \oplus \mathbb{Z}$.

Example: 0) $\Delta = C = \operatorname{tail}(\Delta)$, i.e. $\Delta^c = \{0\} \Rightarrow \mathbb{P}(C) \subseteq \mathbb{A}^H$, and this equals $\mathbb{TV}(C^\vee) := \operatorname{Spec} k[C \cap M]$. The embedding is induced by $H : C^\vee \to \mathbb{Q}^H_{\geq 0}$.

1) ∇ has the vertices $v^i = e^i$ and the generators of the tail cone $w^i = e^i$. The basic dependencies are $(v^i, 1) + (w^j, 0) = (v^j, 1) + (w^i, 0)$; they lead to the equations of (17.1). Thus, $\boxed{\text{blowing up means cutting off (elementary) corners}}$ of polyhedra.

2) Blowing up $\mathbb{P}^2$ in two points equals blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ once, say $\mathbb{P}^2_{(2)} = (\mathbb{P}^1 \times \mathbb{P}^1)_{(1)}$.

17.6. **Cubic surfaces.** Non-degenerate quadrics in $\mathbb{P}^2$, $\mathbb{P}^3$, and $\mathbb{P}^5$ are isomorphic to $\mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\operatorname{Grass}(2,4)$, respectively.

The cubic surface $S := V(x^3 + y^3 + z^3 + w^3) \subseteq \mathbb{P}^3_k$ contains exactly 27 lines: Gauß elimination transforms their equations into $x_0 - (a_2x_2 + a_3x_3) = x_1 - (b_2x_2 + b_3x_3) = 0$; substituting x_0, x_1 in the original cubic, the vanishing of the coefficients leads to

$$a_2^3 + b_2^3 + 1 = a_3^3 + b_3^3 + 1 = 0 \quad \text{and} \quad a_2^2a_3 + b_2^2b_3 = a_3^2a_2 + b_3^2b_2 = 0.$$

Considering $c_i := a_i/b_i$ shows that (w.l.o.g.) $b_2 = a_3 = 0$, hence the equations for lines inside S turn into $x_0+\omega^i x_2 = x_1+\omega^j x_3 = 0$ with $\omega = \sqrt[3]{1}$ (plus permutations).

Let $L_1, L_2 \subseteq \mathbb{P}^3$ be disjoint lines on a *general* smooth cubic $S = V(g) \subseteq \mathbb{P}^3 \rightsquigarrow f : \mathbb{P}^3 \setminus (L_1 \cup L_2) \to L_1 \times L_2$ such that p, $f_1(p) \in L_1$, $f_2(p) \in L_2$ are collinear, i.e. $f_2 = \pi_{L_1} : \mathbb{P}^3 \setminus L_1 \to L_2$. This gives a morphism $f_2 : S \to L_2$ via $f_2(p) := T_pS \cap L_2$ for $p \in L_1$; using coordinates: $L_1 = (*\,*\,0\,0)$, $L_2 = (0\,0\,*\,*) \Rightarrow f_2 : (x_0 : x_1 : x_2 : x_3) \mapsto (x_2 : x_3)$ and $T_pS \cap L_2 = (-\frac{\partial g}{\partial x_3} : \frac{\partial g}{\partial x_2})$, see Problem **??**.

The map $f : S \to L_1 \times L_2$ is invertible except in the points $(p, q) \in L_1 \times L_2$ with $\overline{p,q} \subseteq S$ – here, the entire line $\overline{p,q}$ forms the preimage. There are exactly five those points (at least in the above example), hence $f : S \to L_1 \times L_2 = \mathbb{P}^1 \times \mathbb{P}^1$ is the blowing up of five points or, alternatively, the blowing up of six points in $\mathbb{P}^2$. In particular, $\boxed{S = \mathbb{P}^2_{(6)}}$ is rational.

Recovering of the 27 lines in the blowing up $\mathbb{P}^2_{(6)} \to \mathbb{P}^2$: six exceptional divisors, 15 strict transforms of connecting lines, six strict transforms of quadrics through five points. A toric analogon is $\mathbb{P}^2_{(3)}$ – after starting with $\nu_3(\mathbb{P}^2)$ one sees the six toric lines as the six edges of length one of the polytope.

18. Sheaves

1.6.22 (41)

18.1. **Presheaves.** X = topological space $\rightsquigarrow$ "*Presheaf* on X":= contravariant functor $\mathcal{F} : \mathcal{O}pen(X)^{\mathrm{opp}} \to \mathcal{A}b/\mathcal{R}ings$; they form a category via $\mathrm{Hom}_{\mathcal{P}reSh}(\mathcal{F}, \mathcal{G}) :=$ {natural transformations $\mathcal{F} \to \mathcal{G}$}.

Examples: function sheaves, constant (pre-)sheaf, sections in bundles, restriction $\mathcal{F}|_U$ of presheaves, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ with $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$.

For an open $U \subseteq X$ and a point $P \in X$ we obtain functors $\mathcal{P}reSh(X, \mathcal{A}b) \to \mathcal{A}b$

$$\mathcal{F} \mapsto \Gamma(U, \mathcal{F}) := \mathcal{F}(U) \text{ ("\textit{sections}") and } \mathcal{F}_P := \varinjlim_{U \ni P} \mathcal{F}(U) \text{ ("\textit{stalk}" in } P)$$

Example: $\mathcal{O}^{\mathrm{an}}_{\mathbb{R},0} = \mathbb{R}[|x|]$, but $\mathcal{C}^\infty_{\mathbb{R},0}$ is much bigger.

For sections $s \in \mathcal{F}(U)$ we call $\boxed{s_P \in \mathcal{F}_P \text{ the \textit{germ}}}$ of s in $P \in U$; der *support* $\mathrm{supp}\, s := \{P \in U \mid s_P \neq 0\}$ is automatically closed in U. Further operations among presheaves are, e.g., $\ker(\mathcal{F} \to \mathcal{G})$, im, coker, $\mathcal{F}/\mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \otimes \mathcal{G}$; the obvious definitions of injectivity and surjectivity work; $\mathcal{P}reSh(X, \mathcal{A}b)$ becomes an abelian category making the two above functors $\mathcal{P}reSh \to \mathcal{A}b$ exact.

18.2. **Sheaves.** $\mathcal{F}|X$ is called a *sheaf* :$\Leftrightarrow$ $\mathcal{F}(\emptyset) = 0$ and for open $U_i \subseteq X$ the sequence $0 \to \mathcal{F}(\bigcup_i U_i) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$ is exact; $\mathcal{S}h(X, \mathcal{A}b) \overset{\iota}{\hookrightarrow} \mathcal{P}reSh(X, \mathcal{A}b)$ is defined as a full subcategory. If $\mathcal{F}, \mathcal{G} \in \mathcal{S}h$, then $\ker(\mathcal{F} \to \mathcal{G}) \in \mathcal{S}h$; similarily $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G}) : U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ stay sheaves.

The essence of $\mathcal{S}h(X)$: [$s \in \mathcal{F}(U)$ vanishes $\Leftrightarrow \forall P \in U$: $s_P = 0$] and [$f : \mathcal{F} \to \mathcal{G}$ is zero/injective/isom $\Leftrightarrow f_P$ is zero/injective/isom $\forall P \in X$].

The major problem of $\mathcal{S}h(X)$: The $\mathcal{P}re\mathcal{S}h$ notions $\mathrm{im}(\mathcal{F} \to \mathcal{G})$ (and coker and $\otimes$) drop out of $\mathcal{S}h$. *Solution:* Keep ker, but redefine im and coker in (18.6) such that $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ becomes exact $\Leftrightarrow \mathcal{F}_P \to \mathcal{G}_P \to \mathcal{H}_P$ is exact for all P. Now, the original problem manifests as the only left-exactness of ι or the section functors $\Gamma(U, \bullet)$.

18.3. **Sheafification.** Let $\mathcal{U} \subseteq \mathcal{O}pen(X)$ be a basis of the topology, i.e. every open 8.6.22 (43)
subset $U \subseteq X$ is a union of some $U_i \in \mathcal{U}$. The notions of (18.1) make also sense for a functor $\mathcal{F} : \mathcal{U}^{\mathrm{opp}} \to \mathcal{A}b$. To any such $\mathcal{F}$ we associate the $\boxed{\text{sheaf } \mathcal{F}^a}$ defined as

$$\mathcal{F}^a(U) := \{s \in \prod_{P \in U} \mathcal{F}_P \mid \text{locally } s \text{ comes from } s_i \in \mathcal{F}(U_i) \text{ for } U_i \in \mathcal{U}\}$$

and coming with natural isomorphisms $\alpha : \mathcal{F}_P \xrightarrow{\sim} \mathcal{F}^a_P$. (If $P \in U \in \mathcal{U}$, then in the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}^a(U) \\ \downarrow & \swarrow_{\mathrm{pr}_P} & \downarrow{\scriptstyle p_U} \\ \mathcal{F}_P & \overset{\sim}{\underset{\alpha}{\dashrightarrow}} & \mathcal{F}^a_P \end{array}$$

the α from the universal property of $\mathcal{F}_P$ makes the quadrangle commute; by the local surjectivity of $\mathcal{F}(U) \to \mathcal{F}^a(U)$, everything commutes; hence α is an isomorphism.)

There are two special cases: (1) If $\mathcal{F}|\mathcal{U}$ has the sheaf property of (18.2), but limited to $\mathcal{U}$, then $\mathcal{F}^a$ becomes the unique sheaf with $\mathcal{F}^a|_{\mathcal{U}} = \mathcal{F}$ (the $\mathcal{U}$-sheaf homomorphism $\mathcal{F}^a|_{\mathcal{U}} \leftarrow \mathcal{F}$ is an isomorphism on the stalks).
(2) If $\mathcal{U} = \mathcal{O}pen(X)$, then $\mathcal{F}^a = a(\mathcal{F})$ is called the sheafification of $\mathcal{F}$; it does not change sheaves ($a \circ \iota = \mathrm{id}_{\mathcal{S}h}$), and it comes with natural maps $\mathcal{F} \to \mathcal{F}^a$ making $\boxed{a \dashv \iota}$ into adjoint functors, i.e. $\mathrm{Hom}_{\mathcal{S}h}(\mathcal{F}^a, \mathcal{G}) = \mathrm{Hom}_{\mathcal{P}re\mathcal{S}h}(\mathcal{F}, \iota\mathcal{G})$.
Example: The constant sheaf $\underline{A} = (\underline{A}^{\mathrm{pre}})^a$ assigns $U \mapsto A^{\pi_0(U)}$.

18.4. **Famous sheaves.** Famous ring sheaves in the classical topology are $\underline{\mathbb{C}} \subseteq \mathcal{O}^{\mathrm{an}} \subseteq \mathcal{C}^\infty$ or the sheaf of meromorphic funtions $\mathcal{M}^{\mathrm{an}}$ on $\mathbb{C}^n$ (total quotient sheaf of $\mathcal{O}^{\mathrm{an}}$). "(Locally) ringed spaces" $(X, \mathcal{O}_X)$, cf. (19.1). Then, $\mathcal{O}^* \subseteq \mathcal{O}$ (units in $\mathcal{O}$) is a sheaf abelian groups.

On $\mathbb{C}$ there are famous sequences: $0 \to \underline{\mathbb{C}} \to \mathcal{O} \xrightarrow{d} \mathcal{O} \to 0$ with $d : f(z) \mapsto f'(z)$ being locally (on the stalks) surjective: $\int_{0 \rightsquigarrow z} f(z)dz$ is a preimage of f; but there is no global preimage "$\log z$" of $1/z \in \Gamma(\mathbb{C}^*, \bullet)$.

The "exponential sequence" $\boxed{0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0}$ with $\exp : f(z) \mapsto e^{f(z)}$; here $\log(g(z))$ yields the local preimage of $g \in \mathcal{O}^*$, but $g(z) = z$ has no global one on $\mathbb{C}^*$.

Examples of *invertible sheaves* from (17.2): $\mathcal{O}(-1) :=$ sheaf of regular sections of $\widetilde{\mathbb{A}^n_k} \to \mathbb{P}^{n-1}$; locally $\mathcal{O}(-1)|_{D_+(z_i)} \cong \mathcal{O}|_{D_+(z_i)}$, but $\Gamma(\mathbb{P}^{n-1}, \bullet)$ yields 0 and $\mathbb{C}$, respectively.

18.5. **Sheaves on** $\operatorname{Spec} A$. The *structure sheaf* $\mathcal{O}_{\operatorname{Spec} A} := \widetilde{A}$ is a special case of the sheaf $\widetilde{M}$ for A-modules M given by $\boxed{\Gamma(D(f), \widetilde{M}) := M_f}$ with the natural restriction maps; $\widetilde{M}_P = M_P$. According to (18.3)(1) we check the restricted sheaf properties:

Proposition 54. *$\widetilde{M}$ is a sheaf on* $\operatorname{Spec} A$.

Proof. *"Injectivity" sheaf property:* If $m \in M$ vanishes in M_{f_i} for a covering of $D(f_i)$, then $m/1 = 0$ in all M_P, hence $m = 0$ (consider $\operatorname{Ann}(m)$).
"Surjectivity" sheaf property: $m_i/f_i \in M_{f_i}$ (note that $M_{f_i} = M_{f_i^n}$) with $m_i/f_i = m_j/f_j$ in $M_{f_if_j} \Rightarrow m_i f_i^N f_j^{N+1} - m_j f_i^{N+1} f_j^N = (f_if_j)^N(m_if_j - m_jf_i) = 0$ in M for all i, j. The $D(f_i^{N+1})$ cover $\operatorname{Spec} A \Rightarrow 1 = \sum_j \ell_j f_j^{N+1}$ for some $\ell_j \in A \rightsquigarrow m := \sum_j \ell_j m_j f_j^N$ yields $m/1 = (mf_i^{N+1})/f_i^{N+1} = (m_if_i^N)/f_i^{N+1} = m_i/f_i$. □

$\widetilde{M} \oplus \widetilde{N} = \widetilde{M \oplus N}$ and, if M is finitely presented, $\mathcal{H}om(\widetilde{M}, \widetilde{N}) = \widetilde{\operatorname{Hom}(M, N)}$ (compare both sides on the open subsets $D(f)$).

Analogously: $\boxed{\widetilde{M} \text{ on } \operatorname{Proj} S}$ for graded S-modules M. If $f \in S$ is homogeneous of positive degree, then, via $D_+(f) = \operatorname{Spec} S_{(f)}$ from (16.9) and Problem **??**, $\widetilde{M}|_{D_+(f)} = \widetilde{M_{(f)}}$. Special cases are $\mathcal{O}_{\operatorname{Proj} S}(k) := \widetilde{S(k)}$. If $\deg f = 1$, then $M_{(f)} \xrightarrow{\cdot f^k} M(k)_{(f)}$ is an isomorphism.

15.6.22 (45)

18.6. **The abelian category of sheaves.** Operations with sheaves are the usual ones among presheaves with $\boxed{\text{subsequent sheafification}}$, e.g. $\mathcal{F} \otimes_{\mathcal{O}}^{\mathcal{S}h} \mathcal{G} := (\mathcal{F} \otimes_{\mathcal{O}}^{\mathcal{P}re\mathcal{S}h} \mathcal{G})^a$ leads to a canonical $\mathcal{F} \otimes_{\mathcal{O}}^{\mathcal{P}re\mathcal{S}h} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}}^{\mathcal{S}h} \mathcal{G}$ inducing isomorphisms on the stalks. Further examples are $\operatorname{im}(\mathcal{F} \to \mathcal{G})$, $\operatorname{coker}(\mathcal{F} \to \mathcal{G})$, $\mathcal{F}/\mathcal{G}$. Composing several operations gets along with a single sheafification at the end.

Example: For $X = \operatorname{Spec} A$, the presheaves $\widetilde{M} \otimes_{\mathcal{O}}^{\text{pre}} \widetilde{N}$ and $\widetilde{M \otimes_A N}$ coincide on the sets $D(f)$, hence $M \mapsto \widetilde{M}$ commutes with $\otimes$. The same holds true for graded S-modules and the associated sheaves on $\operatorname{Proj} S$; in particular, $\mathcal{O}_{\operatorname{Proj} S}(a) \otimes \mathcal{O}_{\operatorname{Proj} S}(b) = \mathcal{O}_{\operatorname{Proj} S}(a + b)$.

Lemma 55. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups. Then*

(k) *$\mathcal{K} \to \mathcal{F}$ is isomorphic to $\ker \varphi \Leftrightarrow \forall P \in X$: $0 \to \mathcal{K}_P \to \mathcal{F}_P \to \mathcal{G}_P$ is exact;*
(c) *$\mathcal{G} \to \mathcal{C}$ is isomorphic to $\operatorname{coker} \varphi \Leftrightarrow \forall P \in X$: $\mathcal{F}_P \to \mathcal{G}_P \to \mathcal{C}_P \to 0$ is exact;*
(i) *$\operatorname{coker}(\ker \varphi) = \ker(\operatorname{coker} \varphi)$ has $\operatorname{im} \varphi_P$ as its stalks.*
(e) *$\mathcal{S}h(X)$ is an abelian category, and $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact $\Leftrightarrow \forall P \in X$: $\mathcal{F}_P \to \mathcal{G}_P \to \mathcal{H}_P$ is exact.*
(s) *On $X = \operatorname{Spec} A$, the functor $M \mapsto \widetilde{M}$ is exact. Moreover, $\Gamma(\operatorname{Spec} A, \bullet)$ is exact on "quasi coherent" sheaves, i.e. those of type $\widetilde{M}$.*

Proof. (c, $\Rightarrow$) $\mathcal{F} \mapsto \mathcal{F}_P$ is exact on $\mathcal{P}re\mathcal{S}h$; sheafification does not change the stalks.
(c, $\Leftarrow$) $\mathcal{F} \to \mathcal{G} \to \mathcal{C}$ is zero $\rightsquigarrow$ there is a map $\operatorname{coker}^{\text{pre}} \varphi \to \mathcal{C}$ inducing isomorphisms on the stalks. □

In general, both the section functors and ι are left exact functors on $\mathcal{S}h(X)$. If $\mathcal{R}$ is a ring sheaf, then tensorizing with locally free sheaves is exact; isomorphism classes of invertible sheaves (with respect to $\mathcal{R}$) form a group under $\otimes_{\mathcal{R}} \rightsquigarrow \operatorname{Pic}(X, \mathcal{O}_X)$.

18.7. **Changing the topological space.** $f: X \to Y$ continous $\rightsquigarrow f_*: \mathcal{P}re\mathcal{S}h(X) \to \mathcal{P}re\mathcal{S}h(Y)$ and $f_*: \mathcal{S}h(X) \to \mathcal{S}h(Y)$ via $(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$. This functor is left exact, but is has no good description on the level of stalks.

On the other hand, $f^{-1}: \mathcal{P}re\mathcal{S}h(Y) \to \mathcal{P}re\mathcal{S}h(X)$, $(f^{-1}\mathcal{G})(U) := \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$ is exact; it requires sheafifying to $f^{-1}: \mathcal{S}h(Y) \to \mathcal{S}h(X)$, but since $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$ it stays exact at the sheaf level.

$\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$, since both mean a system of compatible maps $\mathcal{G}(V) \to \mathcal{F}(U)$ for $f(U) \subseteq V$, i.e. $U \subseteq f^{-1}(V)$. Hence, $f^{-1} \dashv f_*$.

19. Schemes

19.1. **Locally ringed spaces.** $f = (f, f^*): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a *morphism of locally ringed spaces* $:\Leftrightarrow$ $f: X \to Y$ is continuous and $f^*: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is local, i.e. $f_P^*: \mathcal{O}_{Y,f(P)} \to (f_*\mathcal{O}_X)_{f(P)} \to \mathcal{O}_{X,P}$ satisfies $f_P^*(\mathfrak{m}_{f(P)}) \subseteq \mathfrak{m}_P$. (The latter means $f^*(\varphi) = \varphi \circ f$ if the ring sheaves consist of true functions into the base field; counter example: $\operatorname{Spec} k(x) \xrightarrow{\eta \mapsto 0} \operatorname{Spec} k[x]_{(x)}$).

Proposition 56. *The full subcategory affSch* $= \{(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = (\operatorname{Spec} A, A)\}$ *coincides with this from* (1.7)*, i.e. with* $\mathcal{R}ings^{\text{opp}}$*; similarily* $\boxed{\mathit{affSch}_k^{\text{opp}} \xrightarrow{\sim} \mathcal{A}lg_k}$.

Proof. $f: (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A) \rightsquigarrow \varphi := \Gamma(\operatorname{Spec} A, f^*): A \to B \rightsquigarrow g := (\operatorname{Spec} \varphi): \operatorname{Spec} B \to \operatorname{Spec} A$ with $g^{-1}(D_A(a)) = D_B(\varphi(a))$ and $g^*: \mathcal{O}_A \to g_*\mathcal{O}_B$ via $\varphi: A_a \to B_{\varphi(a)}$. Since, for $Q \in \operatorname{Spec} B$, the homomorphism $\varphi: A_{\varphi^{-1}(Q)} \to B_Q$ is clearly local, it remains to check that $(f, f^*) = (g, g^*)$:

The original f gives rise to local $A_{f(Q)} \to B_Q$ compatible with $\varphi: A \to B$. Hence $\varphi(A \setminus f(Q)) \subseteq B \setminus Q$ and $\varphi(f(Q)) \subseteq Q$, i.e. $f(Q) = \varphi^{-1}(Q)$. Moreover, since $f^*: A_a \to B_{\varphi(a)}$ is compatible with $\varphi = \Gamma(\operatorname{Spec} A, f^*)$, it equals $\varphi = g^*$. □

Using $\mathcal{O}_Y \to f_*\mathcal{O}_X$, the push forward functor f_* becomes $f_*: \mathcal{S}h_{\mathcal{O}_X} \to \mathcal{S}h_{\mathcal{O}_Y}$. On the other hand, if $\mathcal{G}$ is a $\mathcal{O}_Y$-module, then $f^{-1}\mathcal{G}$ is just a $f^{-1}\mathcal{O}_Y$-module, and we use $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ to define $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ (including sheafifying again). It remains just right exact, but we still have $f^* \dashv f_*$. 22.6.22 (47)

19.2. **Definition of schemes.** A locally ringed space $(X, \mathcal{O}_X)$ is called *scheme* $:\Leftrightarrow$ $X = \bigcup_i U_i$ with affine schemes $\boxed{(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})}$; gluing maps $\rightsquigarrow$ $\operatorname{Hom}_{\mathcal{S}ch}((X, \mathcal{O}_X), \operatorname{Spec} A) = \operatorname{Hom}_{\mathcal{R}ings}(A, \Gamma(X, \mathcal{O}_X))$.

Example: $\operatorname{Proj} S = \bigcup_{f \in S_{d \geq 1}} \operatorname{Spec} S_{(f)}$ with $\mathcal{O}_{\operatorname{Proj} S} = \widetilde{S}$.

Lemma 57. *$\operatorname{Spec} A$, $\operatorname{Spec} B \subseteq X$ open $\Rightarrow \exists$ covering $\{U_\nu\}$ of $(\operatorname{Spec} A) \cap (\operatorname{Spec} B)$ such that U_ν equals both $\operatorname{Spec} A_{f_\nu}$ and $\operatorname{Spec} B_{g_\nu}$ (for some $f_\nu \in A$, $g_\nu \in B$).*

Proof. w.l.o.g. $\operatorname{Spec} A \subseteq \operatorname{Spec} B$ (consider an affine covering $\{\operatorname{Spec} C_\nu\}$ of the intersection and intersect $(\forall \nu)$ both coverings of $\operatorname{Spec} C_\nu$); then, if $\operatorname{Spec} B_g \subseteq \operatorname{Spec} A$, we have that $\operatorname{Spec} B_g = (\operatorname{Spec} A) \times_{\operatorname{Spec} B} (\operatorname{Spec} B_g) = \operatorname{Spec} A_g$. □

19.3. **Constructions with schemes.** We recall a couple of basic properties mostly being treated in the previous sections for the affine case or in the exercises:

19.3.1. *Morphisms and regular functions.* A is considered the ring of regular functions on $\operatorname{Spec} A$ via $(a \in A)(P \in \operatorname{Spec} A) := \overline{a} \in A/P \subseteq \operatorname{Quot}(A/P) =: K(P)$. If $\varphi : A \to B$ gives rise to $(f = \varphi^\#) : \operatorname{Spec} B \to \operatorname{Spec} A$, then for a $Q \in \operatorname{Spec} B$ and $P := f(Q) = \varphi^{-1}(Q) \subseteq A$ we obtain the commutative diagram

$$\begin{array}{ccc} A/P & \hookrightarrow & B/Q \\ \downarrow & & \downarrow \\ K(P) & \hookrightarrow & K(Q), \end{array}$$

i.e. for $a \in A$ we have $a(f(Q)) = a(P) = \varphi(a)(Q) \in K(Q)$ implying that $\varphi(a) = a \circ f$ with both sides understood as maps on the spectra. However, an element $b \in B$ is determined by its values on $\operatorname{Spec} B$ only up to the nilradical $\sqrt{0}$.

19.3.2. *Closed embeddings.* $\varphi^\# : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding $:\Leftrightarrow \varphi : A \twoheadrightarrow B$ is surjective; the special case $A_{\text{red}} := A/\sqrt{0}$ yields a homeomorphism $(\operatorname{Spec} A)_{\text{red}} := \operatorname{Spec} A_{\text{red}} \hookrightarrow \operatorname{Spec} A$ (the "*reduced structure*" on $\operatorname{Spec} A$). (Counterexample: $k \subset K$ fields, but $\operatorname{Spec} K \to \operatorname{Spec} k$ is not a closed embedding.)
Non affine closed embeddings $\iota : Y \hookrightarrow X$ are defined locally on the target; the kernel of $A \twoheadrightarrow B$ is replaced by ideal sheaf $\mathcal{J} = \ker(\mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Y)$.

19.3.3. *Open embeddings.* $\varphi^\#$ is dominant $\Leftarrow \varphi : A \hookrightarrow B$ is injective. The standard open embeddings are $\operatorname{Spec} A_f = D(f) \subseteq \operatorname{Spec} A$. For an open embedding $j : U \hookrightarrow X$ we have $\mathcal{O}_U = j^* \mathcal{O}_X = \mathcal{O}_X|_U$.

19.3.4. *Fiber product.* In the category of affine schemes $\operatorname{Spec} A \times_{\operatorname{Spec} S} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_S B)$ is the fiber product. $\mathbb{A}^m \times_{\mathbb{Z}} \mathbb{A}^n = \mathbb{A}^{m+n}$ has *not* the product topology. $\mathbb{A}^n_A = \mathbb{A}^n \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A = \mathbb{A}^n_k \times_{\operatorname{Spec} k} \operatorname{Spec} A$ (the latter for k-algebras only).
Beyond the affine case, in $\mathcal{S}ch$, fiber products $\boxed{X \times_S Y}$ do also exist – they arise from glueing the affine construction, c.f. Problem **??**.

29.6.22 (49)

19.3.5. *Preimages.* $f : \operatorname{Spec} A \to \operatorname{Spec} B \Rightarrow f^{-1}(\operatorname{Spec} B/J) = \operatorname{Spec}(A \otimes_B B/J) = \operatorname{Spec} A/JA$ and $f^{-1}(\operatorname{Spec} B_g) = \operatorname{Spec}(A \otimes_B B_g) = \operatorname{Spec} A_{\varphi(g)}$.

19.3.6. *Scheme theoretic image.* $f : \operatorname{Spec} A \to \operatorname{Spec} B$ with $\varphi : B \to A$ induces $B \twoheadrightarrow B/\ker\varphi \hookrightarrow A$, hence $\operatorname{Spec} A \stackrel{\text{domin}}{\longrightarrow} V(\ker\varphi) \subseteq \operatorname{Spec} B$. Thus, $V(\ker\varphi) = \overline{f(\operatorname{Spec} A)}$, and $\operatorname{Spec}(B/\ker\varphi)$ is the "smallest" scheme structure on $V(\ker\varphi)$ such that f factors through.

19.3.7. *Closure.* $\operatorname{Spec}(A/(0:f^\infty)) = \overline{D(f)} \subseteq \operatorname{Spec} A$ is the scheme theoretic image of $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$. Generalization (for noetherian A) to $\overline{\operatorname{Spec} A \setminus V(J)} = \bigcup_{f\in J} \overline{D(f)} = \bigcup_{f\in J} V(0:f^\infty) = V(\bigcap_{f\in J}(0:f^\infty)) = \operatorname{Spec}(A/(0:J^\infty))$.

19.3.8. *Elimination.* $p: V(I) \subseteq \mathbb{A}^{m+n} \twoheadrightarrow \mathbb{A}^n$ corresponds to $p^*: k[\mathbf{y}] \hookrightarrow k[\mathbf{x},\mathbf{y}] \twoheadrightarrow k[\mathbf{x},\mathbf{y}]/I \Rightarrow \overline{p(V(I))} = \operatorname{Spec} k[\mathbf{y}]/\ker p^* = \operatorname{Spec} k[\mathbf{y}]/I \cap k[\mathbf{y}]$.

19.3.9. *K-rational points.* $X = \operatorname{Spec} A$; $K = \text{field} \Rightarrow X(K) \overset{\text{Yoneda}}{:=} \operatorname{Hom}(\operatorname{Spec} K, X) = \operatorname{Hom}(A,K) = \{(P,i) \mid P \in \operatorname{Spec} A,\ i: K(P) \hookrightarrow K\}$. If $A = k$-algebra and $K \supseteq k$ is an extension field, then $X_k(K) = \operatorname{Hom}_k(A,K) = \{(P,i) \mid k \subseteq K(P) \hookrightarrow K\}$. If $[K:k] < \infty \overset{\text{Prop 24(2)}}{\Longrightarrow} P \in \operatorname{MaxSpec} A$. In particular, $X_k(k) = \{\mathfrak{m} \in \operatorname{MaxSpec} A \mid A/\mathfrak{m} = k\}$, e.g. $\mathbb{A}^n_k(k) = k^n$.

19.3.10. *Tangent directions.* $A = k$-algebra, $X = \operatorname{Spec} A \Rightarrow \operatorname{Hom}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, X) = \operatorname{Hom}_k(A, k[\varepsilon]/\varepsilon^2) = \{P \in X(k)$ with tangent directions, i.e. derivation $d: A \to k\}$ ($d(fg) = f(P)d(g) + d(f)g(P)$ by the multiplicativity of $f \mapsto f(P) + \varepsilon\, d(f)$).

If $k = \overline{k}$ and $(A,\mathfrak{m})$ = local with $k \xrightarrow{\sim} A/\mathfrak{m}$, then $T_\mathfrak{m} := \operatorname{Der}_k(A,k) = \operatorname{Hom}_A(\mathfrak{m},k) = \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, in particular, $\mathfrak{m}/\mathfrak{m}^2 = T^*_\mathfrak{m}$ is the cotangent space. Thus, $(A,\mathfrak{m})$ is regular $\Leftrightarrow \dim_k T_\mathfrak{m} \geq \dim A$ becomes an equality.

19.4. **Finiteness assumptions.** Special properties of schemes and morphisms are:

(i) (Locally) *noetherian schemes* X, i.e. there is a [finite] open covering $X = \bigcup_i \operatorname{Spec} A_i$ with noetherian $A_i \Rightarrow$ every affine open $\operatorname{Spec} A \subseteq X$ has A = noetherian [and X is quasi compact].

This property is bequeathed to open and closed subschemes, and noetherian schemes imply that the underlying topological space is noetherian, i.e. that increasing chains of open subsets terminate.

(ii) $f: X \to Y$ is (locally) of *finite type* $:\Leftrightarrow f$ locally (on X as well as on Y) equals $f: \operatorname{Spec} A \to \operatorname{Spec} B$ with $B \to A$ being finitely generated algebras [and f is quasi compact]. (For those f, "(locally) noetherian" is beqeathed from Y to X.) 6.7.22 (51)

(iii) $f: X \to Y$ is *affine* $:\Leftrightarrow$ the preimages of (a covering of) open, affine $\operatorname{Spec} B \subseteq Y$ are affine open subschemes $\operatorname{Spec} A \subseteq X$.

(iv) $f: X \to Y$ is *finite* $:\Leftrightarrow f$ is affine with $B \to A$ being finite homomorphisms, i.e. A becomes a finitely generated B-module.

19.5. **Integral schemes and varieties.** X is *reduced* $:\Leftrightarrow$ all (or a cover of) open $\operatorname{Spec} A \subseteq X$ satisfy $\sqrt{0} = 0$; X is *integral* if it is, additionally, *irreducible*, i.e. if all (or a cover of) open $\operatorname{Spec} A \subseteq X$ are dense with A being integral domains.

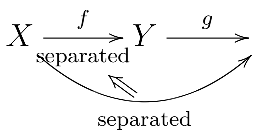
Integral schemes X have a unique *generic point* η_X (sitting in every non-empty open subset) and give rise to a function field $K(X) := \mathcal{O}_{X,\eta} = \varinjlim_{U\subseteq X} \mathcal{O}_X(U) = \operatorname{Quot} A$ for every such $\operatorname{Spec} A \subseteq X$. If $X = \operatorname{Proj} S$ (with an integral, graded ring S), then $K(X) = S_{((0))}$.

A scheme $X = (X, \mathcal{O}_X)$ is called a *variety* over k :$\Leftrightarrow$ X is integral, of finite type over $\operatorname{Spec} k$, and separated (the intersection of affine $U, V \subseteq X$ is again affine, and $\Gamma(U, \mathcal{O}), \Gamma(V, \mathcal{O})$ generate $\Gamma(U \cap V, \mathcal{O})$ as rings). Separation of a morphism $X \to S$ means that the diagonal $\Delta : X \to X \times_S X$ is a closed embedding.

20. Separated morphisms

20.1. **Simulating Hausdorff.** $f : X \to Y$ is called "*separated*" :$\Leftrightarrow$ $\Delta : X \hookrightarrow X \times_Y X$ is a closed embedding $\Leftrightarrow$ $\Delta(X) \subseteq X \times_Y X$ is a closed subset (everything is local on Y, for affine X, Y the first (and stronger) fact is always true, and for non-affine X, we can cover $X \times_Y X$ by $U_i \times_Y U_i$ and $(X \times_Y X) \setminus \Delta(X)$). *Counter example:* $[\mathbb{A}^1_k$ with double origin$] = \mathbb{TV}\big([0, \infty) \cup_{\{0\}} [0, \infty)\big)$, instead of $\mathbb{P}^1 = \mathbb{TV}\big((-\infty, 0] \cup_{\{0\}} [0, \infty)\big)$.

Properties: Closed and open *embeddings* are separated ($f : Z \hookrightarrow Y$ is affine; $U \stackrel{\Delta}{\to} U \times_Y U$ is an isomorphism); invariance under *base change*; the *composition* $X \stackrel{f}{\to} Y \stackrel{g}{\to} Z$ of separated f, g is separated $(X \times_Y [Y \stackrel{\Delta}{\to} Y \times_Z Y] \times_Y X = [X \times_Y X \to X \times_Z X])$.



$[gf$ separated $\Rightarrow f$ separated$]$: $X \hookrightarrow X \times_Y X \hookrightarrow X \times_Z X$ is altogether a closed embedding, and the latter map is injective (test $\operatorname{Spec} K \to \ldots$).

"Varieties over k" :$\Leftrightarrow$ separated schemes $X \to \operatorname{Spec} k$ of finite type.

20.2. **Intersection of affine sets.** For the absolute separateness (over $\operatorname{Spec} \mathbb{Z}$ or, for k-schemes, over $\operatorname{Spec} k$), there is the following criterion:

Proposition 58. *$X \to \operatorname{Spec} B$ is separated $\Leftrightarrow$ for open, affine $U, V \subseteq X$ the set $\boxed{U \cap V \text{ is again affine}}$, and $\Gamma(U, \mathcal{O}_X) \otimes_B \Gamma(V, \mathcal{O}_X) \twoheadrightarrow \Gamma(U \cap V, \mathcal{O}_X)$ is surjective.*

Proof.
$$\begin{array}{ccc} U \cap V & \longrightarrow & U \times_B V \\ \big\downarrow{\scriptstyle \text{open}} & & \big\downarrow{\scriptstyle \text{open}} \\ X & \xrightarrow[\text{closed?}]{\Delta} & X \times_B X \end{array}$$

$U \times_B V = \operatorname{pr}_1^{-1}(U) \cap \operatorname{pr}_2^{-1}(V)$ as open subschemes of $X \times_B X$, hence $(U \cap V) = \Delta^{-1}(U \times_B V)$. On the other hand, $U \times_B V$ is affine, and closedness is a local property. □

Consequence: $\mathbb{TV}(\Sigma, N)$, thus in particular $\mathbb{P}^n$, is separated.

20.3. **Maximal domains of definition.** Let $f, g : [X = \text{reduced}] \to [Y = \text{separated}]$
20.7.22 (53) over S with $f = g$ on a dense, open $U \subseteq X$ $\Rightarrow$ $f = g$ on X. In particular, rational maps have always a maximal domains of definition:

$F|_U$ factorizes over $\Delta(Y)$ $\Rightarrow$ $X' \subseteq X$ is a closed subscheme containing U.

$$\begin{array}{ccc} X & \xrightarrow[(f,g)]{F} & Y \times_S Y \\ \big\uparrow & & \big\uparrow \\ X' := F^{-1}(\Delta(Y)) & \longrightarrow & \Delta(Y) \end{array}$$

$X, Y = k$-varieties $\rightsquigarrow$ $\{$dominant rational maps $f : X \dashrightarrow Y\ \} = \{k$-embeddings $K(Y) \hookrightarrow K(X)\}$: If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then $\operatorname{Quot}(B) \to \operatorname{Quot}(A)$ lifts to $B \hookrightarrow A_f$. Birational $\Leftrightarrow K(Y) = K(X)$.

k = perfect $\Rightarrow$ for each field extension $K = k(\alpha_1, \ldots, \alpha_m) \supseteq k$ there is an $e \subseteq \{\alpha_1, \ldots, \alpha_m\}$ with $K \supseteq k(e) \supseteq k$ (separable|transzendent), cf. [ZS, ch. II, Th 30+31, S.104]. "Satz vom primitiven Element" $\Rightarrow$ d-dimensional k-varieties are birational equivalent to hypersurfaces in $\mathbb{P}^{d+1}$.

↗ §12

21. Quotient singularities and resolutions

21.1. **Simplicial cones.** G = [finite abelian group] acts via $\deg : \mathbb{Z}^n \to B := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$ (characters of G) linearly on $\mathbb{A}^n_{\mathbb{C}}$, i.e. $b_i := \deg(e_i) \rightsquigarrow g(x_i) = b_i(g) \cdot x_i$.

$\mathbf{x}^r \in \mathbb{C}[\mathbb{Z}^n]$ is G-invariant $\Leftrightarrow \forall g \in G: \; g(\mathbf{x}^r) = \mathbf{x}^r \Leftrightarrow \forall g \in G: \; (\deg r)(g) = 1 \Leftrightarrow \deg r = 1$; i.e. $M := \ker(\deg : \mathbb{Z}^n \to B)$ yields $\mathbb{C}[M] = \mathbb{C}[\mathbb{Z}^n]^G \subseteq \mathbb{C}[\mathbb{Z}^n]$. In particular, $\mathbb{A}^n_{\mathbb{C}}/G = \mathrm{Spec}[\mathbb{Z}^n_{\geq 0}]^G = \mathrm{Spec}\, \mathbb{C}[\mathbb{Q}^n_{\geq 0} \cap M]$.

Let $0 \to M \to \mathbb{Z}^n \to B \to 0$ be exact; dualizing $\rightsquigarrow 0 \to \mathbb{Z}^n \to N \to \mathrm{Ext}^1_{\mathbb{Z}}(B, \mathbb{Z}) \to 0$; the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ shows that $\mathrm{Ext}^1_{\mathbb{Z}}(B, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{C}^*) = G$, hence $0 \to \mathbb{Z}^n \xrightarrow{p} N \to G \to 0$. (If deg is not surjective, the we replace B by the image and change G accordingly.)

$M_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}^n$ and $p : \mathbb{Q}^n \xrightarrow{\sim} N_{\mathbb{Q}}$ are isomorphisms; $(\mathbb{Q}^n_{\geq 0})^\vee = \mathbb{Q}^n_{\geq 0} \rightsquigarrow \sigma := p(\mathbb{Q}^n_{\geq 0}) \subseteq N_{\mathbb{Q}}$ is simplicial (spanned by the $p(e^i)$) and $\boxed{\mathbb{A}^n_{\mathbb{C}}/G = \mathbb{TV}(\sigma, N)}$; on the other hand, all simplicial cones lead to abelian quotient singularities.

Example 59. $\mu_r \subseteq \mathbb{C}^*$ acts on $\mathbb{C}^n$ via $\xi \mapsto \mathrm{diag}(\xi^{a_1}, \ldots, \xi^{a_n})$ with $\mathbf{a} \in \mathbb{Z}^n$ such that $\gcd(\mathbf{a}, r) = 1$. With $\mathrm{Hom}_{\mathbb{Z}}(\mu_r, \mathbb{C}^*) = \mathbb{Z}/r\mathbb{Z}$ this yields $0 \to M \to \mathbb{Z}^n \xrightarrow{\mathbf{a}} \mathbb{Z}/r\mathbb{Z} \to 0$, hence $N = \langle \mathbb{Z}^n, \frac{1}{r}\mathbf{a} \rangle_{\mathbb{Z}} \subseteq \mathbb{Q}^n$ with $\frac{1}{r}\mathbf{a} \mapsto 1 \in \mathbb{Z}/r\mathbb{Z}$. Denote this particular $\mathbb{A}^n_{\mathbb{C}}/\mu_r =: \frac{1}{r}\mathbf{a}$.

Using coordinates in dimension two: $\mu_n \subseteq \mathbb{C}^*$ acts on $\mathbb{C}^2$ via $\xi \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$; this yields $0 \to \left(M = \mathbb{Z} \begin{bmatrix} -q \\ 1 \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} n \\ 0 \end{bmatrix} \right) \to \mathbb{Z}^2 \xrightarrow{(1,q)} \mathbb{Z}/n\mathbb{Z} \to 0$, hence the map $\mathbb{Z}^2 \to N = \mathbb{Z}^2$ is given by the matrix $\begin{pmatrix} -q & 1 \\ n & 0 \end{pmatrix}$, i.e. $X_{n,q} := \frac{1}{n}(1, q) = \mathbb{C}^2/\mu_n = \mathbb{TV}(\sigma, \mathbb{Z}^2)$ with $\boxed{\sigma := \langle (1,0), \, (-q, n) \rangle} \subseteq \mathbb{Q}^2$.

21.2. **CQS in dimension two.** Let $q \in (\mathbb{Z}/n\mathbb{Z})^*$ with $0 \leq q < n$; cone $\sigma := \langle (1,0), \, (-q, n) \rangle \subseteq \mathbb{Q}^2 = N_{\mathbb{Q}}$; let $(1,0) = s^0, \ldots, s^{m+1} = (-q, n)$ be the lattice points on the compact edges of $\nabla := \mathrm{conv}\left((\sigma \cap N) \setminus 0 \right) \rightsquigarrow \boxed{s^{i-1} + s^{i+1} = b_i \, s^i}$ with $b_i \in \mathbb{Z}_{\geq 2}$, $i = 1, \ldots, m$ ($0, s^i, s^{i+1}$ are vertices of an elementary triangle $\Rightarrow \{s^i, s^{i+1}\}$ are $\mathbb{Z}$-bases of N).

Definition 60. $c_i \in \mathbb{Z}_{\geq 2} \rightsquigarrow$ continued fraction $[c_1, \ldots, c_\ell] := c_1 - 1/[c_2, \ldots, c_\ell]$.

Proposition 61. $n > 1 \Rightarrow n/q = [b_1, \ldots, b_m]$.

Proof. Since $(1,0)+s^2=b_1\,(0,1)$, one obtains $s^2=(-1,b_1)\Rightarrow s^2$ is the lowest lattice point on $(-1,*)$ above $\mathbb{Q}_{\geq 0}(-1,n/q)$, i.e. $b_1=\lceil n/q\rceil$ $(=\lfloor n/q\rfloor+1$ if $q\neq 1)$. Induction: The cone $\sigma':=\langle(0,1),(-q,n)\rangle$ cut off from σ along s^1 becomes $\sigma'\cong\langle(1,0),(n,q)\rangle$ after the coordinate change $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$; afterwards, the first entry of $(n,q)\Rightarrow(-q',n')$ will be normalized within $(\mathbb{Z}/q\mathbb{Z})^*$ toward $-q'=n-\lceil n/q\rceil q=n-b_1q\Rightarrow q/(b_1q-n)=[b_2,\ldots,b_m]\Rightarrow 1/[b_2,\ldots,b_m]=q'/n'=(b_1q-n)/q=b_1-n/q$. □

21.3. **Duality.** $\{s^0,\ldots,\,s^{m+1}\}$ is the Hilbert basis of σ (since $\{s^i,s^{i+1}\}$ are $\mathbb{Z}$-bases of N and ∇ is convex); denote by $\{t^0,\ldots,\,t^{k+1}\}$ the Hilbert basis of $\sigma^\vee=\langle[0,1],\,[n,q]\rangle\cong\langle[0,1],\,[n,q-n]\rangle\cong\langle[1,0],\,[q-n,n]\rangle\rightsquigarrow\boxed{t^{j-1}+t^{j+1}=a_jt^j}$ with $n/(n-q)=[a_1,\ldots,a_k]$. $\rightsquigarrow$ equations $z_{j-1}z_{j+1}=z_j^{a_j}$ of $X_{n,q}\subseteq\mathbb{A}^{k+2}$.

$\ddot\partial\nabla:=\partial\nabla\setminus\partial\sigma=$ union of the compact edges of ∇ *without* the two extremal vertices, i.e. $\ddot\partial\nabla\cap N=\{s^1,\ldots,s^m\}$; analogously $\{t^1,\ldots,t^k\}\subset\ddot\partial\Delta\subset\Delta\subset\sigma^\vee$.

Proposition 62. 1) $\mathcal{P}:=\{(i,j)\in[1,m]\times[1,k]\,|\,\langle s^i,t^j\rangle=1\}\subset\big(\mathbb{Z}^2,\,(\leq,\leq)\big)$ *is totally ordered; it forms a path leading from* $(1,1)$ *to* (m,k) *along horizontal or vertical edges only.*
2) *Length of the horizontal edge* $(\bullet,j)$ *in* $\mathcal{P}=(a_j-2)=$ *length of* $\nabla\cap[t^j=1]$.
3) *Length of the vertical edge* $(i,\bullet)$ *in* $\mathcal{P}=(b_i-2)=$ *length of* $\Delta\cap[s^i=1]$.
$\rightsquigarrow$ Riemenschneider's point diagram; $\ddot\partial\Delta/\ddot\partial\nabla$-duality (vertices $\widehat{=}\,a_j/b_i\geq 3$).

Proof. (i) $\overline{s^is^{i+1}}\subseteq$ edge of $\nabla\Rightarrow\overline{s^is^{i+1}}\subset[t=1]$ with $t\in\{t^1,\ldots,t^k\}=\ddot\partial\Delta\cap M$: $\{s^i,s^{i+1}\}$ is basis $\Rightarrow t\in M$; $[t=1]$ meets both σ-edges $\Rightarrow t\in\operatorname{int}\sigma^\vee$; all splittings $t=t'+t''$ in $\sigma^\vee\cap M$ contradict $s^i\in\operatorname{int}\sigma$ (oder $s^{i+1}\in\operatorname{int}\sigma$).

(ii) Every $[t^j=1]$ cuts off a $\ddot\partial\nabla$-face:
Edge $\overline{t^jt^{j+1}}\overset{(i)}{\hookrightarrow}[s^i=1]\Rightarrow s^i\in[t^j=1]$; moreover $[t^j\leq 0]\cap\sigma=\{0\}$.

(i)+(ii) $\Rightarrow$ (1) and [length of the horizontal $\mathcal{P}$-edges] = [length of the ∇-edges].

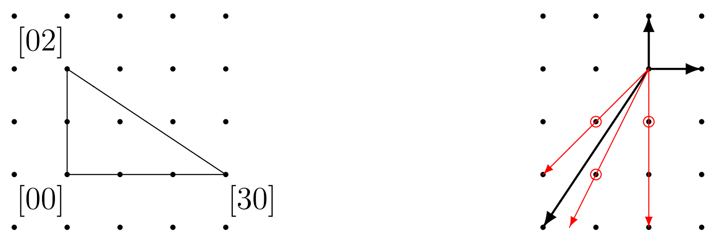
(iii) $\{s^i,\ldots,s^{i+\ell}\}=\nabla\cap[t^j=1]$-edge with $\ell\geq 1$ in the direction $v:=s^{i+1}-s^i\Rightarrow\langle v,t^j\rangle=0\Rightarrow\langle v,t^{j-1}\rangle=1$ ($\{t^{j-1},t^j\}=$ basis) $\Rightarrow\langle s^{i+\ell},t^{j-1}\rangle=\langle s^i,t^{j-1}\rangle+\ell$, hence $0=\langle s^{i+\ell},\,t^{j-1}+t^{j+1}-a_jt^j\rangle=(1+\ell)+1-a_j$. □

Gap between Alg2 and Alg3

21.4. **Weighted projective spaces.** Let $\mathbf{w}\in\mathbb{Z}^{n+1}$ be primitive $\rightsquigarrow\mathbb{P}(\mathbf{w}):=\mathbb{A}^{n+1}\setminus\{0\}/\mathbb{C}^*$ with $t\,(z_0,\ldots,z_n):=(t^{w_0}z_0,\ldots,t^{w_n}z_n)$, i.e. in the language of (21.1), $\deg:\mathbb{Z}^{n+1}\overset{\mathbf{w}}{\to}\mathbb{Z}=\operatorname{Hom}_{\text{alGr}}(\mathbb{C}^*,\mathbb{C}^*)$, i.e. $\mathbb{P}(\mathbf{w})=\operatorname{Proj}\mathbb{C}[\mathbf{z}]$ with this grading. The charts are $D_+(z_i)=\operatorname{Spec}k[\ker\mathbf{w}\cap C_i^\vee]$ where $C_i:=\partial_i\mathbb{Q}^{n+1}_{\geq 0}$. Thus $\mathbb{P}(\mathbf{w})=\boxed{\mathbb{TV}(\pi(\partial\mathbb{Q}^{n+1}_{\geq 0}),\,\mathbb{Z}^{n+1}/\mathbf{w}\mathbb{Z})}$. If $\{\mathbf{w},a^1,\ldots,a^n\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$, then the chart $D_+(z_i)$ has a cyclic quotient singularity of type $\frac{1}{w_i}(a_i^1,\ldots,a_i^n)$.

General procedure: If $\Delta\subseteq M_{\mathbb{Q}}$ is a polyhedron with cone $\Delta:=\mathbb{Q}_{\geq 0}(\Delta,1)\subseteq M_{\mathbb{Q}}\oplus\mathbb{Q}$, then $(\operatorname{cone}\Delta)^\vee \subseteq N_{\mathbb{Q}} \oplus \mathbb{Q}$ projects to the inner normal fan $\mathcal{N}(\Delta)$. In particular, the fan of $\mathbb{P}(\mathbf{w})$ equals the normal fan of $\Delta_{\mathbf{w}} := [\mathbf{w} = 1] \cap \mathbb{Q}^{n+1}_{\geq 0}$ (or integral multiples).

Example 63. The singular charts of $\mathbb{P}(1,2,3)$ are $\operatorname{Spec}\mathbb{C}[z_0^2/z_1,\, z_0z_2/z_1^2,\, z_2^2/z_1^3]$ and $\operatorname{Spec}\mathbb{C}[z_0^3/z_2,\, z_0z_1/z_2,\, z_1^3/z_2^2]$ with an $A_1 = \frac{1}{2}(1,-1)$ and an $A_2 = \frac{1}{3}(1,-1)$-singularity, respectively. Projecting $6\Delta_{\mathbf{w}} = \operatorname{conv}\{[600],[030],[002]\} \subseteq \mathbb{Q}^3 \xrightarrow{\operatorname{pr}_{23}} \mathbb{Q}^2$ yields



$6\Delta_{(1,2,3)} \subseteq M_{\mathbb{Q}}$ (subdivided) Fan of $\mathbb{P}(1,2,3)$ in $N_{\mathbb{Q}}$.

21.5. **Toric Resolutions.** Let $\sigma \subseteq N_{\mathbb{Q}}$ be a full-dimensional polyhedral cone: Hilbert basis $E \subseteq \sigma^\vee \cap M \rightsquigarrow 0 \in \mathbb{TV}(\sigma) \subseteq \mathbb{A}^E$ corresponds to the ideal $\mathfrak{m}_0 = (z_e \mid e \in E) \subseteq k[\mathbf{z}] \twoheadrightarrow k[\sigma^\vee \cap M] \Rightarrow \mathfrak{m}_0/\mathfrak{m}_0^2 = k^E$, but $\dim \mathbb{TV}(\sigma) = \operatorname{rank} N =: n$. In particular, $\mathbb{TV}(\sigma)$ is smooth in $0 \Leftrightarrow (\sigma, N) \cong (\mathbb{Q}^n_{\geq 0}, \mathbb{Z}^n) \Leftrightarrow \mathbb{TV}(\sigma) \cong \mathbb{A}^n$.

(i) If σ is as in (21.2), then the subdivision into the fan Σ with $\Sigma(1) = \{s^0, \ldots, s^{m+1}\}$ yields a resolution $\pi : \mathbb{TV}(\Sigma) \to \mathbb{TV}(\sigma)$ of the isolated singularity $0 \in \mathbb{TV}(\sigma)$, e.g. the red rays in the right figure in Example 63 (with self intersection numbers $(\overline{\operatorname{orb}}(s^i)^2) = -b_i$ similarily to $(E^2) = -1$ in the blow up of $\mathbb{A}^2$).

(ii) Every $\mathbb{TV}(\sigma)$ allows such a resolution: First, subdivide σ into a simplicial fan; afterwards, if $\sigma = \langle a^1, \ldots, a^n \rangle \subseteq \mathbb{Q}^n$ is still not smooth, then there is an $a^* \in \mathbb{Z}^n \cap \sum_{i=1}^n [0,1) a^i$, hence the cones $\tau_i := \langle a^*, a^1, \ldots, \hat{a^i}, \ldots, a^n \rangle \subseteq \sigma$ improve the situation: With $a^* = \sum_{i=1}^n \lambda_i a^i$ we have that $\operatorname{vol}(\tau_i) = \lambda_i \operatorname{vol}(\sigma)$. Eventually, we obtain a $\boxed{\text{“smooth” subdivision } \Sigma \leq \sigma}$.

21.6. **Resolutions via Newton polytopes.** Let $f \in \mathbb{C}[\mathbf{x}]$ with $f(0) = 0 \rightsquigarrow$ the hypersurface $V(f) = \operatorname{Spec}\mathbb{C}[\mathbf{x}]/(f)$ is regular (smooth) in $0 \Leftrightarrow x_1, \ldots, x_n$ are linearly dependent in $(\mathbf{x})/(\mathbf{x}^2, f) \Leftrightarrow f'(0) = (\partial_1 f(0), \ldots, \partial_n f(0)) \neq 0$.

Let $g \in \mathbb{C}[\mathbf{x}]$ and $I \subseteq [n] := \{1, \ldots, n\}$ with $J := [n] \setminus I$. Then $V(g)$ is called *transversal* to the coordinate hyperplane $\mathbb{C}^J = V(x_I)$ in $c = (c_I = 0, c_J) \in V(g) \cap \mathbb{C}^J$ $:\Leftrightarrow V(g|_{\mathbb{Z}^J})$ is smooth in c_J (or in any $(*, c_J)$) $\Leftrightarrow \exists\, \partial_{j(\in J)}(g|_{\mathbb{Z}^J})(c) \neq 0$. Since for $I' \subseteq I$ (hence $J' \supseteq J$) the $\mathbb{C}^J$-transversality implies that with $\mathbb{C}^{J'}$ (all monomials of $g|_{\mathbb{Z}^{J'}}$ not in $g|_{\mathbb{Z}^J}$ yield 0 whenever applied to c), we obtain:

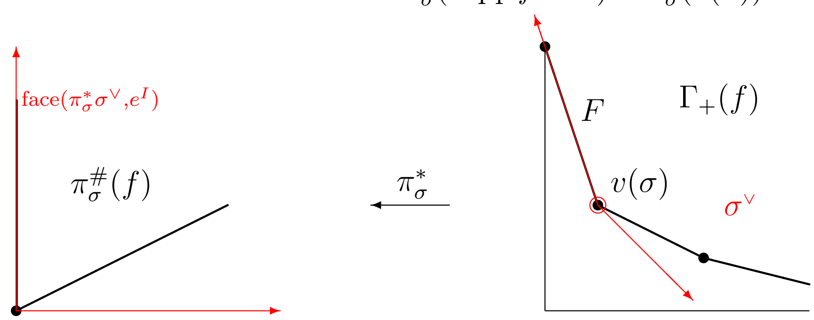
True transversality to the origin ($I = [n]$) is not possible – it can only be obtained via $0 \notin V(g)$, i.e. $g(0) \neq 0$. $V(g)$ is transversal to all coordinate planes in $\mathbb{C}^n$ (hence smooth in $\mathbb{C}^n \setminus (\mathbb{C}^*)^n$) $\Leftrightarrow$ there is no $J \subsetneq [n]$ such that the system $g|_{\mathbb{Z}^J} = \partial_\bullet(g|_{\mathbb{Z}^J}) = 0$ has a solution inside the torus $(\mathbb{C}^*)^n$.

Varchenko's resolution of hypersurfaces: $f \in \mathbb{C}[x_1, \ldots, x_n]$ with $f(0) = 0 \rightsquigarrow$ "Newton polyhedra" $\Gamma(f) := \operatorname{conv}(\operatorname{supp} f) \subseteq \mathbb{Q}^n_{\geq 0}$ and $\Gamma_+(f) := \Gamma(f) + \mathbb{Q}^n_{\geq 0}$; let $\Sigma \leq \mathcal{N}(\Gamma_+(f)) \leq \mathbb{Q}^n_{\geq 0}$ be a smooth subdivision $\rightsquigarrow X := \pi^{\#}(V(f)) =$ strict transform via $\pi : \mathbb{TV}(\Sigma) \to \mathbb{C}^n$.

Proposition 64. *Assume that $0 \in V(f)$ is an isolated singularity and let f be* $\boxed{\textit{non-degenerate}}$ *on the Newton boundary, i.e. for no compact face $F \leq \Gamma_+(f)$, the equations $\partial_\bullet(f|_F) = 0$ have a common solution inside $(\mathbb{C}^*)^n$. Then X is smooth in a neighborhood of $E := \pi^{-1}(0) \subseteq \mathbb{TV}(\Sigma)$, and X is transversal to E.*

Proof. Every $\sigma = \langle a^1, \ldots, a^n \rangle \in \Sigma$ has an associated vertex $v(\sigma) \in \Gamma_+(f)$. The map $\pi_\sigma : \mathbb{C}^n \cong \mathbb{TV}(\sigma) \to \mathbb{TV}(\mathbb{Q}^n_{\geq 0}) = \mathbb{C}^n$ is given on the N-level by $A : (\mathbb{Q}^n_{\geq 0}, \mathbb{Z}^n) \xrightarrow{\sim} (\sigma, \mathbb{Z}^n) \overset{\text{id}}{\hookrightarrow} (\mathbb{Q}^n_{\geq 0}, \mathbb{Z}^n)$, i.e. $A : \mathbb{Z}^n \to \mathbb{Z}^n$ sends $e^i \mapsto a^i$. Pulling back functions means $\pi^*_\sigma(x^r) = x^s$ with $s = A^T r$, i.e. $\langle e^i, s \rangle = \langle a^i, r \rangle$. In particular, $\pi^*_\sigma(\Gamma_+(f)) \subseteq \pi^*_\sigma(v(\sigma)) + \mathbb{N}^n$, i.e. $\pi^*_\sigma(f) = x^{\pi^*_\sigma(v(\sigma))} \pi^{\#}_\sigma(f)$ with $\pi^{\#}_\sigma(f)(0) \neq 0$. Moreover, $\pi^{-1}_\sigma(0) \subseteq \mathbb{C}^n \setminus (\mathbb{C}^*)^n$ and

$$\begin{aligned} \operatorname{face}(\pi^*_\sigma \sigma^\vee, e^I) \cap \operatorname{supp} \pi^{\#}_\sigma(f) &= \pi^*_\sigma\big(\operatorname{face}(\sigma^\vee, A(e^I)) \cap \operatorname{supp} f / x^{v(\sigma)}\big) \\ &= \pi^*_\sigma(\operatorname{supp} f \cap F) - \pi^*_\sigma(v(\sigma)) \end{aligned}$$



for some (compact) face $F \leq \Gamma_+(f)$. Finally, since we just care about solutions in $(\mathbb{C}^*)^n$, we may use that (i) π_σ becomes an automorphism, (ii) the monomial $x^{v(\sigma)}$ does not matter, and (iii) we may replace $\partial_i x^r$ by $x_i \, \partial_i x^r = \langle e^i, r \rangle x^r$. □

Remark: Logarithmic differentials $df/f = d\log(f)$ perform an altogether linear assignment $(r \in M) \mapsto \mathbf{x}^r \mapsto d\mathbf{x}^r/\mathbf{x}^r$, hence involve the same constant matrix describing their coordinate change. Dually, each $a \in N$ provides in a coordinate free way a derivation $\partial_a : \mathbb{C}[M] \to \mathbb{C}[M]$, $\mathbf{x}^r \mapsto \langle a, r \rangle \mathbf{x}^r$.

22. Closed subschemes and quasi coherent sheaves

19.10.22 (57) 22.1. **Pull back of sheaves.** Let $f : X \to Y$ continuous $\rightsquigarrow f_* : \mathcal{S}h(X) \to \mathcal{S}h(Y)$; for ringed spaces this even yields $\mathcal{M}od_{\mathcal{O}_X} \to \mathcal{M}od_{\mathcal{O}_Y}$.

Example: If $f = \operatorname{Spec} \varphi : \operatorname{Spec} A \to \operatorname{Spec} B$, then $(f_* \widetilde{M})(D(b)) = \widetilde{M}(f^{-1} D(b)) = \widetilde{M}(D(\varphi b)) = M_{\varphi(b)} = M_b$ shows that $f_* \widetilde{M} = \widetilde{M}^{(B \to A)}$ where the latter means M understood as a B-module. In general, f_* behaves badly with stalks.

Let $\mathcal{G} \in \mathcal{S}h(Y) \rightsquigarrow \boxed{f^{-1}\mathcal{G} := [U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)]^a}$; since $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$, this functor is exact. Moreover, $f^{-1} \dashv f_*$ on $\mathcal{S}h(X)$ and $\mathcal{S}h(Y)$: If $\mathcal{F} \in \mathcal{S}h(X)$, then elements of $\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$ and $\operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$ both correspond to systems of compatible homomorphisms $\mathcal{G}(V) \to \mathcal{F}(U)$ for $U \subseteq f^{-1}(V)$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces $\Rightarrow \mathcal{O}_Y \to f_*\mathcal{O}_X$ provides $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X \rightsquigarrow$ two further variants of f^{-1}:

a) $\mathcal{G} = \mathcal{O}_Y$-module $\rightsquigarrow f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ becomes an $\mathcal{O}_X$-module with $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$; this functor f^* remains right exact; $\boxed{f^* \dashv f_*}$; there is a canonical $\Gamma(Y, \mathcal{G}) \to \Gamma(X, f^*\mathcal{G})$ (corresponds to $\mathrm{id}_{\mathcal{S}h(Y)} \to f_*f^*$); $f^*(\mathrm{vb}) = \mathrm{vb}$, and $f^*(\text{globally generated}) = [\text{globally generated}]$. If $f = \operatorname{Spec}\varphi$, then $f^*\widetilde{N} = \widetilde{A \otimes_B N}$.
(*Proof:* If $\mathcal{F}|\operatorname{Spec}(A)$, then every A-linear $M \to \Gamma(\mathcal{F})$ provides an $\mathcal{O}_A$-linear map $\widetilde{M} \to \mathcal{F}$, e.g. $\widetilde{\Gamma(\mathcal{F})} \to \mathcal{F}$. This can be used for $A \otimes_B N \to \Gamma(\operatorname{Spec} A, f^*(\widetilde{N}))$; on the stalks in $P \in \operatorname{Spec} A$ this becomes an isomorphism.)

b) $\mathcal{J} \subseteq \mathcal{O}_Y$ *ideal sheaf* $\rightsquigarrow \boxed{f^{-1}\mathcal{J} \cdot \mathcal{O}_X} := \operatorname{im}(f^*\mathcal{J} \to f^*\mathcal{O}_Y = \mathcal{O}_X)$ is an ideal in $\mathcal{O}_X$. If $f = \operatorname{Spec}\varphi$, then $f^{-1}\widetilde{J} \cdot \mathcal{O}_A = \widetilde{JA}$.

Example: In (21.6), let $I := (\operatorname{supp} f) \subseteq k[\mathbf{x}]$ be the smallest monomial ideal with $f \in I$. Using $\pi : \mathbb{TV}(\Sigma) \to \mathbb{C}^n$ we get $\pi_\sigma^{-1}\widetilde{I} \cdot \mathcal{O}_{\mathbb{TV}(\sigma)} = (\pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\mathbb{TV}(\Sigma)})|_{\mathbb{TV}(\sigma)} = \widetilde{(x^{v(\sigma)})}$, i.e. the pull back becomes principal on all charts.

22.2. **Quasi coherent sheaves.** X = scheme $\rightsquigarrow \mathcal{O}_X$-module $\mathcal{F}$ is called *quasi coherent* $:\Leftrightarrow \exists$ open, affine covering by some $U_i = \operatorname{Spec} A_i$ with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for suitable A_i-modules M_i.

Proposition 65. a) $\mathcal{O}_{\operatorname{Spec} A}$*-modules* $\mathcal{F}$ *equal some* $\widetilde{M} \Leftrightarrow$ *for all* $f \in A$ *the maps* $\varphi_f : \Gamma(X, \mathcal{F}) \otimes_A A_f \to \Gamma(D(f), \mathcal{F})$ *are isomorphisms.* (Consider $\varphi : \Gamma(X, \mathcal{F})^\sim \to \mathcal{F}$.)

b) *Kernel, image, and cokernel of quasi coherent* $\mathcal{O}_X$*-modules are quasi coherent.*

c) $f : X \to Y \Rightarrow f^*$ *and* f_* *preserve "quasi coherent".*

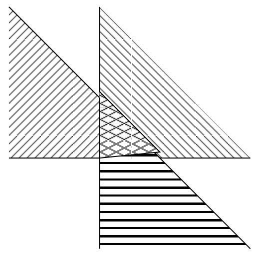
d) $\mathcal{F} =$ *quasi coherent on* $\operatorname{Spec} A \Rightarrow \mathcal{F}$ *equals some* $\widetilde{M}$.

Proof. (c) Y = affine; $\mathcal{F}|_{U_i} = \widetilde{M_i}$ for some covering $\psi_i : U_i = \operatorname{Spec} A_i \hookrightarrow X$; let $\phi_{ij\nu} : V_{ij\nu} = \operatorname{Spec} B_{ij\nu} \hookrightarrow (U_i \cap U_j) \hookrightarrow X$ be an affine covering of the intersections. Then $0 \to \mathcal{F} \to \oplus_i (\psi_i)_*\mathcal{F}|_{U_i} \to \oplus_{i,j,\nu} (\phi_{ij\nu})_*\mathcal{F}|_{\operatorname{Spec} C_{ij\nu}}$ is exact, and one applies f_*.
(d) follows with the same argument for $f = \mathrm{id}$. □

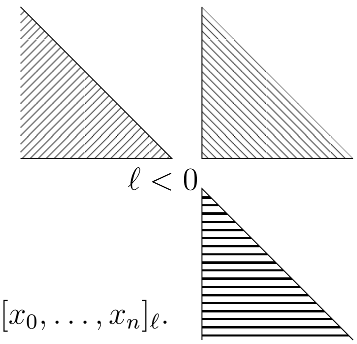
Example: M = graded S-module $\rightsquigarrow \widetilde{M}$ of (18.5), e.g. $\mathcal{O}_{\operatorname{Proj} S}(\ell) = \widetilde{S(\ell)}$ are quasi coherent on $\operatorname{Proj} S$. With the notation of (16.5), we can look at the example $X = \mathbb{P}^n_k$: Locally, $\mathcal{F} := z_0^{-\ell} \cdot \mathcal{O}_{\mathbb{P}^n}(\ell) \subseteq j_*\mathcal{O}_{(k^*)^n} \subseteq K(\mathbb{P}^n)$ is $(z_i/z_0)^\ell \mathcal{O}_{U_i} = k[\ell \cdot f_i + (\sigma_i^\vee \cap M)]$ (with $f_0 = 0$), hence

$$\Gamma(\mathbb{P}^n_k, \mathcal{F}) = k[\cap_i(\ell f_i + \sigma_i^\vee) \cap M] = \begin{cases} k[x_1/x_0, \dots, x_n/x_0]_{\leq \ell} & \text{if } \ell \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\ell \geq 0$



$\ell < 0$



$$\Rightarrow \Gamma(\mathbb{P}^n_k, \mathcal{O}(\ell)) = x_0^\ell \cdot k[x_1/x_0, \dots, x_n/x_0]_{\leq \ell} = k[x_0, \dots, x_n]_\ell.$$

Problem 66. Let E be a locally free $\mathcal{O}_X$-module of rank r on a scheme X, i.e., there exists an affine, open covering $\{U_i\}_{i \in I}$ of X together with isomorphisms $\phi_i : E|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^r$.

a) Show that $E^\vee := \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ is locally free of rank r, too. Moreover, it satisfies $E^{\vee\vee} = E$.

b) Analogously to the same construction on modules over rings, we define

$$(\operatorname{Sym}^d E)(U \subseteq X) := \operatorname{Sym}^d E(U).$$

Thus, we obtain via $\mathcal{A} := \oplus_{d \geq 0} \operatorname{Sym}^d E$ a ring sheaf on X. How does $\mathcal{A}$ look like for the special case $E = \mathcal{O}_X \cdot s_1 \oplus \ldots \oplus \mathcal{O}_X \cdot s_r$ being a free $\mathcal{O}_X$-module?

c) Let $\pi : \operatorname{Spec}_X \mathcal{A} \to X$ be the gluing of the schemes and morphisms $\operatorname{Spec} \mathcal{A}(U_i) \to U_i = \operatorname{Spec} B_i$ where $\{U_i\}_{i \in I}$ is like in (a). Show that π is a *vector bundle*, i.e., it is *locally* isomorphic to $X \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{A}^r_{\mathbb{Z}} \to X$, an the transition maps $U_i \times \mathbb{A}^r \hookleftarrow \pi^{-1}(U_i \cap U_j) \hookrightarrow U_j \times \mathbb{A}^r$ are linear in the fibers (on $U_i \cap U_j$).

d) The sets of sections of π – in the original meaning of this word, i.e., $s_U : U \to \pi^{-1}(U)$ with $\pi \circ s_U = \mathrm{id}_U$) form a sheaf of $\mathcal{O}_X$-modules on X. Accordingly, we denote $\operatorname{Spec}_X \mathcal{A}$ as $\mathbb{A}$(name of this sheaf).

e) For $X = \mathbb{P}^1_k$ and $E = \mathcal{O}_{\mathbb{P}^1}(\ell)$ describe $\pi : \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) \to \mathbb{P}^1$ in the toric language, i.e., via fans. Can you spot the toric among the global sections of π (again, in the original, literal meaning of the word)?

(*Hint:* For the bundle $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{P}^1$ we do already know the result – it has to be the blowing up $\widetilde{\mathbb{A}}^2 \to \mathbb{P}^1$.)

22.3. **Closed embeddings.** A morphism $i : Z \to X$ between noetherian schemes is called a *closed embedding* ("Z is a *closed subscheme* of X") $:\Leftrightarrow$ the following, mutually equivalent conditions are satisfied:

Proposition 67. *$i : Z \hookrightarrow X$ is locally* (with respect to X) *isomorphic to* $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A \Leftrightarrow$ *topologically, i is a closed embedding plus $\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z$ is surjective.*

Proof. First, since this is a special case of finite maps, both local versions agree. Then, for ($\Leftarrow$), the ideal sheaf $\mathcal{I} := \ker(\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z)$ is quasi coherent, and $\mathcal{I}$ or $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$ encode Z completely: $Z' := \operatorname{Spec}\mathcal{O}_X/\mathcal{I}$ and Z equal $\{P \in X \mid (\mathcal{O}_X/\mathcal{I})_P \neq 0\}$ and one applies i^{-1} to $i_*\mathcal{O}_{Z'} = i_*\mathcal{O}_Z$. $\square$

Examples: 1) $\mathbb{P}^{n-1} = V(z_0) \hookrightarrow \mathbb{P}^n$ has $\mathcal{I} \cong \mathcal{O}_{\mathbb{P}^n}(-1)$ as its ideal sheaf. More general, $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{V(F_d)} \to 0$ for a homogeneous $F_d \in \mathbb{C}[\mathbf{z}]_d$.

2) $f : X \to Y$ morphism of schemes, $W \subseteq Y$ closed subscheme with ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Y \Rightarrow f^{-1}(W) := W \times_Y X \subseteq X$ has the ideal sheaf $f^{-1}\mathcal{J} \cdot \mathcal{O}_X$.

3) $(X, \mathcal{O}_X)$ is called (quasi) affine/*projective* $:\Leftrightarrow X$ is (open in a) closed subscheme of $\mathbb{A}^n_k/\mathbb{P}^n_k$.

4) $\boxed{X_{\text{red}} \subseteq X}$ is the smallest closed subscheme on the topological space X.

5) The closed orbits in toric varieties $\mathbb{TV}(\Sigma) = \mathbb{P}(\Delta)$ are $\overline{\operatorname{orb}}(\tau) = \mathbb{TV}(\overline{\Sigma}, N_\tau) = \mathbb{P}(F_\tau)$ with $N_\tau := N/\operatorname{span}(\tau)$, $F_\tau := \operatorname{face}(\Delta, \tau)$, and $\overline{\Sigma} := \{\overline{\sigma} \subseteq (N_\tau)_\mathbb{Q} \mid \Sigma \ni \sigma \supseteq \tau\}$: Seminar 10/

$$\begin{array}{ccccccc}
\mathbb{TV}(0, N_\tau) & = & \operatorname{Spec}\mathcal{C}[\tau^\perp \cap M] & = & \operatorname{orb}(\tau) & \xrightarrow{\text{closed}} & \mathbb{TV}(\tau) \\
\downarrow{\scriptstyle\text{open}} & & \downarrow{\scriptstyle\text{open}} & & \downarrow{\scriptstyle\text{open}} & & \downarrow{\scriptstyle\text{open}} \\
\mathbb{TV}(\overline{\sigma}, N_\tau) & = & \operatorname{Spec}\mathcal{C}[\sigma^\vee \cap \tau^\perp \cap M] & = & \overline{\operatorname{orb}}(\tau) & \xrightarrow{\text{closed}} & \mathbb{TV}(\sigma).
\end{array}$$

Moreover, $\boxed{\mathbb{TV}(\Sigma) = \sqcup_{\sigma\in\Sigma} \operatorname{orb}(\sigma)}$ is a stratification with $\operatorname{orb}(\sigma) \subseteq \overline{\operatorname{orb}}(\tau) \Leftrightarrow \tau \le \sigma$ and $\dim \operatorname{orb}(\sigma) + \dim \sigma = \operatorname{rank} N$. (For the underlying topological spaces we know that $\mathbb{TV}(\sigma) = \operatorname{Hom}_{\text{sGrp}}(\sigma^\vee \cap M, \mathbb{C})$, and each such map $\varphi : \sigma^\vee \cap M \to \mathbb{C}$ gives rise to a face $\tau \le \sigma$ with $\sigma^\vee \cap \tau^\perp \cap M = \varphi^{-1}(\mathbb{C}^*)$, i.e. $\varphi : \tau^\perp \cap M \to \mathbb{C}^*$ and $\varphi(\sigma^\vee \setminus \tau^\perp) = 0$.)

22.4. **The scheme theoretic image.** This is the globalization of (19.3.6). Let $Z \subseteq X$ be a closed subscheme with $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I} \Rightarrow$ the "*scheme theoretic image* $\overline{f(Z)}$" of Z along $f : X \to Y$ is given by the ideal sheaf $\mathcal{J} := (f^*)^{-1}(f_*\mathcal{I} \subseteq f_*\mathcal{O}_X) \subseteq \mathcal{O}_Y$; it provides the $\boxed{\text{“smallest” scheme}}$ structure P on $\overline{f(Z)} \subseteq Y$ such that $f|_Z$ factors through it. If Z is reduced, then so is $\overline{f(Z)}$.

Proof:

$$\begin{array}{ccccc}
\mathcal{O}_Y & \xrightarrow{f^*} & f_*\mathcal{O}_X & & \\
\downarrow & & \downarrow & & \\
\mathcal{O}_{\overline{f(Z)}} := \mathcal{O}_Y/\mathcal{J} & \hookrightarrow & f_*\mathcal{O}_X/f_*\mathcal{I} & \hookrightarrow & f_*(\mathcal{O}_X/\mathcal{I})
\end{array}$$

$\Rightarrow \mathcal{J} := (f^*)^{-1}(f_*\mathcal{I})$ is the maximal ideal sheaf possible for such a P

Locally on $Y = \operatorname{Spec} B$: $X = \bigcup_i \operatorname{Spec} A_i \rightsquigarrow A := \prod_i A_i$ and $\operatorname{Spec} A = \coprod_i \operatorname{Spec} A_i \to X \to \operatorname{Spec} B$ via $\varphi : B \to A$; thus $P = \operatorname{Spec} B/\varphi^{-1}(\prod_i I_i)$. Since $\overline{B} \hookrightarrow \overline{A}$ is injective, we see once more that $\operatorname{Spec}\overline{A} \to \operatorname{Spec}\overline{B}$ is dominant. $\square$

Examples: $f : X \hookrightarrow Y$ open embedding $\rightsquigarrow X \subseteq \overline{X} \subseteq Y$; the closure of graphs Γ_f of rational maps $f : X \dashrightarrow Y$ in $X \times Y$ ($\Gamma_{(\mathbb{A}^n \setminus 0) \to \mathbb{P}^{n-1}} \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1}$ redefines the blowing up); $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$ (this is used for the strict transforms).

26.10.22 (57)

22.5. **The universal property of the blowing up.** Generalizing (17.4) to arbitrary ideals $I \subseteq A \rightsquigarrow$ "Rees-ring" $\oplus_{d\geq 0} I^d$ from (11.5) $\rightsquigarrow$ $\boxed{\mathrm{Bl}_A(I) := \mathrm{Proj} \oplus_{d\geq 0} I^d}$ $= \mathrm{Proj} \oplus_{d\geq 0} I^d t^d \xrightarrow{\pi} \mathrm{Spec}\, A$. With $I = (g_1, \ldots, g_n)$ we have

$$\mathrm{Bl}_A(I) \supseteq D_+(g_i) = \mathrm{Spec}(\oplus_{d\geq 0} I^d t^d)_{(g_i \in It)} = \mathrm{Spec}\, A[g_1/g_i, \ldots, g_n/g_i] \xrightarrow{\pi_i} \mathrm{Spec}\, A$$

where the previous rings are understood as $A[\mathbf{g}/g_i] \subseteq A_{g_i}$ i.e. $f(\mathbf{g}/g_i) = 0$ in $A[\mathbf{g}/g_i]$ $\Leftrightarrow$ $g_i^{\gg 0} f(\mathbf{g}/g_i) = 0$ in A. In particular, the ideal sheaf $\pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\mathrm{Bl}_A(I)}$ is locally $I \cdot A[g_1/g_i, \ldots, g_n/g_i] = (g_i)$, i.e. $\boxed{\text{principal; } g_i \text{ is a non-zero divisor}}$ since it is a unit in A_{g_i}. Globally, $\mathcal{O}_{\mathrm{Bl}_A(I)}(-E) = \pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\mathrm{Bl}_A(I)} = \mathcal{O}_{\mathrm{Bl}_A(I)}(1)$.

Proposition 68. *Let* $J \subseteq I \subseteq A$. *Then* $\mathrm{Bl}_{A/J}(I/J) \hookrightarrow \mathrm{Bl}_A(I) \to \mathrm{Spec}\, A$ *equals the strict transform of* $\mathrm{Spec}\, A/J \hookrightarrow \mathrm{Spec}\, A$.

Proof. Locally, E corresponds to the ideal $(g_i) \subseteq A[\mathbf{g}/g_i]$, the full preimage $\pi^{-1}V(J)$ is defined by the ideal $\mathfrak{a} := J \cdot A[\mathbf{g}/g_i]$, and $\mathrm{Bl}_{A/J}(I/J)$ is the vanishing locus of $\mathfrak{b} := \ker\big(A[\mathbf{g}/g_i] \twoheadrightarrow (A/J)[\mathbf{g}/g_i]\big) = \{f \in A[\mathbf{g}/g_i] \mid g_i^{\gg 0} f \in J\}$. Now, the claim follows from $\mathfrak{b} = (\mathfrak{a} : g_i^\infty)$, cf. (19.3.7) and (22.4). □

Theorem 69. *Every* $f : X \to \mathrm{Spec}\, A$ *with an invertible ideal sheaf* $f^{-1}\widetilde{I}$ *factors uniquely via* $\mathrm{Bl}_A(I)$. In particular, everything glues to blowing ups in ideal sheaves.

Proof. If $\varphi : A \to B$ has $\varphi(I)B = (\varphi(g_1), \ldots, \varphi(g_n)) = (b)$ with a non-zero divisor $b \in B$, then $\varphi(g_i) = bs_i$ implies $\bigcup_i D(s_i) = \mathrm{Spec}\, B$, and on $D(s_i)$ we have $\varphi(\mathbf{g})/\varphi(g_i) \in B_{s_i}$ providing $A \to A[\mathbf{g}/g_i] \to B_{s_i}$. □

Alternativer Beweis mit den Methoden von §25: $I = (g_1, \ldots, g_n) \Rightarrow \mathrm{Bl}_A(I) \subseteq \mathbb{P}^{n-1}_A$ ist gegeben durch die homogenen Gleichungen $G(\mathbf{x}) \in A[\mathbf{x}]$ mit $G(\mathbf{g}) = G(g_1, \ldots, g_n) = 0 \in A$, z.B. $G_{ij}(\mathbf{x}) := g_i x_j - g_j x_i$.

$$\begin{array}{ccccc} X & \overset{F}{\dashrightarrow} & \mathrm{Bl}_A(I) & \hookrightarrow & \mathbb{P}^{n-1}_A \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} & \swarrow & \\ & & \mathrm{Spec}\, A & & \end{array}$$

Falls F vorhanden, so ist $F^{-1}\mathcal{O}_{\mathrm{Bl}_A(I)}(1) \cdot \mathcal{O}_X = F^{-1}(\pi^{-1}I \cdot \mathcal{O}_{\mathrm{Bl}_A(I)}) \cdot \mathcal{O}_X = \mathcal{L}$, also existiert eine Surjektion $\rightsquigarrow$ Isomorphismus $F^*\mathcal{O}(1) \xrightarrow{\sim} \mathcal{L}$; dabei entsprechen sich die globalen Schnitte $F^*(x_i) = f^*(g_i)$.

Umgekehrt definieren $\mathcal{L}$ mit $f^*(g_i) \in \Gamma(X, \mathcal{L})$ genau einen Morphismus $F : X \to \mathbb{P}^{n-1}_A$, und dieser geht über $\mathrm{Bl}_A(I)$: Falls G Gleichung, wie oben mit $\deg G = d$, so folgt $G(f^*(g_1), \ldots, f^*(g_n)) = f^*G(g_1, \ldots, g_n) = 0$ in $\Gamma(X, \mathcal{L}^{\otimes d}) = \Gamma(X, \mathcal{L}^d)$.

23. Weil and Cartier divisors

23.1. **Normal rings.** A = noetherian domain; $I \subseteq \mathrm{Quot}(A)$ fractional ideal (finitely generated A-submodule) $\rightsquigarrow I^\vee := \mathrm{Hom}_A(I, A) \subseteq \mathrm{Quot}(A)$ with $I \cdot I^\vee \subseteq A$.

Lemma 70. 1) *$a \in A \setminus 0$, $P \in \mathrm{Ass}(A/a) \Rightarrow P^\vee \supsetneq A$.*
2) $A =$ normal, $P^\vee \supsetneq A \Rightarrow P \cdot P^\vee \neq P$.
3) (A, P) local, $P \cdot P^\vee = A \Rightarrow P$ is principal ($\rightsquigarrow A$ is regular, 1-dimensional).

Proof. (1) $A/P \hookrightarrow A/a$, $1 \mapsto b$ means $P = ((a) : (b))$; hence $b/a \in P^\vee \setminus A$.
(2) If $PP^\vee = P \Rightarrow P(P^\vee)^n = P \subseteq A$, every $x \in P^\vee$ implies $A[x] \subseteq \frac{1}{p}A$ for some $p \in P \Rightarrow x$ is integral over $A \Rightarrow x \in A$.
(3) Nakayama $\Rightarrow \exists a \in P \setminus P^2 \rightsquigarrow$ ideal $aP^\vee \subseteq A$; from $aP^\vee \not\subseteq P$ (otherwise $a \in a(P^\vee P) \subseteq P^2$) we derive $aP^\vee = A$, hence $(a) = a(P^\vee P) = (aP^\vee)P = P$. □

Proposition 71. *1) (A, P) local, normal, 1-dimensional $\Rightarrow$ regular* ($\Rightarrow$ normal).
2) A normal, $a \in A \setminus 0 \Rightarrow$ all $P \in \mathrm{Ass}(A/a)$ are minimal, i.e. $\mathrm{ht}(P) = 1$.

Proof. (0) Lemma 70(1,2) $\rightsquigarrow P\,P^\vee \supsetneq P$ whenever $P \in \mathrm{Ass}(A/a)$ for some $a \in A \setminus 0$.
(1) $\dim A = 1 \Rightarrow \forall a \in P$: $P \in \mathrm{Ass}(A/a) \overset{(0)}{\Rightarrow} PP^\vee = A \rightsquigarrow$ Lemma 70(3) applies.
(2) $P \in \mathrm{Ass}(A/a) \overset{(0)}{\Rightarrow} PP^\vee = A$. Lemma 70(3) on A_P yields $\dim(A_P) = 1$. □

Corollary 72. *For normal rings A we obtain $A = \bigcap_{\mathrm{ht}(P)=1} A_P$. In particular, $A = \{f \in \mathrm{Quot}(A) \mid \mathrm{div}(f) \geq 0\}$.*

Proof. $A = \bigcap_{a \in A\setminus 0} \bigcap_{P \in \mathrm{Ass}(A/a)} A_P$: Let $b/a \in \mathrm{Quot}(A) \rightsquigarrow I := \{x \in A \mid x \cdot b/a \in A\} = ((a) : (b)) = \mathrm{Ann}(b \in A/a)$. If $I \subsetneq A$ is not prime $\rightsquigarrow \exists x, y \notin I$, $xy \in I \Rightarrow \mathrm{Ann}(b \in A/a) \subsetneq \mathrm{Ann}(xb \in A/a) \subsetneq A$. This continues until we obtain a prime ideal P of this form, i.e. $I \subseteq P \in \mathrm{Ass}(A/a)$. □

23.2. **The class group.** Let X be an n-dimensional variety over k. Prime divisors = 1-codimensional, integral subschemes $D \subset X \rightsquigarrow \mathcal{O}_{X,\eta(D)} =$ 1-dimensional, local (integral) domain; Weil divisors $\mathrm{Div}\, X := Z_{n-1}(X) := \mathbb{Z}^{\oplus\{\text{prime div}\}} \subseteq \mathrm{Div}_{\mathbb{Q}}\, X$; effective Weil divisors $D \geq 0$ on X.

$D =$ prime divisor, $f \in K(X) = \mathrm{Quot}\,\mathcal{O}_{X,\eta(D)} \rightsquigarrow \mathrm{ord}_D(f) \in \mathbb{Z}$ via extension of $\mathrm{ord}_D(f \in \mathcal{O}_{X,\eta(D)}) := \ell(\mathcal{O}_{X,\eta(D)}/f)$; additivity: $0 \to \mathcal{O}/f \overset{g}{\to} \mathcal{O}/fg \to \mathcal{O}/g \to 0$; for *regular* $\mathcal{O}_{X,\eta(D)}$ we have $f = t^{\mathrm{ord}_D(f)} \cdot [\text{unit}]$ with $(t) = \mathfrak{m}_{X,\eta(D)} \subseteq \mathcal{O}_{X,\eta(D)}$.

2.11.22 (58+59)

$f \in K(X)^* \rightsquigarrow$ principal divisors $\mathrm{PDiv}(X) := \{\mathrm{div}(f) := \sum_D \mathrm{ord}_D(f) \cdot D\} \subseteq \mathrm{Div}(X)$; $\mathrm{Cl}(X) := \mathrm{Div}(X)/\mathrm{PDiv}(X)$, i.e. $K(X)^* \overset{\mathrm{div}}{\to} \mathrm{Div}(X) \to \mathrm{Cl}(X) \to 0$. *Example:* $\mathrm{Cl}(\mathbb{P}^1) = \mathbb{Z}$.

Proposition 73. *$X = \mathrm{Spec}\, A \rightsquigarrow [A$ is factorial $\Leftrightarrow \mathrm{Cl}(X) = 0$ and A is normal$]$.*

Proof. ($\Rightarrow$) "factorial" $\Rightarrow$ "normal"; $D = V(f \in A)$ prime divisor $\Rightarrow D = \mathrm{div}(f)$.
($\Leftarrow$) $P \subseteq A$ of height 1 $\rightsquigarrow$ prime divisor $D = \mathrm{div}(f \in \mathrm{Quot}\, A)$. Apply Corollary 72 twice: $\mathrm{ord}_\bullet(f) \geq 0 \Rightarrow f \in A$, and $g \in P \subseteq A \Rightarrow \mathrm{ord}_\bullet(g/f) \geq 0 \Rightarrow P = (f)$. □

23.3. **Cartier divisors.** Let X be a variety. It gives rise to the exact sequence $1 \to \mathcal{O}_X^* \to K(X)^* \to K(X)^*/\mathcal{O}^* \to 1$. Global sections $D \in \Gamma(X, K(X)^*/\mathcal{O}^*)$ are called Cartier divisors $D \in \operatorname{CaDiv}(X)$; they are represented by pairs (U_i, f_i) for some open covering $\{U_i\}$ of X and $f_i \in K(X)^*$ with $f_i/f_j \in \mathcal{O}^*(U_{ij})$. The associated invertible sheaf

$$\mathcal{O}_X(D) := \bigcup_i 1/f_i \cdot \mathcal{O}_{U_i} \subseteq K(X)$$

corresponds to the 1-cocycle $\delta(D) \in \check{Z}^1(\{U_i\}, \mathcal{O}_X^*)$, and all invertible subsheaves of $K(X)$ arise in this way. Principal divisors are Cartier via $\operatorname{PDiv}(X) \hookrightarrow \operatorname{CaDiv}(X)$, $\operatorname{div}(f) \mapsto (X, f)$. Since $D \sim D' \Leftrightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$, this leads to the identification $\operatorname{CaDiv}(X)/\operatorname{PDiv}(X) = \operatorname{Pic}(X)$.

On the other hand, the maps $\operatorname{div} : K(U_i)^* \to \operatorname{Div} U_i$ glue into $\operatorname{div} : \operatorname{CaDiv}(X) \to \operatorname{Div}(X)$. For normal X, this is injective, leading to $\operatorname{Pic}(X) \subseteq \operatorname{Cl}(X)$. For factorial (e.g. regular) X, it is surjective, too.

Example: (25.7) $\rightsquigarrow$ Weil divisors being not Cartier.

16.11.22 (60+61)

23.4. **Divisors in toric geometry.** If $X = \mathbb{TV}(\Sigma) \supseteq T$, then $\operatorname{Div}_T(X) := \mathbb{Z}^{\Sigma(1)}$ is generated by the T-invariant prime divisors $\overline{\operatorname{orb}}(a)$ with $a \in \Sigma(1)$.

Proposition 74. $\boxed{\operatorname{div}(\mathbf{x}^r) = \sum_{a\in\Sigma(1)} \langle a, r\rangle \cdot \overline{\operatorname{orb}(a)}}$. *Moreover, if* $\Sigma \leq \mathcal{N}(\Delta)$, *then* Δ *represents a Cartier divisor with associated sheaf* $\mathcal{O}_X(\Delta)$ *and associated Weil divisor* $\operatorname{div}(\Delta) = -\sum_{a\in\Sigma(1)} \min\langle a, \Delta\rangle \cdot \overline{\operatorname{orb}(a)}$.

Proof. Since $\mathbf{x}^r \in k[M]^*$ it remains to check that $\operatorname{ord}_{\overline{\operatorname{orb}(a)}}(\mathbf{x}^r) = \langle a, r\rangle$. Do this in the generic point $\eta_{\overline{\operatorname{orb}(a)}} \in \mathbb{TV}(\mathbb{Q}_{\geq 0} \cdot a) \subseteq \mathbb{TV}(\Sigma)$; one may assume $a = (1, \underline{0})$. $\square$

In particular (if Σ is full-dimensional), we obtain the commutative diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{\ \Sigma(1)^*\ } & \mathbb{Z}^{\Sigma(1)} & \xrightarrow{\ \pi\ } & \operatorname{Cl}(X) & \longrightarrow & 0 \\
\downarrow{\scriptstyle r\mapsto \mathbf{x}^r} & & \downarrow{\scriptstyle a\mapsto \overline{\operatorname{orb}}(a)} & & \| & & \\
K(X)^* & \xrightarrow{\ \operatorname{div}\ } & \operatorname{Div}(X) & \xrightarrow{\ D\mapsto [D]\ } & \operatorname{Cl}(X) & \longrightarrow & 0
\end{array}$$

One uses that $k[M]$ is factorial and, moreover, that $k[M]^* = k^* \cdot \{\mathbf{x}^r \mid r \in M\}$. It says that $\operatorname{Cl}(\mathbb{TV}(\Sigma))$ is Gale-dual to $\mathbb{Z}^{\Sigma(1)} \to N$. Define the following cones in $\operatorname{Cl}(X)_{\mathbb{Q}} := \operatorname{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\operatorname{Eff}(X) := \overline{\mathbb{Q}_{\geq 0} \cdot \{[D] \in \operatorname{Cl}(X) \mid D \geq 0\}} \quad (\text{“pseudo effective cone”})$$

and

7.12.22 (62+63)

$$\operatorname{Amp}(X) := \mathbb{Q}_{\geq 0} \cdot \{[D] \in \operatorname{Pic}(X) \mid D \text{ is ample}\} \quad (\text{“ample cone”}).$$

Proposition 75. *Let* $\pi : \mathbb{Z}^{\Sigma(1)} \twoheadrightarrow \operatorname{Cl}(X)$ *as in the above diagram. Then* $\operatorname{Eff}(X) = \pi(\mathbb{Q}_{\geq 0}^{\Sigma(1)})$ *and, if additionally* $\Sigma = \mathcal{N}(\Delta)$ ($\Rightarrow \operatorname{Pic}(X)_{\mathbb{Q}} = C(\Delta) - C(\Delta)$ by (25.6)), *then* $\operatorname{Amp}(X) = \operatorname{int} C(\Delta) = \operatorname{Pic}(X)_{\mathbb{Q}} \cap \bigcap_{\sigma\in\Sigma} \operatorname{int} \pi(\mathbb{Q}_{\geq 0}^{\Sigma(1)\setminus\sigma(1)})$.

Proof. (Eff) $D \geq 0$ on $X \Rightarrow$ choose $f \in K(X)^*$ with $D = \operatorname{div}(f)$ on T and $1 \in \operatorname{supp} f$. Since $\operatorname{ord}_a(f) \leq 0$ for all $a \in \Sigma(1)$, we have $D - \operatorname{div}(f) \in \pi(\mathbb{Z}^{\Sigma(1)}_{\geq 0})$.
(Amp, 1) $D = \operatorname{div}(\Delta')$ with $\Delta' \in \operatorname{int} C(\Delta)$ is ample, and for every $\sigma \in \Sigma$ it can be shifted such that $\Delta'(\sigma) = 0$, i.e, $\operatorname{div}(\Delta') \in \pi(\mathbb{Z}^{\Sigma(1)\setminus\sigma(1)}_{>0})$.
(Amp, 2) Let $h(\sigma, \sigma') = r(\sigma) - r(\sigma') = t(\sigma, \sigma') \cdot (\Delta(\sigma) - \Delta(\sigma'))$ be the 1-cocycle of a Cartier divisor D (σ, σ' adjacent, top-dimensional). If $[D] \in \operatorname{int} \pi(\mathbb{Q}^{\Sigma(1)\setminus\sigma(1)}_{\geq 0}) = \pi(\mathbb{Q}^{\Sigma(1)\setminus\sigma(1)}_{>0})$, then $r(\sigma) = 0$ and $\langle a, r(\sigma')\rangle < 0$ for $a \in \sigma'(1) \setminus \sigma(1)$. The same becomes true for $\Delta(\sigma)$ and $\Delta(\sigma')$ by shifting Δ. Hence, $t(\sigma, \sigma') > 0$.
(Amp, 3) Finally, if D is ample, then some multiple $\mathcal{O}_X(kD)$ is globally generated, and the corresponding Δ' yields $\Sigma = \mathcal{N}(\Delta')$. □

24. Smooth and regular schemes

Section §14 characterizes regular local rings – (14.1) via the tangent cone, and (14.2) via the existence of finite free resolutions. Corollary 49 shows that localizations of regular local rings (in prime ideals) stay regular. In (24.1) we have introduced the cotangent sheaves; now we combine both approaches. Let k be a $\boxed{\text{perfect field}}$, cf. (20.3); we will use local k-algebras $(A, \mathfrak{m})$ with $k \xrightarrow{\sim} A/\mathfrak{m}$, i.e. we deal with local rings of k-rational points.

24.1. **(Co) Tangent sheaves.** Let $A \to B$ be an algebra and M a B-module $\rightsquigarrow$ $\operatorname{Der}_A(B, M) := \{A\text{-linear derivations } d : B \to M\}$, i.e. $d(bb') = b\,d(b') + b'd(b) \rightsquigarrow$ universal A-derivation $\boxed{B \to \Omega_{B|A}}$ characterized by $\operatorname{Hom}_B(\Omega_{B|A}, M) \xrightarrow{\sim} \operatorname{Der}_A(B, M)$.
Example: $\Omega_{k[\mathbf{x}]|k} = \oplus_i\, k[\mathbf{x}]\, dx_i$. 4.1.23 (65)
This construction is compatible with localizations ($\Omega_{B_b|A} = \Omega_{B|A} \otimes_B B_b$ and $\Omega_{B|A_a} = \Omega_{B|A}$ if $a \in B^*$) hence glue to a quasi coherent $\mathcal{O}_X$-module $\Omega_{X|Y}$ for a given morphism $X \to Y$ of schemes, e.g. $\Omega_X := \Omega_{X|\operatorname{Spec} k}$ for every k-scheme X.
Example: $\Omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$, but $\Omega_E = \mathcal{O}_E$ for smooth $E = \overline{V(y^2 - f_3(x))} \subseteq \mathbb{P}^2$.
The fundamental exact sequences for the universal differentials $\Omega_\bullet$ become $f^*\Omega_{Y|S} \to \Omega_{X|S} \to \Omega_{X|Y} \to 0$ for $X \xrightarrow{f} Y \to S$ and, for closed subschemes $Z \xhookrightarrow{\iota} Y$,

$$\big(\mathcal{I}/\mathcal{I}^2 = \iota^*\mathcal{I}_{Z\subseteq Y}\big) \to \big(\Omega_{Y|S} \otimes \mathcal{O}_Z = \iota^*\Omega_{Y|S}\big) \to \Omega_{Z|S} \to 0, \quad \text{cf. (22.3).}$$

Dually, with $\mathcal{T} := \mathcal{H}om_{\mathcal{O}}(\Omega, \mathcal{O})$ denoting the "*tangent sheaves*", one obtains the exact sequences $0 \to \mathcal{T}_{X|Y} \to \mathcal{T}_{X|S} \to f^*\mathcal{T}_{Y|S}$ and $0 \to \mathcal{T}_{Z|S} \to \mathcal{T}_{Y|S} \otimes \mathcal{O}_Z \to \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$.

24.2. **The toric** Euler **sequence.** Let Σ be a smooth fan; its rays $\Sigma(1)$ gives rise to a surjection $f : \mathbb{Z}^{\Sigma(1)} \to N$. Now, to describe $\boxed{\Omega_X \text{ for } X = \mathbb{TV}(\Sigma)}$, we build the commutative diagram ahead: 11.1.23 (66)
(i) $\operatorname{Cl}(\Sigma)^* := \ker f$ ($\cong \mathbb{Z}$ for $\Sigma = \mathbb{P}^n$) followed by $\otimes_{\mathbb{Z}}\mathcal{O}_X$ gives the central row;

(ii) the codimension one orbits $H_a := \overline{\mathrm{orb}}(a)$ provide the central column.

(iii) Locally on $\mathbb{C}^n = \mathbb{TV}(\sigma \in \Sigma)$, the H_a are the coordinate hyperplanes, and we define $\Omega_{\mathbb{C}^n}(\log H) := \oplus_i \mathcal{O}_{\mathbb{C}^n}\, dx_i/x_i \supseteq \Omega_{\mathbb{C}^n}$. The assignment $r \mapsto dx^r/x^r$ shows that $M \otimes_{\mathbb{Z}} \mathcal{O}_X = \Omega_X(\log H)$.

(iv) The cokernel of the left hand column is checked locally via $\oplus_i \mathbb{C}[\mathbf{x}]\, dx_i/x_i \twoheadrightarrow \oplus_i \mathbb{C}[\mathbf{x}]/(x_i)\, e_i$ sending $dx_i/x_i \mapsto \overline{e_i}$ (residuum map).

(v) The top line follows by diagram chasing and is called the toric Euler sequence. For $X = \mathbb{P}^n$, it turns into the exact sequence $0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_X & \xrightarrow{dx^r \mapsto \sum_a \langle a,r\rangle x^r e_a} & \oplus_{a\in\Sigma(1)}\mathcal{O}_X(-H_a) & \longrightarrow & \mathrm{Cl}(\Sigma)\otimes\mathcal{O}_X \to 0 \\
 & & \downarrow & & \downarrow & & \| \\
0 & \to & \big(M\otimes\mathcal{O}_X = \Omega_X(\log H)\big) & \xrightarrow{r \mapsto \sum_a \langle a,r\rangle e_a} & \mathcal{O}_X^{\Sigma(1)} & \longrightarrow & \mathrm{Cl}(\Sigma)\otimes\mathcal{O}_X \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \oplus_{a\in\Sigma(1)}\mathcal{O}_{H_a} & = & \oplus_{a\in\Sigma(1)}\mathcal{O}_{H_a} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

In particular, $\omega_X := \det \Omega_X = \otimes_{a\in\Sigma(1)}\mathcal{O}_X(-H_a)$, e.g. $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. (This again shows that $\Omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ and, using adjunction, $\Omega_E = \mathcal{O}_E$ from (24.1).)

24.3. **Regular implies factorial.** While regular rings are automatically integral (Problem 65) and "factorial" means "regular in codimension one", we have

Proposition 76. *Regular local rings are factorial.*

Proof. Let P be a height one prime in the regular $(A, \mathfrak{m})$. If $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $A/(x)$ is regular, hence a domain, hence $(x) \subseteq A$ is prime.

If $x \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup P)$, then $P_x \subseteq A_x$ is locally free of rank 1: Every $Q \in \operatorname{Spec} A_x$ over P leads to a factorial A_Q (induction), hence $P_Q A_Q$ is principal. Now, if $F_\bullet \to P$ is a free A-resolution of P, then $P_x = \otimes_i (\det F_i)^{\pm 1} = f \cdot A_x$. If $f \in A \setminus (x)$, then this implies $P = (f)$ (if $p \in P$ satisfies that $p\,x^k = fg$, then the primality of x implies $x|g$, and one can lower k). □

18.1.23 (67) 24.4. **The cotangent space.** Let $(A, \mathfrak{m})$ be a local k-algebra with $k \xrightarrow{\sim} A/\mathfrak{m}$ being a perfect field. Then $\boxed{\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \Omega_{A|k} \otimes_A k}$ becomes an isomorphism ("cotangent space"): Injectivity follows from the surjectivity of $\mathrm{Der}_k(A, M) \twoheadrightarrow \mathrm{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, M)$ for $A/\mathfrak{m}$-modules M (extend $\mathfrak{m}/\mathfrak{m}^2 \to M$ by $(c \in k) \mapsto 0$).

Theorem 77. 1) *$\Omega_{A|k}$ is a free A-module of rank* $\dim A$ $\Leftrightarrow$ *A is a regular ring.*
2) *Let X be a variety over $k = \overline{k}$. Then $\Omega_{X|k}$ is locally free of rank* $\dim(X)$ ("X is non-singular") $\Leftrightarrow$ *all local rings of X are regular* ("X is regular").

Proof. We are using Problem 28. Since k is perfect, $Q(A)|k$ is separably generated, hence $\dim_{Q(A)} \Omega_{Q(A)|k} = \text{tr-deg}_k Q(A) = \dim A$: If $k \hookrightarrow (K = k(x_1, \ldots, x_d)) \hookrightarrow$

$(L = K(s))$ is a tower of fields, then $0 \to \Omega_{K|k} \otimes_K L \to \Omega_{L|k} \to \Omega_{L|K} \to 0$ is exact, and the latter vanishs because of $m_s'(s)\, ds = d\, m_s(s) = 0$. $\square$

Corollary 78. *If X is a $\overline{k}$-variety, then $X_{\text{smooth}} \subseteq X$ is open and dense.*

Proof. Consider $\Omega_{K(X)|k} = (\Omega_{X|k})_{\eta(X)}$. $\square$

24.5. **Smooth subvarieties.** Let X be an n-dimensional, smooth $(k = \overline{k})$-variety; let $Y \subseteq X$ be an irreducible closed subscheme of codimension r with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$.

Proposition 79. *Y is smooth $\Leftrightarrow$ $\Omega_{Y|k}$ is locally free, and the conormal sequence $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X|k} \otimes \mathcal{O}_Y \stackrel{\varphi}{\to} \Omega_{Y|k} \to 0$ of (24.1) is left exact, too.*
In this case, $\mathcal{I}$ is locally generated by r elements, and $\mathcal{I}/\mathcal{I}^2$ is locally free of rank r.

Proof. $q := \operatorname{rank} \Omega_{Y|k} \Rightarrow \ker \varphi$ is locally free of rank $n - q$.
($\Leftarrow$) Nakayama $\rightsquigarrow$ $\mathcal{I}$ is locally generated by $n - q$ elements ($I/I^2 \twoheadrightarrow I/\mathfrak{m}I$), hence $\dim Y \geq q$. On the other hand, for $y \in Y$, $q = \dim_k \mathfrak{m}_y/\mathfrak{m}_y^2 \geq \dim Y$.
($\Rightarrow$) Denote $A := \mathcal{O}_{X,y}$ with $y \in Y$ and $I = \mathcal{I}_y$. Since $\Omega_{A/I}$ is free of rank $q = n - r$, we have a split embedding $\ker \varphi \cong (A/I)^r \hookrightarrow (A/I)^n$. Let $\mathcal{I}' := \langle x_1, \ldots, x_r \rangle \subseteq \mathcal{I}$ locally generate the $\ker \varphi$ and $Y' := V(\mathcal{I}') \supseteq Y$, Nakayama implies that we can lift the splitting

$$\begin{array}{ccccccc} (A/I')^r & & & (A/I')^r & & & \\ & \searrow & \nearrow & & \searrow & & \\ & & I'/I'^2 & \longrightarrow & \left[(A/I')^n = \Omega_A \otimes A/I'\right] & \longrightarrow \Omega_{A/I'} \longrightarrow & 0, \end{array}$$

hence $I'/I'^2 \stackrel{\sim}{\to} (A/I')^r$ and $\Omega_{A/I'}$ is free, too. In particular, $Y \subseteq Y'$ are both smooth of the same dimension, hence equal. $\square$

In this situation, as in (24.1), we can dualize the above sequence into the exact $0 \to \mathcal{T}_Y \to \mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{N}_{Y|X} \to 0$ as well as take the determinant to obtain the adjunction formula $\omega_Y = \omega_X \otimes \det \mathcal{N}_{Y|X}$. 25.1.23 (68)

24.6. **Geometric genus.** We define the geometric genus of a projective, smooth k-variety X (with $k = \overline{k}$) as $p_g(X) := \dim_k \Gamma(X, \omega_X)$.

Proposition 80. *Let X, X' be two birationally equivalent, smooth, projective k-varieties. Then $p_g(X) = p_g(X')$.*

Proof. If $U \subseteq X$, then the restriction map $\Gamma(X, \omega_X) \hookrightarrow \Gamma(U, \omega_X)$ is injective; if $\operatorname{codim}_X(X \setminus U) \geq 2$, then it is even bijective (to be checked locally, since X is normal: $A = \bigcap_{\operatorname{ht} P = 1} A_P$, cf. Corollary 72).
On the other hand, the range of definition of $X \dashrightarrow X'$ is of the latter type $X \supseteq U \stackrel{f}{\to} X'$ (see Problem 135). Since $f = \operatorname{id}$ on a smaller $U \supseteq W \subseteq X'$, the pull back map $\Gamma(X', \omega_{X'}) \to \Gamma(U, \omega_U) = \Gamma(X, \omega_X)$ (induced from $f^*\omega_{X'} \to \omega_U$) takes place inside $\Gamma(W, \omega_W)$, hence is injective. $\square$

25. Invertible sheaves

On affine schemes: Invertible sheaves $\leftrightarrow$ projective modules of rank one.

14.12.22 (64) 25.1. **Morphisms by sections.** Fix a base ring A and an $A-$scheme $X \rightsquigarrow$ [A-morphisms $X \stackrel{\varphi}{\to} \mathbb{A}^n_A$] $\widehat{=}$ $[A[\mathbf{x}] \to \Gamma(X, \mathcal{O}_X)]$ $\widehat{=}$ $[\varphi_1, \ldots, \varphi_n \in \Gamma(X, \mathcal{O}_X)]$.

Proposition 81. [A-morphisms $X \stackrel{\varphi}{\to} \mathbb{P}^n_A$] $\widehat{=}$ [Invertible sheaves $\mathcal{L}|X$ with generating sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$]/Iso. *The ideal sheaf of the scheme theoretical image of such a* φ *is then induced from* $\ker\big(A[s_0, \ldots, s_n] \to \oplus_{d\geq 0}\Gamma(X, \mathcal{L}^d)\big)$.

Proof. $\varphi : X \to \mathbb{P}^n_A \rightsquigarrow \mathcal{L} := \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$, $s_i := \varphi^*(z_i)$. Conversely, we define $X \to \mathbb{P}^n_A$ by $P \mapsto (s_0(P) : \ldots : s_n(P))$. The local description uses trivializations like $\mathcal{L}|_{X_s} = s \cdot \mathcal{O}_{X_s}$ on $X_s \subseteq X$ for $s \in \Gamma(X, \mathcal{L})$. In particular, this yields $X_{s_\nu} \to D_+(z_\nu) \subseteq \mathbb{P}^n_A$ via $z_i/z_\nu \mapsto s_i/s_\nu \in \Gamma(X_{s_\nu}, \mathcal{O}_X)$. $\square$

Special situations: (i) $\mathcal{L}$ is called very ample, if there are s_i such that $X \to \mathbb{P}^n_A$ becomes an immersion as a locally closed subset. An invertible sheaf is called "*ample*" if some power is very ample.
(ii) If $\Gamma(X, \mathcal{L})$ is a finitely generated A-module and $\mathcal{L}$ is globally generated $\rightsquigarrow$ $\Phi_{\mathcal{L}} : X \to \overline{X} \subseteq \mathbb{P}^n_A$, and $\mathcal{L} = \Phi^*(\overline{\mathcal{L}})$ with $\overline{\mathcal{L}}|\overline{X}$ very ample.

Examples: $(\mathbb{P}^n, \mathcal{O}(d))$, $(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(1,1))$; among them $(\mathbb{P}^1, \mathcal{O}(2))$ and $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$ are quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$, respectively.

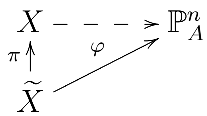
25.2. *Beispiel für ample Garben:* Kubische Kurve = elliptische Kurve; kubische Fläche – $\mathbb{P}^2$ in 6 Punkten aufblasen: Sieht man wenigstens die Geraden? Ja: Auf den exzeptionellen Divisoren ist die Garbe $3H - E$ genau $\mathcal{O}(1)$ (sieht man z.B. torisch); die strikten Transformierten von Verbindungsgeraden gehen auch so: $\mathcal{O}(3-1-1)$. Siehe [GrHa, S.480 ff.]: $3H - E$ ist sehr ample (mit dem üblichen Verfahren der Trennung von Punkten); die globalen Schnitte haben Dimension 4.

25.3. **Automorphisms of $\mathbb{P}^n$.** B = factorial $\Rightarrow$ $\mathrm{Pic}(\mathrm{Spec}\, B) = 0$ (via 1-cocycles on the open covering $\{D(g_i)\}$, cf. Problem 104).
A factorial $\Rightarrow$ $A[\mathbf{z}]$ factorial $\Rightarrow$ $\mathrm{Pic}\, \mathbb{P}^n_A = \mathbb{Z}$ (if $h_{ij} \in A[\mathbf{z}/z_i, \mathbf{z}/z_j]^*$ is a 1-cocycle $\Rightarrow$ $h_{ij} = [u_{ij} \in A^*] \cdot (z_i/z_j)^{k_{ij}}$, and $k_{ij} \in \mathbb{Z}$ cannot depend on i, j).
A-automorphisms of $\mathbb{P}^n_A$: $\varphi \in \mathrm{Aut}_A\, \mathbb{P}^n_A \Rightarrow \varphi^*(\mathcal{O}(1)) = \mathcal{O}(d)$; since $\Gamma(\mathbb{P}^n_A, \mathcal{O}(-1)) = 0$ we know that $\varphi^*(\mathcal{O}(1)) = \mathcal{O}(1)$. Thus, $\varphi^*(x_i) = \sum_j a_{ij}x_j$, i.e. $\mathrm{Aut}_A\, \mathbb{P}^n_A = \mathrm{PGL}(n, A)$.

25.4. **Resolving indeterminancies again.** We describe an instance of the general graph method of (22.4). Let $\mathcal{L}$ be invertible on the A-scheme X; let $\Phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^n_A$ be induced from $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$, i.e. $\Phi_{\mathcal{L}}$ is defined on $\bigcup_i X_{s_i} = X \setminus B$ with $B := V_{\mathbb{P}}(s_0, \ldots, s_n)$; its ideal sheaf is $\mathcal{J} := \sum_i s_i \, \mathcal{L}^{-1} \subseteq \mathcal{O}_X$.

$\mathcal{J} \otimes \mathcal{L} = \sum_i s_i \, \mathcal{O}_X \subseteq \mathcal{L}$ is, by definition, generated by the global sections $s_0, \ldots, s_n$ – but it is not invertible anymore.

Let $\pi : \widetilde{X} \to X$ be the blowing up in $\mathcal{J}$ with $\widetilde{\mathcal{J}} := \pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$; then $\widetilde{\mathcal{L}} := \widetilde{\mathcal{J}} \otimes \pi^*\mathcal{L}$ is invertible, generated by $\pi^*(s_0), \ldots, \pi^*(s_n)$, and, $\widetilde{\mathcal{L}} = \mathcal{L}$ holds true on $\widetilde{X} \setminus \pi^{-1}(B) = X \setminus B$. Hence, we obtain φ with $\varphi^*\mathcal{O}(1) = \widetilde{\mathcal{L}}$.



Example: Projection $\pi : \mathbb{P}^n_k \dashrightarrow \mathbb{P}^{n-1}_k$, $(x_0 : \ldots : x_n) \mapsto (x_1 : \ldots : x_n)$ or, locally in the chart $U_0 \subseteq \mathbb{P}^n_k$, $\pi : \mathbb{A}^n_k \setminus \{0\} \to \mathbb{P}^{n-1}_k$, $(x_1, \ldots, x_n) \mapsto (x_1 : \ldots : x_n)$. Then, $\mathcal{L} = \mathcal{O}_{\mathcal{A}^n}$ and $\mathcal{J} = (x_1, \ldots, x_n) \subseteq k[x_1, \ldots, x_n]$, hence $\pi : \widetilde{\mathbb{A}}^n_k = \mathrm{Bl}_{k[\mathbf{x}]}(\mathbf{x}) \to \mathbb{A}^n_k$ with $\widetilde{\mathcal{J}} = [\mathcal{O}_{\widetilde{\mathbb{A}}^n}(1)$ associated to the closed embedding $\widetilde{\mathbb{A}}^n_k \subseteq \mathbb{P}^{n-1}_{k[\mathbf{x}]} = \mathbb{P}^{n-1}_k \times \mathbb{A}^n_k]$.

14.12.22 (64)

25.5. **The Picard group of affine toric varieties.** Generalizing the case of $\mathbb{A}^n_k$, we show that $\boxed{\mathrm{Pic}\,\mathbb{TV}(\sigma) = 0}$ for *affine* toric varieties (while, e.g., $k[x, y, z]/(xz - y^2)$ is not factorial by $xz = y^2$). Let $S := k[\sigma^\vee \cap M]$.

Lemma 82. *Let $L \subseteq k[M]$ be an S-submodule with $L \otimes_S k[M] = k[M]$ (i.e. L contains monomials). Then $L^\vee$ is M-graded, i.e. it is generated by monomials. Moreover, if L is also invertible $\Rightarrow$ $L = x^r \cdot S$ for a unique $r \in M$.*

Proof. $L^\vee = \mathrm{Hom}_S(L, S) = \{f \in \mathrm{Quot}(S) \mid f \cdot L \subseteq S\} \subseteq k[M]$. Let $L = \langle \ell^1, \ldots, \ell^m \rangle$. For $f \in L^\vee$ and $a \in \sigma$ we know that $\deg_a f + \deg_a \ell^i \geq 0$ (with $\deg_a := \min\langle a, \mathrm{supp}\rangle$) $\Rightarrow$ for every f-monomial x^r we have $x^r \cdot \ell^i \in [a \geq 0] \Rightarrow x^r \cdot \ell^i \in S$, i.e. $x^r \in L^\vee$.

Now, let L be *invertible*. Corollary 7(1) in (2.7) $\rightsquigarrow$ the Nakayama lemma applies also to the graded case $(S, S_+) = (\oplus_{d \geq 0} S_d, \oplus_{d > 0} S_d)$ $\Rightarrow$ as in (8.1) it follows that, if $S_0 = k$, then minimal, homogeneous generating systems of graded, *projective* S-modules of finite presentation are automatically free.

Alternatively: $L = \langle \mathbf{x}^{r^i} \rangle$, $L^{-1} = \langle \mathbf{x}^{s^j} \rangle \Rightarrow \langle \mathbf{x}^{r^i + s^j} \rangle = S \Rightarrow$ w.l.o.g. $r^1 + s^1 = 0$ and $r^i + s^j \in \sigma^\vee$ otherwise $\Rightarrow r^i - r^1,\, s^j - s^1 \in \sigma^\vee$, i.e. $L = \mathbf{x}^{r^1} \cdot S$. □

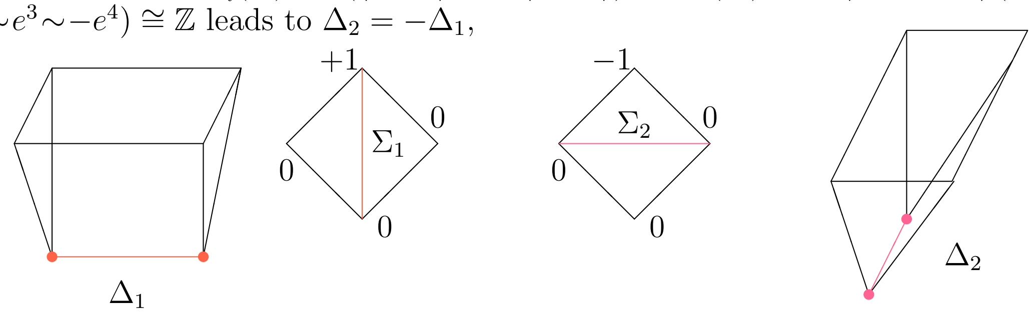
25.6. **The Picard group of general toric varieties.** Σ = fan in $N_{\mathbb{Q}}$; $\Delta \subseteq M_{\mathbb{Q}}$ lattice polyhedron with $\boxed{\Sigma \leq \mathcal{N}(\Delta)}$. The normal fan consists of the linearity regions of $(a \in \mathrm{tail}(\Delta)^\vee) \mapsto \min\langle a, \Delta \rangle$; in particular $|\Sigma| = |\mathcal{N}(\Delta)| = \mathrm{tail}(\Delta)^\vee$, and we have $[\Sigma^{\mathrm{top}} \to \Delta\text{-vertices}, \sigma \mapsto \Delta(\sigma)]$ with $\mathcal{N}_{\Delta(\sigma)}(\Delta) \supseteq \sigma \in \Sigma$, i.e. $\min\langle a, \Delta \rangle = \langle a, \Delta(\sigma) \rangle$ for $a \in \sigma$. Hence, $\Delta(\sigma) - \Delta(\sigma')$ is orthogonal to $\sigma \cap \sigma'$, and we may also define (non-unique) $\Delta(\sigma)$ for non-maximal cones σ. This shows that the globally generated sheaf $\boxed{\mathcal{O}_{\mathbb{TV}(\Sigma)}(\Delta)} := \sum_{\sigma \in \Sigma} x^{\Delta(\sigma)} \cdot \mathcal{O}_{\mathbb{TV}(\Sigma)}$ locally equals $\mathcal{O}_{\mathbb{TV}(\sigma)}(\Delta) = x^{\Delta(\sigma)} \cdot \mathcal{O}_{\mathbb{TV}(\sigma)}$. This sheaf induces the map $\Phi_{\mathcal{O}(\Delta)} : \mathbb{TV}(\Sigma) \to \mathbb{P}(\Delta)$ discussed in (16.5).

Conversely, let $\mathcal{L}$ be invertible on $\mathbb{TV}(\Sigma, N) \rightsquigarrow h$ = 1-cocycle of $\mathcal{O}^*$ with respect to the standard covering $\{\mathbb{TV}(\sigma)\}$ as in Problem 95: $\sigma, \tau \in \Sigma^{\rm top} \rightsquigarrow h_{\sigma\tau} = u_{\sigma\tau} \cdot \mathbf{x}^{h(\sigma,\tau)} \in k[(\sigma \cap \tau)^\vee \cap M]^* \subseteq k[M]^* = k^* \cdot \{\mathbf{x}^M\}$, i.e. h = 1-cocycle $\{h(\sigma,\tau) \in (\sigma \cap \tau)^\perp \cap M\}$. This is equivalent to $h = \{h(\sigma,\tau) \in (\sigma \cap \tau)^\perp \cap M \mid \sigma, \tau \in \Sigma^{\rm top} \text{ adjacent}\}$ with the additional condition that $\sum_i h(\sigma_i, \sigma_{i+1}) = 0$ along cycles around 2-codimensional cones. Moreover, since $k[\sigma^\vee \cap M]^* = k^*$, there are no 1-coboundaries.

On the other hand, since $k[M]$ = factorial, the embedding $j : T \hookrightarrow \mathbb{TV}(\Sigma)$ yields $\mathcal{L} \subseteq j_*\mathcal{L}|_T \xrightarrow{\sim} j_*\mathcal{O}_T$. Hence, by Lemma 82, $L_\sigma := \mathcal{L}(U_\sigma) \hookrightarrow \mathcal{O}_T(U_\sigma \cap T) = k[M]$ provides elements $r(\sigma) \in M$ which locally trivialize $\mathcal{L} \rightsquigarrow h(\sigma,\tau) = r(\sigma) - r(\tau)$, and $\{r(\sigma) \mid \sigma \in \Sigma^{\rm top}\}$ is, up to a common shift along M, uniquely determined. *Example:* $\mathcal{O}_\Sigma(\Delta)$.

Let $\boxed{\Sigma = \mathcal{N}(\Delta)} \rightsquigarrow$ For every $r = r(\mathcal{L})$ the sum $R := r + (N \gg 0) \cdot \Delta$ is a 1-cocycle with $R(\sigma,\tau) \in \mathbb{Q}_{\geq 0} \cdot (\Delta(\sigma) - \Delta(\tau))$ i.e.. it corresponds to Minkowski summand Δ_R of $\mathbb{Q}_{\geq 0} \cdot \Delta$ (Δ is "*ample*") $\rightsquigarrow \mathcal{L} \cong \mathcal{O}(\Delta_R) \otimes \mathcal{O}(\Delta)^{-N}$, and $\operatorname{Pic} \mathbb{P}(\Delta) = \{\Delta' - \Delta''\}$ consists of the lattice points in the so-called Grothendieck group of the convex cone $C(\Delta)$ of $(\mathbb{Q}_{\geq 0} \cdot \Delta)$-Minkowski summands.

25.7. **A toric flop.** Let $\mathbb{TV}(\Sigma_i) \to X = V(xy - zw) = \mathbb{TV}(\langle a^1, \ldots, a^4\rangle)$ be the two small resolutions; $\mathcal{O}_{\Sigma_i}(\Delta) \mapsto (\langle a^1, \Delta\rangle, \ldots, \langle a^4, \Delta\rangle) \in A_2(X) = \mathbb{Z}^4/M = \mathbb{Z}^4/(e^1 \sim -e^2 \sim e^3 \sim -e^4) \cong \mathbb{Z}$ leads to $\Delta_2 = -\Delta_1$,



i.e. under the natural identification $\operatorname{Pic} \mathbb{TV}(\Sigma_1) \xrightarrow{\sim} \mathbb{Z} \xleftarrow{\sim} \operatorname{Pic} \mathbb{TV}(\Sigma_2)$ we obtain that $\mathcal{L}$ is globally generated on $\mathbb{TV}(\Sigma_1) \Leftrightarrow \mathcal{L}^{-1}$ ist globally generated on $\mathbb{TV}(\Sigma_2)$.

26. Weil divisors and reflexive sheaves on normal schemes

26.1. **Reflexive modules and sheaves.** Let A = normal ring; denote $M^\vee := $
25.1.23 (68) $\operatorname{Hom}_A(M, A)$ for A-modules M. A finitely generated A-module L is reflexive $:\Leftrightarrow L = M^\vee$ for some A-module M. This implies that L is torsion free, i.e. $P \in D(f) \overset{j}{\hookrightarrow} X = \operatorname{Spec} A \Rightarrow L \to L_f \to L_P \to L \otimes \operatorname{Quot}(A)$ is injective. All restriction maps in $\widetilde{L}$ are injective, and there is an open $U \overset{j}{\hookrightarrow} X$ with $\operatorname{codim}_X(X \setminus U) \geq 2$ such that $\widetilde{L}|_U$ is locally free.

Proposition 83. (i) *If L is reflexive, then $L = \bigcap_{\operatorname{ht} P=1} L_P$. In particular, if $U \subseteq X$ is any open subset with* $\operatorname{codim}_X(X \setminus U) \geq 2$, *then $\widetilde{L} = j_*j^*\widetilde{L}$.*

(ii) *On the other hand, for finitely generated, torsion free A-modules L with $L = \bigcap_{\mathrm{ht}\,P=1} L_P$ the adjunction map $L \to L^{\vee\vee}$ is an isomorphism.*

Proof. (i) $L = \mathrm{Hom}(M, A) \Rightarrow L_P = \mathrm{Hom}_{A_P}(M_P, A_P) = \mathrm{Hom}_A(M, A_P)$; then use Corollary 72.
(ii) $L \to L^{\vee\vee}$ induces isomorphisms $L_P \to L_P^{\vee\vee}$ for $\mathrm{ht}\,P = 1$. Now, consider

$$\begin{array}{ccc} L & \longrightarrow & L^{\vee\vee} \\ \downarrow & & \downarrow\sim \\ \bigcap_{\mathrm{ht}\,P=1} L_P & \xrightarrow{\sim} & \bigcap_{\mathrm{ht}\,P=1} L_P^{\vee\vee}. \end{array}$$

□

Example: $L = (x, y) \subseteq k[x, y]$ is not reflexive (since $(x, y)^\vee = k[x, y]$).

26.2. **Sheaves for Weil divisors.** Let X = normal; consider affine charts $\mathrm{Spec}\,A \subseteq X$. Weil divisors $D \in \mathrm{Div}\,X \rightsquigarrow \mathcal{O}_X(D) := \{f \in K(X) \mid \mathrm{div}(f) + D \geq 0\}$ is a (coherent) fractional ideal sheaf, i.e. the global version of a fractional ideal from (23.1): $\mathrm{supp}\,D^+|_{\mathrm{Spec}\,A} \subseteq V(g) \Rightarrow g \cdot \mathcal{O}_A(D) = \mathcal{O}_A(D - \mathrm{div}(g)) \subseteq A$ is finitely generated. *Example:* $\mathcal{O}_X(0) = \mathcal{O}_X$ or $\mathcal{O}_X(D)$ from (23.3).

Lemma 84. *Let $X = \mathrm{Spec}\,A$ be affine. If $D = \sum_i \lambda_i D_i$, then $\forall i\ \exists f \in \mathcal{O}_X(D)$: $\mathrm{ord}_{D_i}(f) = -\lambda_i$ (instead just "$\geq$").*

Proof. $i = 1 \rightsquigarrow h \in I(D_1) \setminus I(D_1)^2 \subseteq A$ (assume $\mathrm{supp}(\mathrm{div}(h)) \subseteq \bigcup_j D_j$) and $g_j \in I(D_j) \setminus I(D_1)$ yield $f := h^{-\lambda_1} \prod_{j\geq 2} g_j^{\gg 0}$. □

Proposition 85. (1) *For $D, D' \in \mathrm{Div}\,X$ we have $\mathcal{O}_X(D) \subseteq \mathcal{O}_X(D') \Leftrightarrow D \leq D'$.*
(2) *$\mathcal{O}_X(-D) = \mathcal{O}_X(D)^\vee := \mathcal{H}om_X\big(\mathcal{O}_X(D), \mathcal{O}_X\big)$; in particular, $\mathcal{O}_X(D)$ is reflexive.*
(3) *$\mathcal{O}_X(D + D') = \big(\mathcal{O}_X(D) \cdot \mathcal{O}_X(D')\big)^{\vee\vee}$ = "reflexive hull".*

Proof. (1) $\mathcal{O}_X(\sum_i \lambda_i D_i) \subseteq \mathcal{O}_X(\sum_i \lambda_i' D_i)$ with $\lambda_1 > \lambda_1'$ contradicts Lemma 84.
(2) $\mathcal{H}om_X\big(\mathcal{O}_X(D), \mathcal{O}_X\big) \subseteq K(X)$ via $\varphi \mapsto \varphi(f)/f$ (does not depend on f), i.e. we obtain $\mathcal{H}om_X\big(\mathcal{O}_X(D), \mathcal{O}_X\big) = \{g \in K(X) \mid g \cdot \mathcal{O}_X(D) \subseteq \mathcal{O}_X\}$. Since $g \cdot \mathcal{O}_X(D) = \mathcal{O}_X(D - \mathrm{div}(g))$, the claim follows from (1).
(3) Let $\varphi : \mathcal{O}_X(D) \cdot \mathcal{O}_X(D') \to \mathcal{O}_X$; we check that $\varphi(\mathcal{O}_X(D + D')) \subseteq \mathcal{O}_X$: For $g \in \mathcal{O}_X(D')$ we obtain $g\varphi : \mathcal{O}_X(D) \to \mathcal{O}_X$, hence $g\varphi \in \mathcal{O}_X(-D)$, i.e. altogether one has $\varphi : \mathcal{O}_X(D') \to \mathcal{O}_X(-D)$, hence $\mathcal{O}_X(D' - \mathrm{div}(\varphi)) \subseteq \mathcal{O}_X(-D)$. □

As for Cartier divisors, we still have $D \sim D' \Leftrightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$: The isomorphisms are always given by elements of $K(X)^*$. Note that $1 \in \Gamma(X, \mathcal{O}(D)) \Leftrightarrow D \geq 0$.

26.3. **Effective divisors.** Let X be normal. There is a bijection {(effective) divisors D} $\leftrightarrow$ {reflexive (non-fractional) ideal sheaves $J \subseteq K(X)$}:

Proposition 86. *$D \mapsto \mathcal{O}_X(-D)$ and $J \overset{\mathrm{div}}{\mapsto} \sum_\nu \ell(\mathcal{O}_{X,\eta(D_\nu)}/J)\, D_\nu$ are mutually inverse. In particular, if $D \geq 0$, then $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ is exact.*

Proof. $\ell(\mathcal{O}_{X,\eta(D_\nu)}/J) = \min_{f\in J} \operatorname{ord}_{D_\nu}(f) \Rightarrow \ell(\mathcal{O}_{X,\eta(D_\nu)}/\mathcal{O}_X(-D)_{\eta(D_\nu)}) = \lambda_\nu$ by Lemma 84. Beginning with J, we obviously have that $J \subseteq \mathcal{O}_X(-\operatorname{div} J)$. Moreover, if $\varphi : J \to \mathcal{O}_X$, i.e. if for $g \in J$ we have $\operatorname{ord}_{D_\nu}(\varphi \cdot g) \geq 0$, then $\varphi \in \mathcal{O}_X(\operatorname{div} J)$. □

Another point of view is to fix a reflexive sheaf $\mathcal{F}|X$ of rank 1 and vary the embeddings of $\mathcal{F}$ into $K(X)$; they are parametrized by rational sections of $\mathcal{F}^\vee$:

Proposition 87. *There is a bijection* $\mathcal{F}_\eta \setminus \{0\} \leftrightarrow \{$*embeddings* $\mathcal{F}^\vee \hookrightarrow K(X)\}$, *and* $s, t \in \mathcal{F}_\eta$ *induce the same subsheaf* $\Leftrightarrow$ *they differ by* $\Gamma(\mathcal{O}_X^*)$. *Hence* $(\mathcal{F}_\eta \setminus 0)/\Gamma(\mathcal{O}_X^*) \leftrightarrow \{\text{divisors with } \mathcal{O}_X(D) \cong \mathcal{F}\}$. *If* $\mathcal{F} = \mathcal{O}_X(E)$, *then* $s \mapsto D(s) = \operatorname{div}(s) + E$.
Moreover, $\big(\Gamma(X,\mathcal{F}) \setminus 0\big)/\Gamma(\mathcal{O}_X^*) = \{\mathcal{O}_X \subseteq \mathcal{F}\} = \{\mathcal{F}^\vee \subseteq \mathcal{O}_X\} = \{D \geq 0\}$ *with* $D(s) = \operatorname{supp}(\operatorname{coker} s)$.

Proof. A section $s \in \mathcal{F}(U) \subseteq \mathcal{F}_\eta$ gives $\mathcal{F}^\vee|_U \to \mathcal{O}_U$, hence $\mathcal{F}^\vee \to j_*j^*\mathcal{F}^\vee \to j_*\mathcal{O}_U \subseteq K(X)$. This map is automatical injective, since it is an isomorphism in η. Or, shorter, $s \mapsto [\varphi \mapsto \varphi(s)]$. On the other hand, every $\mathcal{F}^\vee \overset{\iota}{\hookrightarrow} K(X)$ can be represented as some $\mathcal{F}^\vee|_U \to \frac{1}{g}\mathcal{O}_U$ (with $g \in \Gamma(U, \mathcal{O}_X) \subseteq K(X)$), i.e. $g\iota \in \operatorname{Hom}(\mathcal{F}^\vee|_U, \mathcal{O}_U) = \mathcal{F}(U)$. Finally, we note that $\mathcal{O}(-D) = s \cdot \mathcal{O}(-E) = \mathcal{O}(-E - \operatorname{div}(s))$. □

26.4. **Linear systems.** Let X be a complete k-variety; $D =$ Cartier divisor, $\mathcal{L} := \mathcal{O}_X(D) \rightsquigarrow$ the coordinate free version of Proposition 81 in (25.1) is $\Phi_\mathcal{L} : X \dashrightarrow \mathbb{P}(V) := (V^* \setminus 0)/k^*$, $P \mapsto [s \overset{\Phi(P)}{\mapsto} s(P)]$ with $V \subseteq \Gamma(X, \mathcal{O}_X(D))$. We may identify $\Phi(P)$ with $\Phi(P)^\perp = \{s \in V \mid s(P) = 0\} \subseteq V$.

$|D| := \{0 \leq D' \sim D\} = \big(\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}\big)/k^*$ "general linear system" $\rightsquigarrow V$ induces a "special linear system" $|D|_V := (V \setminus 0)/k^* \subseteq |D|$, and $\Phi(P)^\perp = \{D' \in |D|_V \mid D' \ni P\}$. The base locus of V is $\mathcal{B}(V) = \{x \in X \mid \forall D' \in |D|_V \colon x \in D'\}$, i.e. $\Phi(P)^\perp$ is a hyperplane in $|D|_V \Leftrightarrow P \notin \mathcal{B}(V)$.

27. Fractional toric ideals

27.1. **Tailed subsets.** $X = \mathbb{TV}(\sigma, N) \rightsquigarrow$ Lemma 82 shows that $D \in \operatorname{Div}_T(X)$ provide M-graded $\mathcal{O}_X(D) \subseteq j_*\mathcal{O}_T$, i.e. M-graded pieces $\mathcal{O}_{\mathbb{TV}(\sigma)}(D) \subseteq k[M]$.

$J = \sum_i \mathbf{x}^{r^i} k[\sigma^\vee \cap M] \subseteq k[M]$ monomial, fractional ideal $\leftrightarrow$ finitely generated $(\sigma^\vee \cap M)$-"module" $\Delta(J) \subseteq M$; denote $\boxed{J = \mathbf{x}^\Delta}$.

Definition 88. $\Delta \subseteq M$ is called *polyhedral* $:\Leftrightarrow \Delta = P \cap M$ for some lattice polyhedron $P \subseteq M_\mathbb{Q}$ (implying that $P = \operatorname{conv}(\Delta)$, i.e. discrete polyhedral sets correspond 1-to-1 to lattice polyhedra).

Even rational polyhedra P suffice to make $\Delta = P \cap M$ polyhedral. Moreover, Δ becomes then a $\boxed{\text{finitely generated}}$ (tail(Δ)-module where $\operatorname{tail}(\Delta) := \{r \in M \mid r + \Delta \subseteq \Delta\}$ is the lattice points of the tail cone: $\Delta = M \cap (P^c + \operatorname{tail}_{\leq 1}(P)) + \operatorname{tail}(\Delta)$.

Example: If $D = \sum_a \lambda_a \overline{\operatorname{orb}(a)} \in \operatorname{Div}_T X \rightsquigarrow \Delta_\sigma^\mathbb{Q}(D) := \{r \in M_\mathbb{Q} \mid \langle a, r\rangle \geq$

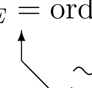
$-\lambda_a$ for $a \in \sigma^{(1)}\}$ does not need to be a lattice polyhedron. Nevertheless, $\Delta_\sigma(D) := \Delta^{\mathbb{Q}}_\sigma(D) \cap M$ is polyhedral, and it provides $\mathcal{O}_X(D) = \mathbf{x}^{\Delta_\sigma(D)}$.

$\{$Polyhedra $\Delta \subseteq M_{\mathbb{Q}}$ with tail $\sigma^\vee\} \leftrightarrow \{j : \sigma \to \mathbb{Q} \,|\,$ fanwise linear, concave$\}$ via $\Delta \mapsto j(\Delta) := \min\langle \bullet, \Delta\rangle$ and $j \mapsto \Delta_{\geq j} := \{r \in M_{\mathbb{Q}} \,|\, \langle \bullet, r\rangle \geq j$ on $\sigma\}$. Taking into account the lattice structure, one obtains $\{J = \mathbf{x}^\Delta \subseteq k[M]$ with tail $= \sigma^\vee\} \leftrightarrow \{j : \sigma \cap N \to \mathbb{Z} \,|\,$ fanwise linear, concave$\}$ via $\boxed{\operatorname{ord}_J(a) := \min_{\mathbf{x}^r \in J}\langle a, r\rangle}$ (similarily to $\operatorname{ord}_{D_\nu}(J)$ from (26.3)).

Weil divisors: $D = \sum_a \lambda_a \overline{\operatorname{orb}(a)} \Rightarrow \operatorname{ord}_{\mathcal{O}_X(D)}(a \in \sigma^{(1)}) = -\lambda_a$, and ord_D is the smallest concave fanwise linear function with these boundary values ("concave interpolation") – this characterizes "reflexive".
Example: The non-reflexive $(x, y) \subseteq k[x, y]$ yields $\operatorname{ord}(1, 0) = \operatorname{ord}(0, 1) = 0$ and $\operatorname{ord}(1, 1) = 1$, but the concave interpolation equals 0.

Cartier divisors correspond to $\Delta_\sigma(D) = R_\sigma + (\sigma^\vee \cap M)$; for simplicial cones we have $\operatorname{Div}_T X \subseteq \operatorname{CaDiv}_{\mathbb{Q}} X$. (Examples $k[x, y, z]/(xz - y^2)$ and $k[x, y, z]/(xz - yw)$.)

27.2. **Pulling back fractional ideals.** Let $\mathcal{J} \subseteq j_*\mathcal{O}_T = k[M]$ be a monomial sheaf of fractional ideals ($j^*\mathcal{J} = \mathcal{O}_T$); locally it corresponds to fractional $(\sigma^\vee \cap M)$ ideals $J_\sigma = \mathbf{x}^{\Delta_\sigma(J)} \subseteq k[M]$; the global sections are $\Gamma(\mathbb{TV}(\Sigma), \mathcal{J}) \mathrel{\widehat{=}} \Delta(J) := \bigcap_{\sigma \in \Sigma} \Delta_\sigma(J)$. Note that $0 \in \Delta(D) := \Delta(\mathcal{O}_X(D)) \Leftrightarrow D \geq 0$.

Let $f : (\Sigma', N') \to (\Sigma, N)$; via $f : T' \to T$ we obtain $j'^*(f^*\mathcal{J}) = f^*(j^*\mathcal{J}) = \mathcal{O}_{T'}$. Define $\mathcal{J}' = f^{-1}\mathcal{J} \cdot j'_*\mathcal{O}_{T'} := \operatorname{im}(f^*\mathcal{J} \to j'_*\mathcal{O}_{T'})$ ($= f^*\mathcal{J}$ for invertible $\mathcal{J}$: A-linear surjections $A \twoheadrightarrow A$ are isomorphisms).
Locally, with $f(\sigma') \subseteq \sigma$, this means $J'_{\sigma'} = (J_\sigma \subseteq k[M]) \odot_{k[\sigma^\vee \cap M]} k[\sigma'^\vee \cap M'] \subseteq k[M']$ is generated by $\{\mathbf{x}^{f^* r}\}$ with $\{\mathbf{x}^r\}$ generating J_σ, i.e. $\Delta_{\sigma'}(J') = f^*\Delta_\sigma(J) + (\sigma'^\vee \cap M')$. Alteratively, the order functions glue to maps $|\Sigma| \cap N \to \mathbb{Z}$. Then, $\boxed{\operatorname{ord}_{\mathcal{J}'} = \operatorname{ord}_{\mathcal{J}} \circ f}$.
Example: Blowing up $\mathcal{J}$ via subdividing σ into the linearity regions of $\operatorname{ord}_{\mathcal{J}}$. For instance, $J = (x, y)$ leads to the invertible $\mathcal{J} := \pi^{-1}\mathfrak{m} \cdot \mathcal{O}_{\widetilde{\mathbb{A}^2}}$, and $\operatorname{ord}_{\mathcal{J}} : (1, 0), (0, 1) \mapsto 0,\ (1, 1) \mapsto 1$ shows that $\mathcal{J} = \mathcal{O}_{\widetilde{\mathbb{A}^2}}(-E)$. Concavity of $\operatorname{ord}_{-E} = \operatorname{ord}_{\mathfrak{m}} \Rightarrow \mathcal{O}_{\widetilde{\mathbb{A}^2}}(-E)$ is globally generated; $\Delta(-E)$ has the vertices $[0, 1]$ and $[1, 0]$  $\rightsquigarrow \Phi_{\mathcal{O}(-E)} : \widetilde{\mathbb{A}^2} \to \mathbb{P}^1 \times \mathbb{A}^2$ yields the embedding; on the other hand, $\Phi_{\mathcal{O}}$ provides the contraction $\widetilde{\mathbb{A}^2} \to \mathbb{A}^2$.

27.3. **The canonical sheaf.** From (24.2) we know that $\omega_X = \mathcal{O}_X(-\sum_{a \in \Sigma(1)} H_a)$ for smooth toric varieties $X = \mathbb{TV}(\Sigma)$ (with $H_a = \overline{\operatorname{orb}}(a)$). In general, we define $\boxed{\omega_X := (\det \Omega_X)^{\vee\vee}} \subseteq j_*j^*\omega_X = j_*\omega_T$. On T we know that $\Omega_T = k[M] \otimes_{\mathbb{Z}} M$ with $d\mathbf{x}^r = \mathbf{x}^r \otimes r$, hence $\omega_T = k[M] \otimes_{\mathbb{Z}} \det M$ with $\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n} = 1 \otimes (e_1 \wedge \ldots \wedge e_n)$. This is coordinate independent, and ω_X ist T-invariant.

Proposition 89. $\boxed{\operatorname{ord}_{\omega_X} = 1 \text{ on } \Sigma^{(1)}}$, *i.e.* $\boxed{K_X = -\sum_{a \in \Sigma^{(1)}} \overline{\operatorname{orb}}(a)}$ *is a* (the nicest) *so-called canonical divisor on* X.

Proof. The fact that $\mathbf{x}^r \cdot (d\mathbf{x}/\mathbf{x})^{\wedge n} \in \bigcap_{a\in\Sigma^{(1)}} j_{a*}j_a^*\omega_X \Leftrightarrow \operatorname{ord}_a \mathbf{x}^r \geq 1$ for all $a \in \Sigma^{(1)}$ follows as in Proposition 74: $\omega_{\mathbb{TV}((1,\underline{0})\cdot\mathbb{Q}_{\geq 0})}$ is generated by $dx_1 \wedge \frac{dx_2}{x_2} \wedge \ldots \wedge \frac{dx_n}{x_n} = x_1\,(d\mathbf{x}/\mathbf{x})^{\wedge n}$. □

1.2.23 (69) In particular, $\boxed{\Delta_\sigma(K_X) = (\operatorname{int}\sigma^\vee) \cap M}$. X is "*Gorenstein*" $:\Leftrightarrow \omega_X$ is invertible (K_X is Cartier) $\Leftrightarrow \forall\sigma \in \Sigma\ \exists m_\sigma \in M$ with $(\operatorname{int}\sigma^\vee) \cap M = m_\sigma + (\sigma^\vee \cap M) \Leftrightarrow \ldots \langle\sigma(1), m_\sigma\rangle = 1 \Leftrightarrow$ all σ are cones over lattice polytopes (in height 1, namely in $[m_\sigma = 1]$). For the condition "$\mathbb{Q}$-*Gorenstein*" one relaxes the previous condition by asking just for $m_\sigma \in M_\mathbb{Q}$.

X is (weakly) $\boxed{\text{CY}}$ $:\Leftrightarrow \omega_X \cong \mathcal{O}_X \Leftrightarrow \exists m \in M\colon \forall\sigma \in \Sigma \ldots \Leftrightarrow$ the above $m_\sigma = m$ do not depend on $\sigma \Leftrightarrow \Sigma$ is the cone over a complex of lattice polyhedra (in height 1).

Example: There are no complete toric CYs; the easiest non-affine (and smooth) one is the small resolution of cone$(\mathbb{P}^1 \times \mathbb{P}^1)$, cf. (25.7).

Theorem 90 (Serre duality)**.** *If X is a d-dimensional, smooth, projective variety and $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, then* $\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^{d-i}(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X)^\vee$.

27.4. **Toric Fano varieties.** While $\Delta^\mathbb{Q}(K_X) := \bigcap_{\sigma\in\Sigma} \Delta_\sigma^\mathbb{Q}(K) = \emptyset$ for complete toric varieties $X = \mathbb{TV}(\Sigma)$, we will investigate $-K_X$ instead. Assume that $|\Sigma| =$ convex; then $\Delta^\mathbb{Q}(-K_X) = \{r \in M_\mathbb{Q} \mid \langle\Sigma(1), r\rangle \geq -1\}$ contains 0 as an interior lattice point.

Definition 91. X is Fano $:\Leftrightarrow -K_X$ is ample (i.e. in particular $\mathbb{Q}$-Cartier). In the toric situation, this means that $\mathcal{N}(\Delta_\mathbb{Q}(-K_X)) = \Sigma$.

For rational polyhedra there is a duality theory extending this of polyhedral cones. Defining $P^\vee := \{a \mid \langle a, P\rangle \geq -1\}$, this construction interchanges tail P and $\mathbb{Q}_{\geq 0} \cdot P$. The main tool for investigating this duality is the equality of the polyhedral cones $\overline{\mathbb{Q}_{\geq 0} \cdot (P^\vee, 1)} = \overline{\mathbb{Q}_{\geq 0} \cdot (P, 1)}^\vee$. Note that taking the closure of $\mathbb{Q}_{\geq 0} \cdot (P, 1)$ means to add the cone $(\operatorname{tail} P, 0)$.

In particular, if Σ is complete, then $\Delta_\mathbb{Q}(-K_X)^\vee = \operatorname{conv}(\Sigma(1)) \ni 0$.

Lemma 92. *For a polytope P mit $0 \in \operatorname{int} P$, the normal fan $\mathcal{N}(P)$ equals the face fan of $P^\vee$.*

Proof. $\operatorname{facefan}(P^\vee) = \operatorname{pr}\left[\partial\,\mathbb{Q}_{\geq 0}(P^\vee, 1)\right] = \operatorname{pr}\left[\partial\,\mathbb{Q}_{\geq 0}(P, 1)^\vee\right] = \mathcal{N}(P)$. □

Hence, a complete toric $\boxed{\mathbb{TV}(\Sigma) \text{ is Fano} \Leftrightarrow \operatorname{facefan}(\operatorname{conv}(\Sigma(1))) = \Sigma}$. In particular, we may construct all toric Fano varieties as follows: Let $P :=$ lattice polytope with $0 \in \operatorname{int} P$ and primitive vertices $\rightsquigarrow \Sigma := \operatorname{facefan}(P)$.

27.5. **Reflexive polytopes.** A polytope P is called *reflexive* $:\Leftrightarrow P$ and $P^\vee$ are lattice polytopes. If so, then 0 is the only interior lattice point of both.

Classification: In every dimension there are only finitely many; there are exactly 16 two-dimensional ones.

A toric Fano $\mathbb{TV}(\Sigma)$ is Gorenstein $\Leftrightarrow \Delta^\mathbb{Q}(-K_X) = \operatorname{conv}(\Sigma(1))^\vee$ is a lattice polytope

$\Leftrightarrow \Delta := \Delta^{\mathbb{Q}}(-K_X)$ (or $\operatorname{conv}(\Sigma(1))$) is reflexive. Recall that then $\Sigma = \mathcal{N}(\Delta)$.

If $f \in k[M]$ is a Laurent polynomial with $\Delta := \operatorname{conv}(\operatorname{supp} f))$ and $\Sigma \leq \mathcal{N}(\Delta)$, then $Y := \overline{V(f) \subseteq T} \subseteq \mathbb{TV}(\Sigma) =: X$ is locally a hypersurface. If $\mathcal{J} \subseteq \mathcal{O}_X$ denotes its ideal sheaf, then $\mathcal{J} = f \cdot \mathcal{O}_X(-\Delta)$ (locally on $\mathbb{TV}(\sigma)$, we have $\mathcal{J} = (f/\mathbf{x}^{\Delta(\sigma)}) \subseteq \mathcal{O}_{\mathbb{TV}(\sigma)}$). If the adjunction formula applies (e.g. if Y and X are smooth), then $\omega_Y = \omega_X \otimes \mathcal{O}_Y(\Delta)$. In particular, if Δ is reflexive, then $\omega_Y \cong \mathcal{O}_Y$.

27.6. **Discrepancies.** To make pull backs of canonical divisors possible, we always assume the $\mathbb{Q}$-Gorenstein property in this section.

Definition 93. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities; we denote $K_{\widetilde{X}} = \pi^* K_X + \sum_i \lambda_i E_i$ with $\{E_i\}$ being the exceptional divisors. Then, the $\{\lambda_i\}$ are called the *discrepancies* of π, and the singularities of X are called *canonical/terminal* $:\Leftrightarrow$ $\lambda_i \geq 0/\ \lambda_i > 0$ for all i.

Proposition 94. *Let σ be a $\mathbb{Q}$-Gorenstein cone; let $m_\sigma \in M_{\mathbb{Q}}$ with $\langle \sigma(1), m_\sigma \rangle = 1$.*
1) For a subdivision $\Sigma \leq \sigma$ we have $K_\Sigma = \pi^* K_\sigma + \sum_{a \in \Sigma(1) \setminus \sigma(1)} \big(\langle a, m_\sigma\rangle - 1\big)\, \overline{\operatorname{orb}(a)}$.
2) $\mathbb{TV}(\sigma)$ has canonical singularites $\Leftrightarrow \sigma \cap N \subseteq [m_\sigma = 1] \cup \{0\}$; $\mathbb{TV}(\sigma)$ has terminal singularities $\Leftrightarrow \sigma \cap N \subseteq \sigma(1) \cup \{0\}$.

Proof. (1) For $a \in \Sigma(1)$ we have $\operatorname{ord}_{K_\Sigma}(a) = 1$, but $\operatorname{ord}^{\mathbb{Q}}_{\pi^* K_\sigma}(a) = \langle a, m_\sigma \rangle$.
(2) If a is (properly) below $[m_\sigma = 1]$, then one may consider a subdivision involving a. Alternatively, every smooth subdivision contains vertices below $[m_\sigma = 1]$. $\square$

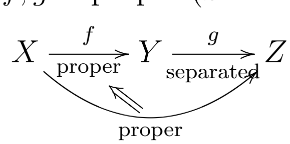
Let $X = \mathbb{TV}(\operatorname{facefan}(P))$ be Fano as in (27.4). Then X has at most *canonical* singularities $\Leftrightarrow$ 0 is the only interior lattice point of P (being equivalent to "reflexive" in dim $= 2$, but strictly weaker than it in dim ≥ 3). X has at most *terminal* singularities $\Leftrightarrow$ the vertices and $0 \in P$ are the only lattice points of P.

28. Proper morphisms

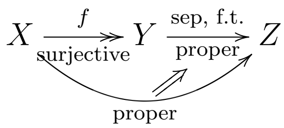
8.2.23 (70)

28.1. **Simulating compactness.** $f : X \to Y$ is called "*proper*" $:\Leftrightarrow$ separated, finite type. and universally closed; this notion is local on Y. *Example:* closed embeddings, finite morphisms; *counter example*: open embeddings. Absolute version of "proper": "*complete*".

Properties: Invariance under *base change*; the *composition* $X \xrightarrow{f} Y \xrightarrow{g} Z$ of proper f, g is proper ($S \to Z \Rightarrow X \times_Z S \to Y \times_Z S \to S$ are closed);



[gf proper, g separated $\Rightarrow f$ proper]: $S \to Y$ new basis $\Rightarrow X \times_Y S \hookrightarrow X \times_Z S$ is a closed embedding, and $X \times_Z S \to S$ is closed;



[gf proper, f surjective, g separated, finite type $\Rightarrow$ g proper]: f remains surjective under $S \to Y$, since $\operatorname{Spec} K \to S \to Y$ induces points $(P \in X) \mapsto (Q \in Y)$ with $K \supseteq K(Q) \subseteq K(P)$, i.e. $\operatorname{Spec} K(P) \otimes_{K(Q)} K \to \operatorname{Spec} K$ provides pre-images.

28.2. **The projective space.** The standard example for complete varieties is

Proposition 95. $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ *is proper.*

Proof. It remains to show that $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map; let $Z = (f_1, \ldots, f_m)$ with homogeneous $f_i \in A[\mathbf{x}]$ of degree d_i. For $d \gg 0$ the map $\beta_d : \oplus_i f_i \cdot A[\mathbf{x}]_{d-d_i} \to A[\mathbf{x}]_d$ is given by a matrix with entries in A; denote by $m_\nu(d) \in A$ the $\binom{d+1}{n}$-minors and by $m(d) \subseteq A$ the ideal generated by them. For $P \in \operatorname{Spec} A$ we obtain:
$P \notin \pi(Z) \Leftrightarrow \emptyset = Z \cap \pi^{-1}(P) = V(f_1, \ldots, f_m)$ in $\mathbb{P}^n_{K(P)} \Leftrightarrow \exists d\colon (f_1, \ldots, f_m) \supseteq (\mathbf{x})^d$ in $K(P)[\mathbf{x}] \Leftrightarrow \exists d: \ \beta_d \otimes_A K(P)$ is surjective $\Leftrightarrow \exists d, \nu: \ m_\nu(d) \neq 0$ in $K(P)$, i.e. $m_\nu(d) \in A_P^*$ $\Leftrightarrow \exists d\colon m(d) = (1)$ in $A_P \Leftrightarrow \exists d\colon P \notin V(m(d)) \subseteq \operatorname{Spec} A$. □

In particular, "projective morphisms" $Z \overset{\text{abg}}{\hookrightarrow} \mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} Y \overset{\text{pr}}{\to} Y$ are proper.
Example: Blowing up $\operatorname{Proj} \oplus_{d \geq 0} I^d \to \operatorname{Spec} A$.

28.3. **Toric situation.** In toric geometry we can exactly describe the difference between projective and complete k-varieties:

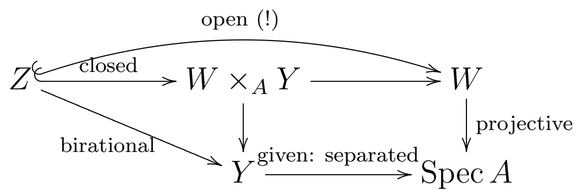
Proposition 96. 1) $\mathbb{TV}(\Sigma, N)$ *is projective over* $k \Leftrightarrow \exists$ *polytope* $\Delta \subseteq M_{\mathbb{R}}$ *with* $\Sigma = \mathcal{N}(\Delta)$*;* $\mathbb{TV}(\Sigma, N)$ *is proper over* $k \Leftrightarrow |\Sigma| = N_{\mathbb{Q}}$ (whirlpool example).
2) *An equivariant morphism* $\mathbb{TV}(\Sigma, N) \to \mathbb{TV}(\Sigma', N')$ *is proper* $\Leftrightarrow \varphi : N \to N'$ *satisfies* $\boxed{\varphi^{-1}|\Sigma'| = |\Sigma|}$.

Proof. (1) $\Sigma = \mathcal{N}(\Delta) \Rightarrow \mathbb{TV}(\Sigma) = \mathbb{P}((\gg 0) \cdot \Delta) \subseteq \mathbb{P}^N$ is projective; the reverse implication follows from knowing the ample sheaves on $\mathbb{TV}(\Sigma) \rightsquigarrow$ *coming up soon!*
$|\Sigma| \subsetneq N_{\mathbb{Q}} \Rightarrow \exists\, \Sigma \hookrightarrow \overline{\Sigma} \rightsquigarrow$ open embedding $\mathbb{TV}(\Sigma) \hookrightarrow \mathbb{TV}(\overline{\Sigma})$, and this is non-proper. However, if $|\Sigma| = N_{\mathbb{Q}}$, then one extends all codimension one walls to hyperplanes $\rightsquigarrow \Sigma \geq \Sigma' = \text{HypPlArrangement} = \mathcal{N}(\text{Zonotop}) \Rightarrow$ projective $= \mathbb{TV}(\Sigma') \to \mathbb{TV}(\Sigma)$ birational, proper (by 28.1; "Chow-Lemma") $\Rightarrow$ surjektive $\Rightarrow \mathbb{TV}(\Sigma)$ is complete (again by 28.1).
(2) The latter arguments do also work in the relative case, i.e. for non-compact polyhedra. □

Example: (Toric) resolutions of singularities in (21.5) were proper and birational.

28.4. **Non-toric Chow Lemma.** The "Chow Lemma" argument of the prevoius proof does also have a non-toric, i.e. a general version:

15.2.23 (71) **Lemma 97.** *$Y|\operatorname{Spec} A$ irreducible, separated $\Rightarrow \exists\, W|\operatorname{Spec} A$ projective, $Z \subseteq W \times_A Y$ closed: $Z \stackrel{\pi_W}{\hookrightarrow} W$ is an open embedding, and $\pi : Z \stackrel{\pi_Y}{\to} Y$ is birational (and proper).*

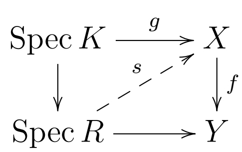


Proof. $Y = \bigcup_i Y_i$ open, affine covering, $Y_i \hookrightarrow \overline{Y}_i$ projective; $U := \bigcap_i Y_i$ affine $\Rightarrow$ $U \stackrel{\text{cl}}{\hookrightarrow} U \times_A \ldots \times_A U \stackrel{\text{op}}{\hookrightarrow} \prod_i Y_i \stackrel{\text{op}}{\hookrightarrow} \prod_i \overline{Y}_i \Rightarrow U \stackrel{\text{op}}{\hookrightarrow} \overline{U} =: W;\ Z := \overline{\Gamma_{U \hookrightarrow Y}} \subseteq W \times_A Y$. $Z \supseteq U \subseteq Y \Rightarrow \operatorname{pr}_Y$ is birational. Moreover, let $Z_i := Z \cap (W \times_A Y_i) \stackrel{\text{op}}{\hookrightarrow} Z$; since $Z_i = \overline{\Gamma_{U \hookrightarrow Y_i}} \subseteq \Delta_i \subseteq (\overline{Y}_1 \times \ldots \times Y_i \times \ldots \times \overline{Y}_m) \times Y_i$ (i-th diagonal), we have $Z_i \stackrel{\sim}{\to} W_i := W \cap (\overline{Y}_1 \times \ldots \times Y_i \times \ldots \times \overline{Y}_m) \stackrel{\text{op}}{\hookrightarrow} W$. $\square$

Corollary 98 (Chow-Lemma)**.** *Y as above + proper $\Rightarrow \operatorname{pr}_W$ proper $\Rightarrow Z = W$ is projective over A.*

28.5. **Valuative criteria.** Let $R = \nu^{-1}(A_{\geq 0}) \subseteq K$ be a valuation ring, i.e. $\nu : K^* \to$ [ordered, abelian group A] with $\nu(xy) = \nu(x) + \nu(y)$ and $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$; denote $\operatorname{Spec} K = \{\eta\}$, $\operatorname{Spec} R = \{\eta, \xi\}$, i.e. $\operatorname{Spec} R$ is an abstract curve, but ξ should rather be seen as a codimension one subvariety in variety with function field K.

Proposition 99.



Let f be proper $\Rightarrow$ there exists a unique $s : \operatorname{Spec} R \to X$ making the diagram commutative.

Proof. The uniquenes follows from (20.3); for proving the existence, we may suppose that $Y = \operatorname{Spec} R$ and g = dominant (base change/schemetheoretic image). Let $f(x) = \xi$; the local rings form a chain

$$R \;=\; \mathcal{O}_{\operatorname{Spec} R, \xi} \stackrel{f^*}{\hookrightarrow} \mathcal{O}_{X,x} \;\subseteq\; \operatorname{Quot} \mathcal{O}_{X,x} \;=\; K(X) \;\subseteq\; K.$$

Since $K = \operatorname{Quot} R$ we have $K(X) = K$; the valuation ν ensures $R = \mathcal{O}_{X,x}$ ($q \in \mathcal{O}_{X,x} \setminus R \Rightarrow \nu(q) < 0$, i.e. $\forall x \in K\ \exists N \gg 0 : \nu(x/q^N) \geq 0$, i.e. $x/q^N \in R \rightsquigarrow x \in \mathcal{O}_{X,x}$). Hence, we obtain s out of $\operatorname{Spec} R = \operatorname{Spec} \mathcal{O}_{X,x} \to X$. $\square$

Remark. By [Hart, Theorem II.4.7], the opposite direction is true, too. I.e. a map f is proper if all (!) codimension one gaps in lifting of maps $Z \to Y$ toward $Z \to X$ can be filled.

28.6. **Global functions on proper schemes.** Let X be scheme over $k = \overline{k}$. We are going to generalize Proposition 53.

Proposition 100. *X = reduced, connected, complete $\Rightarrow \Gamma(X, \mathcal{O}_X) = k$.*

Proof. $g \in \Gamma(X, \mathcal{O}_X) \Rightarrow (\mathrm{id}, g^{-1}) : X_g \xrightarrow{\sim} V(g\,t-1) \subseteq X \times \mathbb{A}^1_k$ is a closed embedding ($\mathcal{O}_X[t] \twoheadrightarrow \mathcal{O}_X[t]/(gt-1) \xrightarrow{\sim} \mathcal{O}_{X_g}$), but does also factorize over $X \times (\mathbb{A}^1_k \setminus 0)$ via $t^{-1} \mapsto g$. $X =$ complete $\Rightarrow X \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ is closed $\Rightarrow$ the image of X_g is a point in $\mathbb{A}^1_k \setminus 0$, i.e. understood as a function, g is constant. Since closed points of $\mathbb{A}^1_k$ are k-rational, we are done. □

28.7. **Applications of properness.** Rather shortly, we mention the following important applications:

28.7.1. *Rigidity.* (Georg Heins Gast-VL 5.7.2011): Rigidity theorem; commutativity of complete group schemes, all morphisms among them with $1_G \to 1_H$ are group homomorphisms. Example: Group structure on elliptic curves.

28.7.2. *Direct images stay coherent.* In the spirit of Proposition 100, one has the very general theorem: $\boxed{f : X \to Y \text{ proper} \Rightarrow f_*(\text{coherent}) \text{ is coherent}}$.

28.7.3. *Stein factorization.* a) $f : X \to Y$ birational, proper; everything noetherian and integral; Y normal $\Rightarrow f_*\mathcal{O}_X = \mathcal{O}_Y$ (follows from (28.7.2).

b) $f : X \to Y$ proper, $f_*\mathcal{O}_X = \mathcal{O}_Y \Rightarrow$ all $\boxed{\text{fibers are connected}}$. (“ZMT”: [Hart, Th III.11.1/3] for projective morphisms).

c) $f : X \to Y$ proper $\Rightarrow$ it factorizes into $X \xrightarrow{g} \operatorname{Spec} f_*\mathcal{O}_X \xrightarrow{h} Y$, where the proper g has connected fibers, and h is finite.

28.7.4. *Quasifiniteness.* Quasifinite, proper morphisms $f : X \to Y$ are finite: Stein factorization $\rightsquigarrow f_*\mathcal{O}_X = \mathcal{O}_Y$, and f is bijective.

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Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

1. Aufgabenblatt zum 27.10.2021

Problem 1. Show that $\mathbb{R}[x]/(x^2+1)$ is isomorphic to $\mathbb{C}$ as an $\mathbb{R}$-algebra.

Problem 2. a) An ideal P in a ring R is called prime (ideal) if and only if the set $R \setminus P$ is closed under multiplication. Show directly that (0) and (3) are prime ideals in $R = \mathbb{Z}$ and that (10) is not.

b) Show that an ideal $P \subseteq R$ is prime if and only if R/P is a domain, i.e. lacks zero-divisors. Revisit the three examples of (a) under this aspect.

c) Let $I, J \subseteq R$ be ideals and let P be a prime ideal in R. Show that $[P \supseteq I$ or $P \supseteq J]$ if and only if $P \supseteq I \cap J$ if and only if $P \supseteq IJ$.

Problem 3. Show that (a) the sum of two nilpotent elements is again nilpotent and (b) that the sum of a nilpotent element and a unit is a always unit.

Problem 4. a) Recall (or consult a textbook or wikipedia) the notion of a category $\mathcal{C}$. Roughly speaking, it is a collection of objects $\mathrm{Ob}(\mathcal{C})$ (e.g. sets or groups or rings), and for every $A, B \in \mathrm{Ob}(\mathcal{C})$ there is a set $\mathrm{Mor}(A, B)$ of so-called morphisms with a couple of axioms. In particular, there is always provided a distinguished element $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ and a so-called composition map $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$, $f, g \mapsto g \circ f$.
In any category there is a well defined notion of isomorphisms. Moreover, $f \in \mathrm{Mor}(A, B)$ are often written as $f : A \to B$.
b) Call an $A \in \mathrm{Ob}(\mathcal{C})$ to be an *initial object*, if for any $B \in \mathrm{Ob}(\mathcal{C})$ the set $\mathrm{Mor}(A, B)$ consists of exactly one element. Check if the category of sets, the category of abelian groups, or the category of commutative rings with 1 have initial objects.
c) While initial objects might not exist at all (example?), show that whenever they exist they are uniquely determined. I.e. show that if $A, B \in \mathcal{C}$ are two initial objects, then there exists a unique isomorphism $f \in \mathrm{Mor}(A, B)$.
d) Let $\mathcal{C}$ be the category with

$$\mathrm{Ob}(\mathcal{C}) := \{(R, r) \mid R = \text{commutative ring with 1, and } r \in R\}.$$

A morphism $f \in \mathrm{Mor}\big((R, r), (S, s)\big)$ is defined to be a ring homomorphism $f : R \to S$ with $f(r) = s$. Determine the initial object in $\mathcal{C}$ (if it exists at all).

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

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FU Berlin, Winter 2021/22

2. Aufgabenblatt zum 3.11.2021

Problem 5. Let $Z \subseteq \mathbb{A}^n_k$ be a closed algebraic subset. Give a clean proof for the following claim discussed in class: Z is a point if and only if $I(Z) \subseteq k[x_1, \ldots, x_n]$ is a maximal ideal.

Problem 6. a) Show that the Zariski topology on $\mathbb{A}^2_k = k^2$ is not equal to the *product topology* (consult a textbook or Wikipedia if necessary) of the Zariski topologies on both factors k^1.

b) Let $Z \subseteq k^n$ be a Zariski closed subset; let $f \in A(Z) := k[x_1, \ldots, x_n]/I(Z)$. Show that $f : Z \to \mathbb{A}^1_k$ is a continous function with respect to the Zariski topology on both sides. (Note that the Zariski topology on $Z \subseteq k^n$ is defined as the topology being induced from the Zariski topology on k^n – consult a textbook or Wikipedia to see what this means).

c) Prove or disprove (by giving a counter example): Every bijective map $\varphi : k^1 \to k^1$ is continous with respect to the Zariski topology on both sides.

Problem 7. A topological space X is called irreducible if it cannot be written as $X = X_1 \cup X_2$ with some proper closed subsets $X_i \subsetneq X$ $(i = 1, 2)$. Show that this is equivalent to the fact that all non-empty open subsets $U \subseteq X$ are dense in X, i.e. fulfill $\overline{U} = X$.

Problem 8. Recall (or consult a textbook or wikipedia) the notion of covariant and contravariant functors between categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a (covariant) functor between two categories; let $A, A_i \in \mathrm{Ob}(\mathcal{A})$ $(i = 1, 2)$.

a) Show that if $f : A_1 \to A_2$ is an iomorphism, then $F(f) : F(A_1) \to F(A_2)$ is an iomorphism, too.

b) Show that $\mathrm{Aut}(A) \to \mathrm{Aut}(F(A))$ is a group homomorphism (where $\mathrm{Aut}(A) := \{\varphi \in \mathrm{Hom}_{\mathcal{A}}(A, A) \mid \varphi \text{ is an isomorphism}\}$).

c) Assume that F is *fully faithful*, i.e. $\mathrm{Hom}(A_1, A_2) \to \mathrm{Hom}(FA_1, FA_2)$ is bijective for all $A_1, A_2 \in \mathrm{Ob}\mathcal{A}$. Show that then the reverse implication of (a) is true, too. That is, if $F(f)$ is an isomorphism, then so is f.

d) Provide an example showing that in (c) the injectivity of

$$\mathrm{Hom}(A_1, A_2) \to \mathrm{Hom}(FA_1, FA_2)$$

does not suffice.

e) ("Yoneda-Lemma") Let $\mathcal{C}$ be a category. Show that the functor

$$\begin{aligned} \Phi: \quad \mathcal{C} &\longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathcal{S}et) \\ Y &\longmapsto \mathrm{Hom}_{\mathcal{C}}(\bullet, Y) \end{aligned}$$

is fully faithful.

(The latter contains the covariant functors $\mathcal{C}^{\mathrm{opp}} \to \mathcal{S}et$, i.e. the contravariant functors $\mathcal{C} \to \mathcal{S}et$ as objects and the natural transformations between them as morphisms. The functors $F = \Phi(Y)$ are called "represented by Y". They come with a distinguished element $\xi \in F(Y)$.)

Hint: Show $\mathrm{Hom}_{\mathrm{Fun}}(\Phi Y, F) = F(Y)$ for any contravariant functor $F : \mathcal{C}^{\mathrm{opp}} \to \mathcal{S}et$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

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3. Aufgabenblatt zum 10.11.2021

Problem 9. a) Construct two non-trivial, open subsets $D(f), D(g) \subseteq \mathbb{A}^2_{\mathbb{C}}$, such that $D(f) \cup D(g) = \mathbb{A}^2_{\mathbb{C}}$.
b) Construct an open covering of the $\mathbb{A}^2_{\mathbb{C}}$ by three subsets $D(f), D(g), D(h)$ such that any choice of only two of them does not cover the whole plane.

Problem 10. Let $f \in k[x_1, \ldots, x_n] =: k[\mathbf{x}]$. Then, we have obtained in Subsection (1.3) the bijective map $p : Z_f \to D(f)$. We are going to show that it is a homeomorphism, i.e. that both p and p^{-1} are continous (with respect to the Zariski topologies on both sides):

a) Denote by ι_Z and ι_D the embeddings $Z \hookrightarrow k^{m+1}$ and $D \hookrightarrow k^m$, respectively. Then $\iota_D \circ p = \mathrm{pr} \circ \iota_Z$ is a continous map $Z_f \to k^m$. Conclude that then p has to be continous, too.
(*Reminder:* A map between topological spaces is continous if the preimages of closed subsets are closed.)

b) It remains to show that the map $\phi : D(f) \to k^{m+1}$, $\mathbf{x} \mapsto (\mathbf{x}, t := 1/f(\mathbf{x}))$ is continous, too. Let $J \subseteq k[\mathbf{x}, t]$ be an ideal. For each $g \in k[\mathbf{x}, t]$ we define $\widetilde{g} \in k[\mathbf{x}]$ to be

$$\widetilde{g}(\mathbf{x}) := f(\mathbf{x})^N \cdot g(\mathbf{x}, \tfrac{1}{f(\mathbf{x})})$$

where $N \gg 0$ is sufficiently large such that $\widetilde{g}$ becomes a polynomial. Note that N depends on g and that it is not uniquely determined at all – just choose and fix one for each g.

Finally, we define $\widetilde{J} := \{\widetilde{g} \mid g \in J\}$ – or likewise the ideal generated from this set. Then show that $\phi^{-1}(V(J)) = V(\widetilde{J}) \cap D(f)$.

Problem 11. Let k be an algebraically closed field, i.e. you may use the HNS saying that $I(V(J)) = \sqrt{J}$ for ideals $J \subseteq k[\mathbf{x}] := k[x_1, \ldots, x_n]$. Show that for Zariski closed subsets $Z_i \subseteq k^n$ one has then $I(\bigcap_i Z_i) = \sqrt{\sum_i I(Z_i)}$.

Problem 12. A k-algebra $k \to R$ is called finitely generated if there are finitely many elements $r_1, \ldots, r_n \in R$ such that there is no proper subalgebra $k \to S \subsetneq R$ containing $r_1, \ldots, r_n$, i.e. $r_1, \ldots, r_n \in S$.

a) Show that $k \to R$ is a f.g. k-algebra if and only if it is of the form, i.e. isomorphic to $k[x_1, \ldots, x_n]/J$ for some ideal $J \subseteq k[\mathbf{x}]$. In particular, there is then a surjection $k[\mathbf{x}] \twoheadrightarrow R$ of k-algebras.

b) Find such a representation for $R = k[t^2, t^3] = k \oplus t^2 \cdot k[t]$.

c) If $f : R \to S$ is a k-algebra-homomorphism between f.g. k-algebras, i.e. f is compatible with the "structure homomorphisms $k \to R$ and $k \to S$, then we know from (a) that there are k-algebra surjections $k[\mathbf{x}] \twoheadrightarrow R$ and $k[\mathbf{y}] \twoheadrightarrow S$. Show that there is a k-algebra homomorphism $F : k[\mathbf{x}] \to k[\mathbf{y}]$ such that

$$\begin{array}{ccc} k[\mathbf{x}] & \xrightarrow{F} & k[\mathbf{y}] \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & S \end{array}$$

commutes. Is F uniquely determined?

d) Do (c) explicitely for $R = k[t^2, t^3] \hookrightarrow k[t] = S$.

e) What is the geometric counterpart of (c) and (d)?

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

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4. Aufgabenblatt zum 17.11.2021

Problems 68(a)-(c) and 14 are supposed to be uploaded on Whiteboard until 11/17, 4pm. This has to be done with a single pdf-file consisting of exactly 2 pages.

Problem 13. Let $\varphi : A \to B$ be a ring homomorphism. Show that

a) the associated $(f = \operatorname{Spec}\varphi) : \operatorname{Spec} B \to \operatorname{Spec} A$ (defined via $f : Q \mapsto \varphi^{-1}Q$) is continous. That is, with respect to the Zariski topology on both sides, show that the pre-images of closed subsets are closed.

b) Give an alternative proof of (a) by showing that the pre-image of the so-called elementary open subsets $D(f) \subseteq \operatorname{Spec} A$ (for $f \in A$) are open in $\operatorname{Spec} B$. Why does it suffice to consider these special open subsets instead of all ones?

c) Recall that, for every $P \in \operatorname{Spec} A$, we denote by $K(P) := \operatorname{Quot}(A/P)$ the associated residue field of P. Show that φ and f from (a) provide a natural embedding $\overline{\varphi} : K\big(f(Q)\big) \hookrightarrow K(Q)$ for each $Q \in \operatorname{Spec} B$.

d) Recall that elements $a \in A$ can be understood as functions on $\operatorname{Spec} A$ via assigning each P its residue class $\overline{a} \in K(P)$. Show that, in this context, the map $\varphi : A \to B$ can be understood as the pull back map (along f) for functions, i.e. that, under use of (c), $\varphi(a) \mathrel{\widehat{=}} a \circ f$.
(A maybe confusing remark: Making the last correspondence more explicit – but maybe less user friendly – one is tempted to write $\varphi(a) = \overline{\varphi} \circ a \circ f$. However, this is even less correct, since there is no "general map" $\overline{\varphi}$; even the domain and the target of $\overline{\varphi}$ depend on Q.)

Problem 14. a) Let A be a ring. Describe the set of elements $a \in A$ with $D(a) = \emptyset$.

b) Let $\varphi : A \to B$ be a surjective ring homomorphism. Show that $\operatorname{Spec}\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is injective.

c) Let $\varphi : A \to B$ be an injective ring homomorphism. Show that $\operatorname{Spec}\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is dominant, i.e. that the image is dense.
(*Hint:* You might use that a subset $S \subseteq X$ of a topological space X is *not* dense iff there exists a non-empty open $U \subseteq X$ being disjoint to S.)

d) Give an example for the situation of (c) where $\operatorname{Spec}\varphi$ is *not* surjective.

Problem 15. Show that $\operatorname{Spec} A$ is quasicompact, i.e. that every open covering admits a finite subcovering. (Note that we avoid the name "compact" for this property because $\operatorname{Spec} A$ is not Hausdorff.)
(*Hint:* Try to use the "elementary" open subsets $D(a)$ whenever you can.)

Problem 16. Let $R_1, \ldots, R_m$ be (commutative) rings (with 1) and denote by $R := \prod_i R_i$ their product.

a) Show that the units $1_i \in R_i$ induce so-called "orthogonal idempotents" $e_i \in R$, i.e. elements having the property $e_i e_j = \delta_{i,j} e_i$. Moreover, show that each choice of orthogonal idempotents $\{e_1, \ldots, e_m\}$ in a ring R gives rise of a decomposition $R = \prod_i R_i$ of R into a product of rings.

b) Do we have natural ring homomorphisms $\varphi_i : R_i \to R$ or $\psi_i : R \to R_i$? Show that the right choice induces a homeomorphism between the topological spaces $\coprod_i \operatorname{Spec} R_i$ and $\operatorname{Spec} R$. What is the geometric interpretation of φ_i / ψ_i when $\operatorname{Spec} R$ is identified with $\coprod_i \operatorname{Spec} R_i$?

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

5. Aufgabenblatt zum 24.11.2021

Problem 17. a) Let $\varphi : A \to B$ be a ring homomorphism where A and B are even fields. Show that φ is then automatically injective.

b) Give counter examples for the cases that either A or B is not a field.

Problem 18. In Problem 14(c) it had to be exploited that injective ring homomorphisms $\varphi : A \to B$ send non-nilpotent elements to non-nilpotent elements. Do those φ also send non-zero divisors to non-zero divisors? (Proof/counter example)

Problem 19. a) Let $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ be an exact sequence of R-modules. Show this sequence is a split exact sequence (i.e. it is isomorphic to the sequence $0 \to K \to K \oplus M \to M \to 0$) $\Leftrightarrow$ the map g has a section, i.e. if there is an (R-linear) map $s : M \to L$ such that $gs = \mathrm{id}_M$.

b) In class we have shown that the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ does not split. Give an alternative proof of this via using (a).

c) Show that short exact sequences of vector spaces (i.e. R is a field) do always split.

Problem 20. Give an example of an injection $M \hookrightarrow M'$ of abelian groups, i.e. $\mathbb{Z}$-modules, and an abelian group N such that

$$M \otimes_{\mathbb{Z}} N \neq 0 \qquad \text{but} \qquad M' \otimes_{\mathbb{Z}} N = 0.$$

Hint: Do your search among the usual suspects $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/2\mathbb{Z}$...

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

6. Aufgabenblatt zum 1.12.2021

Problem 21. Let $F : R\text{-mod} \to S\text{-mod}$ be a covariant, additive functor from the category of R-modules into the category of S-modules. (Additivity here just means that for R-modules M, N the map $F : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(FM, FN)$ is additive, i.e. $\mathbb{Z}$-linear.)

a) Check the additivity of the functors $F =$ tensor $\otimes_R N$, $\mathrm{Hom}(M, \bullet)$, and localization $M \mapsto S^{-1}M$.

b) Show that F preserves the exactness of arbitrary sequences (“*exact functors*”) $\Leftrightarrow$ of sequences of the form $M' \to M \to M''$ $\Leftrightarrow$ of “short” exact sequences $0 \to M' \to M \to M'' \to 0$.
(*Hint:* Decompose $M' \to M \to M''$ into two “short” exact sequences.)

c) F preserves the exactness of sequences of the form $0 \to M' \to M \to M''$ (“*left exact functors*”) $\Leftrightarrow$ short exact sequences $0 \to M' \to M \to M'' \to 0$ yield exact sequences $0 \to FM' \to FM \to FM''$.

Problem 22. Calculate the Dehn invariant $D(S) = \sum_{e \in S_1} \ell(e) \otimes a(e) \in \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Q})$ for the following three solids (all of volume 1):

(i) $S_1 =$ unit cube,

(ii) $S_2 =$ prism $[0, 1] \times A$ where A is a triangle with angles α, β, γ and area 1, and

(iii) $S_3 =$ is a regular tetrahedron with edge length s that $\mathrm{vol}(S_3) = 1$ (what is s?).

Finally, check which of the results in (i), (ii), (iii) are equal, and which differ from each other. (*Hint:* Use that for k-vector spaces V, W with bases $B \subset V$ and $C \subset W$, the set $B \otimes C := \{b \otimes c \mid b \in B,\ c \in C\}$ forms a basis of $V \otimes_k W$.)

Problem 23. a) Recall that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$. What about $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/8\mathbb{Z}$? Can you generalize this into a description of $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z}$?

b) Detemine a basis of the $\mathbb{Q}$-vector space $V = \mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{Q}^3$. What is the difference of this space to the abelian group $\mathbb{Q}^2 \otimes_{\mathbb{Z}} \mathbb{Q}^3$?

c) Detemine a basis of the $\mathbb{C}$-vector space $V = \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^3$. What is it dimension as an $\mathbb{R}$-vector space? What is its difference to the $\mathbb{R}$-vector space $\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C}^3$?

d) What is $R/I \otimes_R R/J$?

e) Determine $\mathbb{R}[x, y] \otimes_{\mathbb{R}} \mathbb{C},\ \mathbb{R}[x, y]/(y^2 - x^3) \otimes_{\mathbb{R}} \mathbb{C},\ \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y],\ \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[x]$.

Problem 24. a) Let $F : \mathcal{C} \to \mathcal{D}$ und $G : \mathcal{D} \to \mathcal{C}$ be functors. We call F to be “*left adjoint*” to G (or G to be “*right adjoint*” to F; written as $F \dashv G$) if $\mathrm{Hom}_{\mathcal{D}}(FA, B) = \mathrm{Hom}_{\mathcal{C}}(A, GB)$ for A, B being objects of $\mathcal{C}$ and $\mathcal{D}$, respectively.

Here the equality sign means a bijection that is functorial in both arguments. Explain what is meant by the last sentence.

b) In the situation of (a) show that $F \dashv G$ is equivalent to the existence of the so-called adjunction maps, i.e. of natural transformations $FG \to \mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{C}} \to GF$ with certain compatibility properties (describe them). How do these maps look like for all examples of mutually adjoint functors you have heard of in the past?

c) Let $\varphi : R \to T$ be a ring homomorphism, i.e. let T be an R-algebra. Then there are the following functors between the module categories $F : \mathcal{M}od_R \to \mathcal{M}od_T$, $M \mapsto M \otimes_R T$ and $G : \mathcal{M}od_T \to \mathcal{M}od_R$, $N \mapsto N$, where N becomes an R-module via $rn := \varphi(r)n$. Show that $F \dashv G$.

d) Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between two categories of modules over some rings. Show that the existence of a right adjoint for F implies right exactness of F. Does the existence of a left adjoint have a comparable impact?

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

7. Aufgabenblatt zum 8.12.2021

Problem 25. a) Determine the localizations $(\mathbb{Z}/6\mathbb{Z})_2$, $(\mathbb{Z}/6\mathbb{Z})_3$, $(\mathbb{Z}/6\mathbb{Z})_{(2)}$, $(\mathbb{Z}/6\mathbb{Z})_{(3)}$. Is there respective localization maps $\mathbb{Z}/6\mathbb{Z} \to \ldots$ injective or surjective?

b) Let M, N be two R-modules. Show that $M \oplus N$ is flat over $R \Leftrightarrow$ both M and N are flat R-modules.

c) Give two different proofs for the flatness of $\mathbb{Z}/6\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. (A ring homomorphism $f : R \to S$ is called flat if S becomes, via f, a flat R-module.)

Problem 26. a) Let R be a (commutative) ring and $f : R^m \to R^n$ an R-linear map given by a matrix A with R-entries. If $\varphi : R \to S$ is a ring homomorphism, then R-modules M turn into S-modules $M \otimes_R S$. Since $R^m \otimes_R S = S^m$, the map f turns into $(f \otimes_R \mathrm{id}_S) : S^m \to S^n$. What is the associated matrix over S?

b) Let R be a (commutative) ring and $f : R^m \twoheadrightarrow R^n$ a surjective, R-linear map. Show that $m \geq n$.

c) Let $g : R^m \hookrightarrow R^n$ be injective. Under the assumption that R is an integral domain, show that $m \leq n$. Does this claim still hold true if R has zero divisors?

Problem 27. a) Let $R = (R, \mathfrak{m})$ be a local ring; let $f : M \to N$ be R-linear. Decide which of the possible four implications ($\Rightarrow$/$\Leftarrow$) holds true: $f : M \to N$ is injective/surjective $\Leftrightarrow \overline{f} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective/surjective? Is it important whether M, N are finitely generated?

b) If $I \subseteq J \subseteq R$ are ideals, then show that the ideal $(J/I)^2 \subseteq R/I$ equals $(J^2 + I)/I$.

c) Let $\mathfrak{m} := (x, y, z) \subseteq R$ with

$$R := \{f/g \mid f, g \in \mathbb{C}[x, y, z]/(xyz + x + y + z) \text{ mit } g(0, 0, 0) \neq 0\}.$$

Determine a basis of the $R/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$ and a minimal generating system of the ideal $\mathfrak{m}$. Express $x, y, z \in \mathfrak{m}$ as R-linear combinations of this system.
(*Hint:* Use that the space $\mathfrak{m}/\mathfrak{m}^2$ equals $(x, y, z)/(x, y, z)^2$ where (x, y, z) is understood as an ideal in the ring $\mathbb{C}[x, y, z]/(xyz + x + y + z)$, i.e. one can use (b) now.)

Problem 28. Let $(R, \mathfrak{m})$ be a local integral domain; denote by $k := R/\mathfrak{m}$ and $K := \mathrm{Quot}\, R$ its residue and quotient field, respectively. If M is a finitely generated R-module, then show that M is free $\Leftrightarrow \dim_k(M \otimes_R k) = \dim_K(M \otimes_R K)$. (*Hint:* Choose a surjection $R^n \twoheadrightarrow M$ with minimal n and tensorize.)

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

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8. Aufgabenblatt zum 15.12.2021

Problem 29. For $R := k \oplus x^2\, k[x] \subseteq k[x]$ and $S := k \oplus xy\, k[x,y] \subseteq k[x,y]$ check if they are finitely generated k-algebras, and check if they are noetherian.

Problem 30. Construct a filtration of $R := k[x,y]/(x^2y, x^3)$ where all factors are isomorphic to R/P_i for some $P_i \in \operatorname{Spec} R$. In particular, identify the P_i for all factors.

Problem 31. a) Let $I := (I, \leq)$ be a poset. It turns into a category via objects $:= I$ and $\operatorname{Hom}_I(a,b) := \begin{cases} \{(a,b)\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise} . \end{cases}$ A "*directed system* on I with values in a category $\mathcal{C}$" is a (covariant) functor $I \to \mathcal{C}$; the "direct limit" $\varinjlim X_i$ of such a system $X = (X_i \mid i \in I)$ is defined via the following universal property: $\operatorname{Hom}_{\mathcal{C}}(\varinjlim X_i, Z) = \{\varphi \in \prod_i \operatorname{Hom}(X_i, Z) \mid i \leq j \Rightarrow \varphi_i = \varphi_j \circ [X_i \to X_j]\}$. In particular, there are canonical maps $X_j \to \varinjlim X_i$ (as the image of $\mathrm{id} \in \operatorname{Hom}_{\mathcal{C}}(\varinjlim X_i, \varinjlim X_i)$). Translate the notion of the direct limit into that of an initial object in some category.

b) What is $\varinjlim X_i$ if I contains a maximum? What is $\varinjlim X_i$ if all elements of I are mutually non-comparable, i.e. if $i \leq j \Leftrightarrow i = j$?

c) Let $\mathcal{C} = \mathcal{M}od_R$ be the category of modules over some ring R. For an element $m_j \in M_j$ we will use the same symbol m_j for its canonical image in $M := \oplus_{i \in I} M_i$, too. Using this notation, show that $\varinjlim M_i = M/N$ where the submodule $N \subseteq M$ is generated by all differences $m_j - \varphi_{jk}(m_j)$ with $m_j \in M_j$, $j \leq k$, and $\varphi_{jk} : M_j \to M_k$ being the associated R-linear map.

d) Assume $(I, \leq)$ to be *filtered*, i.e. for $i, j \in I$ there is always a $k = k(i,j) \in I$ with $i, j \leq k$. If $\mathcal{C} = \mathcal{M}od_R$, then $\varinjlim M_i = \coprod_i M_i / \sim$, where $\coprod$ means the disjoint union (as sets) and "$\sim$" is the equivalence relation generated by $[\varphi_{ij}(m_i) \sim m_i$ for $i \leq j]$ (with $\varphi_{ij} : M_i \to M_j$). (*Hint:* First, define an R-module structure of the right hand side. Then check that an element $x \in M_i$ turns into $0 \in \varinjlim M_i$ if and only if there is a $j \geq i$ with $\varphi_{ij}(x) = 0 \in M_j$.)

Problem 32. a) Let $P \in \operatorname{Spec} R$ be a prime ideal and M an R-module. Show that the localisation M_P is the direct limes of modules M_f with distinguished elements $f \in R$. What is the associated poset $(I, \leq)$? Is it filtered?

b) Show that infinite direct sums are filtered direct limits of finite direct sums. What is the underlying poset?

Aufgabenblätter und Nicht-Skript: http://www.math.fu-berlin.de/altmann

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

VL "(Comm) Algebra I"
FU Berlin, Winter 2021/22

9. Aufgabenblatt zum 5.1.2022

Problems 33 and 34 are supposed to be uploaded on Whiteboard until Jan 5 2022, 4pm. This has to be done with a single pdf-file consisting of exactly 2 pages.

Problem 33. Let $M_1, M_2 \subseteq M$ be submodules of a finitely generated module over a noetherian ring R. Show that $\operatorname{Ass}\big(M/(M_1 \cap M_2)\big) \subseteq \operatorname{Ass}(M/M_1) \cup \operatorname{Ass}(M/M_2)$.
Hint: Try to exploit Propsition 13, i.e. to look for exact sequences relating, e.g., $M/(M_1 \cap M_2)$ and M/M_1.

Problem 34. Show that $I := (x, y) \subseteq k[x, y] =: R$ is not "clean", i.e. there is no "nice filtration" (i.e. with factors $\cong R/P_i$) of I (not of R/I) with an exclusive use of primes associated to $I = (x, y)$ (really to I, not to R/I).

Problem 35. In the category of directed systems of R-modules on a poset $I := (I, \leq)$ (cf. Problem 31) define kernels, images, and cokernels in a natural way, e.g.

$$\ker\big(\varphi : (M_i \,|\, i \in I) \to (N_i \,|\, i \in I)\big) := \big(\ker[\varphi_i : M_i \to N_i] \,|\, i \in I\big).$$

This leads to the notion of exact sequences of directed systems.

a) Show that $\varinjlim$ is right exact (by constructing a right adjoint functor).

b) Show that *filtered* direct limits with values in $\mathcal{M}od_R$ are even exact.

c) Consider the set $I := \{m, a, b\}$ with $m < a$ and $m < b$. Show that the direct limit over this I (even with values in $\mathcal{M}od_R$) is not left exact.

Problem 36. Analogous to Problem 31, we define the "*inverse*" limit $\varprojlim M_i$ of a directed system of R-modules as the *terminal* object of a certain category, namely via $\operatorname{Hom}_R(P, \varprojlim M_i) = \{\varphi \in \prod_i \operatorname{Hom}(P, M_i) \,|\, i \leq j \Rightarrow \varphi_j = [M_j \leftarrow M_i] \circ \varphi_i\}$. In particular, there are canonical maps $\varprojlim M_i \to M_i$.

a) Realize $\varprojlim_i M_i$ as a submodule of $\prod_i M_i$ and derive from this that the projective limit is left exact.

b) Let $p \in \mathbb{Z}$ be a prime and $I := \mathbb{N}$. Show that $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^i\mathbb{Z}$ ("p-adic numbers" – not to be confused with the localization $\mathbb{Z}_p$) is a local ring without zero divisors. Show further that it contains $\mathbb{Z}$ and the localization $\mathbb{Z}_{(p)}$.

Merry Christmas and a Happy New Year 2022!

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

VL "(Comm) Algebra I"
FU Berlin, Winter 2021/22

10. Aufgabenblatt zum 12.1.2022

Problem 37. Let R be noetherian and $P \subseteq R$ a prime ideal. Let M be a finitely generated R-module. Show that that M_P is a free R_P-module if and only if there is an element $f \in R \setminus P$ such that M_f is a free R_f-module.

Problem 38. Recall Problem 26(c): For R being a noetherian ring we were given $m, n \in \mathbb{N}$. Then, the existence of an injective R-linear map $g : R^m \hookrightarrow R^n$ had implied that $m \leq n$.

a) Give an alternative proof of this fact in the case of R being an Artinian ring, i.e., if $\ell_R(R) < \infty$.

b) Provide a proof of the general claim (without assuming that R is Artinian) under use of Part (a). (*Hint*: Show and use that, for a minimal prime P, the localization R_P is artinian.)

Problem 39. a) What is the length of the ring $\mathbb{Z}/30\mathbb{Z}$? Provide a composition series and describe the factors.

b) Write down a composition series of the ring $k[t]/t^3$ and identify its factors as fields.

Problem 40. What are the minimal and what are the associated primes P of $R = \mathbb{C}[x,y]/(x^2, xy^2)$? For the latter provide always an embedding $R/P \hookrightarrow R$. Which of the localizations R_P have finite length – and what is this length then? Visualize a monomial base of R and all R/P – how does this reflect the previous information about the lenghths?

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

VL "(Comm) Algebra I"
FU Berlin, Winter 2021/22

11. Aufgabenblatt zum 19.1.2022

Problem 41. Find reduced, i.e. non-redundant primary decompositions of the ideals

$$I = (xy^5, x^3y^4, x^6y^2) \subset k[x, y] \quad \text{and} \quad J = (x^5, x^3yz, x^4z) \subset k[x, y, z].$$

Download the software Singular or Macaulay2 and check the result by one of these computer algebra systems.

> You can learn about the usage of the computeralgebra system Singular by attending the two weeks compact course "Computeralgebra" in early March. It is a BA-course within the so-called ABV part. That is, as master students, you cannot earn any formal credit for this – but, nevertheless, it might be useful. And it is fun, anyway.

Problem 42. Show directly that $\alpha := t^2+1 \in \mathbb{C}[t]$ is integral over the subring $\mathbb{C}[t^3]$.

Problem 43. Let $A \subseteq B$ be two rings and assume that all elements of B are integral over A. Show that this implies $B^* \cap A = A^*$. Is the reverse implication true as well?

Problem 44. A ring homomorphism $f : A \to B$ is called *integral* if all elements $b \in B$ are integral over $f(A)$.

a) Show that the integrality of f implies the integrality of $f \otimes \mathrm{id}_C : C \to B \otimes_A C$ for every A-algebra C. In particular, localizing f via multiplicative subsets $S \subseteq A$ is compatible with integrality.

b) Let $(f_1, \ldots, f_k) = (1)$ in A. Thus, the open subsets $D(f_i) = \operatorname{Spec} A_{f_i}$ cover $\operatorname{Spec} A$. Show that an A-module M is finitely generated if and only if all M_{f_i} are finitely generated A_{f_i}-modules.

c) Assume again that $(f_1, \ldots, f_k) = (1)$ in A. Show that an element $b \in B$ is integral over $A \Leftrightarrow b/1 \in B_{f_i}$ is integral over A_{f_i} for all i.

d) Let M be an A-module such that the localizations M_P are finitely generated over A_P for all $P \in \operatorname{Spec} A$. Show that $M := \oplus_{P \in \mathrm{MaxSpec}\, A} A_P/PA_P$ is an example demonstrating that the original M does not need to be finitely generated though.

e) Show that an element $b \in B$ is integral over $A \Leftrightarrow b/1 \in B \otimes_A A_P$ is integral over A_P for all $P \in \operatorname{Spec} A$. (*Hint:* For each P construct an element $f \notin P$ such that $b/1 \in B_f$ is integral over A_f.)

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

12. Aufgabenblatt zum 26.1.2022

Problem 45. Let R be a domain such that for every $q \in \operatorname{Quot} R$ one has $q \in R$ or $1/q \in R$ (R is called a "valuation ring"). Show that this property implies that R is local and normal, i.e. integrally closed in its quotient field. (*Hint:* Show that $R \setminus R^*$ is an ideal; for the additivity consider x/y for given $x, y \in R \setminus R^*$.)

Problem 46. For a semigroup H with neutral element $0 \in H$ we define the associated "semigroup algebra" $\mathbb{C}[H] := \oplus_{h \in H} \mathbb{C} \cdot \chi^h$ with multiplication $\chi^h \cdot \chi^{h'} := \chi^{h+h'}$ among the basis vectors.

a) Describe $\mathbb{C}[H]$ explicitely for the examples $H = \mathbb{N}$, $H = \mathbb{Z}$, $H = \mathbb{N}^2$, and $H = \mathbb{N} \times \mathbb{Z}$.

b) Assume that $H \subseteq \mathbb{Z}^n$ is finitely generated with $\mathbb{Z}^n = H - H := \{h - h' \mid h, h' \in H\}$. Show that $\mathbb{C}[H]$ is a normal ring if and only if $H = \mathbb{Z}^n \cap (\mathbb{Q}_{\geq 0} \cdot H)$ inside $\mathbb{Q}^n$ ("H is saturated"). Give an example where this condition does not hold true.

(*Hint:* For the part ($\Leftarrow$) write H as an intersection of half spaces. Hence, the claim can be reduced to the special case of $H = \mathbb{N} \times \mathbb{Z}^{n-1}$.)

Problem 47. a) Let $A \subseteq B$ be a finite extension of domains, i.e. the A-algebra B becomes a finitely generated A-*module*. Further denote by $F : \operatorname{Spec} B \to \operatorname{Spec} A$ the associated map on the geometric side. Show that F is quasi-finite, i.e. that F has finite fibers, i.e. that for each prime ideal $P \subset A$ the set $F^{-1}(P)$ is finite.

(*Hint:* Exploit the usual localization/quotient constructions on the A-side to improve the situation.)

b) Determine the fibers of $P = (x, y)$ and of $P' = (x - 1, y - 1)$ with respect to the situation $A = \mathbb{C}[x, y]$ and $B = \mathbb{C}[x, y, z]/(xy - z^2)$.

c) What is the description of F and P, P', Q_i from (b) within the classical geometric language, i.e. understanding $\operatorname{Spec} \mathbb{C}[x, y] = \mathbb{A}^2_{\mathbb{C}}$ as $\mathbb{C}^2$?

Problem 48. a) Let $f : A \to B$ be an integral ring homomorphism, i.e. B is integral over the subring $f(A)$. Show that $\operatorname{Spec}(f) : \operatorname{Spec} B \to \operatorname{Spec} A$ is then a closed map, i.e. the images of closed subsets are always closed.

(*Hint:* Identify first the natural candidate for the closed subset of $\operatorname{Spec} A$ forming the image of some $\operatorname{Spec} B/J = V(J) \subseteq \operatorname{Spec} B$ under $F = \operatorname{Spec}(f)$. Then show that F does indeed map $V(J)$ surjectively onto this candidate.)

b) Show, in the situation of (a), that for A-algebras C, i.e. for ring homomorphisms $A \to C$, the map $\operatorname{Spec}(f \otimes \mathrm{id}) : \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec} C$ is closed, too.

c) Give an example for a (non-integral) $f : A \hookrightarrow B$ and some A-algebra C, such that $\operatorname{Spec}(f) : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed map, but $\operatorname{Spec}(f \otimes \mathrm{id}) : \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec} C$ is not.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

13. Aufgabenblatt zum 2.2.2022

Problem 49. As introduced in Problem 46, denote by $\mathbb{C}[H]$ the semigroup algebra of an (in our case abelian) semigroup H.

a) For $H_1 := \mathbb{N}^2$ and $H_2 := \{(a,b) \in \mathbb{N}^2 \mid b \le 2a\}$ present $\mathbb{C}[H_i]$ as a quotient of polynomial rings by an ideal. Which geometric objects are described by $\operatorname{Spec}\mathbb{C}[H_1]$ and $\operatorname{Spec}\mathbb{C}[H_2]$? Show that both contain $(\mathbb{C}^*)^2 = \operatorname{Spec}\mathbb{C}[\mathbb{Z}^2]$ as an open subset.

b) Verify the Noether normalization lemma explicitely for the example $\mathbb{C}[H_2]$.

c) Do (a) and (b) with the example $H_3 := \{(a,b,c) \in \mathbb{N}^3 \mid a,b \le c\}$, too.

Is it possible to choose the subalgebra $\mathbb{C}[\mathbf{y}] \subseteq \mathbb{C}[H_3]$ (where $\mathbb{C}[H_3]$ is finite over) such that all y_i are monomials in $\mathbb{C}[H_3]$?

Problem 50. Let $R := \mathbb{C}[x,y]/(y^2 - x^3)$. For a point $(a,b) \in \mathbb{C}^2$ let $\mathfrak{m}_{(a,b)} := (x-a,\, y-b) \subseteq R$.

a) For which points is $\mathfrak{m}_{(a,b)} = (1)$?

b) For which points is $\mathfrak{m}_{(a,b)}$ a projective R-module?

c) Draw the curve $E := \{(a,b) \in \mathbb{R}^2 \mid b^2 = a^3\}$ and mark the points where $\mathfrak{m}_{(a,b)}$ is not projective.

Problem 51. Let $I, J \subseteq A$ be ideals.

a) Determine the kernel of f such that

$$0 \to (???) \to I \oplus J \stackrel{f}{\to} I + J \to 0$$

becomes an exact sequence of A-modules.

b) Assume that $I + J = A$. Show that this implies that $IJ \oplus A \cong I \oplus J$.

c) Present explicitely $(2, 1+\sqrt{-5})$ as a direct summand of a free $\mathbb{Z}[\sqrt{-5}]$-module.

Problem 52. a) Let $0 \to A_\bullet \stackrel{f}{\to} B_\bullet \stackrel{p}{\to} C_\bullet \to 0$ be an exact sequence of complexes. Show that the projection $\mathrm{pr}_B : \operatorname{Cone}(f) \to B_\bullet$ (despite it is not a map complexes itself) induces a map complexes $\Phi = (p \circ \mathrm{pr}_B) : \operatorname{Cone}(f) \to C_\bullet$. Show further that Φ is a quasiisomorphism. In particular, we almost obtain a map $\mathrm{pr}_A \circ \Phi^{-1} : C_\bullet \to A_\bullet[1]$. What does the word "almost" refer to?

b) If all sequences $0 \to A_i \to B_i \to C_i \to 0$ from (a) split, then Φ is even a homotopy equivalence. (*Hint:* One constructs the "inverse" Ψ of Φ as $\Psi_i(c_i) := (s(c_i), \ldots)$ where the second entry is chosen such that Ψ commutes with the differentials.) What does this change about the word "almost" from (a)?

c) A sequence $A_\bullet \to B_\bullet \to C_\bullet$ of complexes is called a *distinguished triangle* if it is isomorphic to the sequence $N_\bullet \to \text{Cone}\,(f)_\bullet \to M_\bullet[1]$ obtained from some map of complexes $f : M_\bullet \to N_\bullet$. Assume now that $A_\bullet \to B_\bullet \to C_\bullet$ is such an object in the homotopy category $K(\mathcal{A})$ (for $\mathcal{A} = \mathcal{M}od_R$ or, more general, $\mathcal{A} =$ abelian category). Show that it gives rise to a new distinguished triangle $B_\bullet \to C_\bullet \to A_\bullet[1]$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

14. Aufgabenblatt zum 9.2.2022

Problem 53. Let $f : M_\bullet \to N_\bullet$ be a complex homomorphism and $A_\bullet$ be a bounded complex. Show that

$$\mathrm{Hom}_\bullet(A_\bullet, \mathrm{Cone}\,(f)_\bullet) = \mathrm{Cone}\,(\mathrm{Hom}(A_\bullet, f)),$$

i.e. the Hom functor commutes with the mapping cone construction. (Note that $\mathrm{Hom}(A_\bullet, f)$ denotes the complex homomorphism $\mathrm{Hom}(A_\bullet, M_\bullet) \to \mathrm{Hom}(A_\bullet, N_\bullet)$ being induced from f.)

Problem 54. Consider the free resolution $\mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^2 \xrightarrow{\beta} \mathbb{Z}/3\mathbb{Z}$ given by $\alpha = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ and $\beta = (1\,2)$. Construct a homotopy equivalence between this free (hence projective) resolution and the usual one $\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$. (*Note that the resolved $\mathbb{Z}$-module $\mathbb{Z}/3\mathbb{Z}$ is <u>not</u> part of the resolution.*)

Problem 55. a) Let R be a commutative ring and $a \in R$ a non-zerodivisor. Determine all $\mathrm{Tor}_i^R(R/(a), M)$ ($M = R$-module).
b) Find *free* resolutions of $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}/4\mathbb{Z}$- and as $\mathbb{Z}/6\mathbb{Z}$-Modul, respectively.
c) Compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, $\mathrm{Tor}_i^{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ and $\mathrm{Tor}_i^{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$.
d) Is $\mathbb{Z}/2\mathbb{Z}$ a projective $\mathbb{Z}/4\mathbb{Z}$- or $\mathbb{Z}/6\mathbb{Z}$-module?

Problem 56. Let $I, J \subseteq R$ be ideals. Show then that $\mathrm{Tor}_0^R(R/I, R/J) = R/(I+J)$ and $\mathrm{Tor}_1^R(R/I, R/J) = (I \cap J)/IJ$.

This was the last series of problems at the present semester "Algebra I". I hope you had fun. This class continues at the summer semester 2022 – and I hope that I will meet many of you there.

I was announced that we keep the time and the location for the classes – but not for the exercise session. They are announced for Wednesday, 12pm.

See my homepage for details of how we run the written exam. In short: You may use all your notes and you can choose if you prefer to write it at Arnim 22 or at home via webEx.

1st Exam Algebra I, February 16, 2022

Problem 1. Show that $\mathbb{C}[x]/x^k$ is a local ring. What is its spectrum?

Problem 2. Let $I := (z^2 - xy,\ x^2 + y^2 - 2) \subseteq \mathbb{C}[x, y, z]$. Give two examples for maximal ideals $\mathfrak{m} \subseteq \mathbb{C}[x, y, z]$ containing I.

Problem 3. Let $R = \mathbb{C}[x]/(x^2 - 1)$. Write down some filtration $R = M_0 \supset M_1 \supset \ldots \supset M_k = 0$ with R-modules M_i such that each M_i/M_{i+1} is isomorphic to R/P_i for some $P_i \in \operatorname{Spec} R$ $(i = 0, \ldots, k-1)$. Is your filtration a composition series? Is R an Artinian ring? What is its length?

Problem 4. Let $R := \mathbb{C}[x, y]/(x^3, x^2y, xy^3)$.

(i) Draw a picture visualizing the monomials of R. What is $\dim_{\mathbb{C}} R$?

(ii) Name two examples for anihilators of non-vanishing monomials which are non-prime ideals, and

(iii) name all monomials with anihilators being prime ideals.

(iv) What are the associated primes for R? Does R have embedded, i.e., associated, but non-minimal primes?

Problem 5. Consider $R := \mathbb{C}[x, y]/(x^3 + x^2y + xy^2)$ as a (finitely generated)

(i) $\mathbb{C}[x]$-algebra and

(ii) $\mathbb{C}[y]$-algebra.

That is, consider the ring homomorphisms (i) $\alpha : \mathbb{C}[x] \to R$ and (ii) $\beta : \mathbb{C}[y] \to R$ sending $\alpha : x \mapsto x$ and $\beta : y \mapsto y$, respectively.

a) Which of the algebras (i) and (ii) are finite, which are not?

b) Translate the algebra homomorphisms into geometric maps a and b both running as $V(x^3 + x^2y + xy^2) \hookrightarrow \mathbb{C}^2 \to \mathbb{C}^1$ and ending with projections to the x- or y-coordinate, respectively. Which of them have only finite fibers, which of them have some infinite fibers?

Problem 6. Let $R := \mathbb{C}[x, y]/(xy)$. Calculate all $\operatorname{Tor}_i^R(R/(x), R/(x))$ for $i \geq 0$.

2nd Exam Algebra I, April 6, 2022

Problem 1. Let $(R, \mathfrak{m})$ be a local ring. Show that every element $x \in R$ satisfying $x^2 = x$ ("x is *idempotent*") is 0 or 1.

Problem 2. Denote by $\mathbb{C}(x)$ the quotiend field of $\mathbb{C}[x]$. Then, the embedding $\mathbb{C}[x] \hookrightarrow \mathbb{C}(x)$ induces a map $\iota : \operatorname{Spec} \mathbb{C}(x) \to \operatorname{Spec} \mathbb{C}[x]$.

(i) What is its image?

(ii) Describe the closure of the image (with respect to the Zariski topology).

Problem 3. Let $R = \mathbb{C}[x]/(x^3)$. Write down some filtration $R = M_0 \supset M_1 \supset \ldots \supset M_k = 0$ with R-modules M_i such that each M_i/M_{i+1} is isomorphic to R/P_i for some $P_i \in \operatorname{Spec} R$ $(i = 0, \ldots, k-1)$. Is your filtration a composition series? Is R an Artinian ring? What is its length?

Problem 4. Let $P, Q \subset R$ be two prime ideals with $P \not\subseteq Q$ and $Q \not\subseteq P$. Define $I := P \cap Q$.

(i) Is I always/sometimes/never a primary ideal?

(ii) Is I always/sometimes/never a radical ideal (i.e., $I = \sqrt{I}$)?

Problem 5. a) Show that $\alpha := \frac{\sqrt{13}+1}{2}$ is integral over $\mathbb{Z}$.

b) What about $\beta := \frac{\sqrt{13}}{2}$?

c) Is $\mathbb{Z}[\sqrt{13}]$ a factorial ring?

Problem 6. Let A be an abelian group, i.e. a $\mathbb{Z}$-module, ocurring in the following exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to A \to \mathbb{Z}/3\mathbb{Z} \to 0.$$

Show that this sequence splits.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

1. Aufgabenblatt zum 27.4.2022

Problem 57. In a noetherian ring R we define for ideals $I, J \subseteq R$ the quotient $(I : J) := \{x \in R \,|\, xJ \subseteq I\}$.

a) Show that this yields an increasing (hence terminating) chain of ideals $I = (I : J^0) \subseteq (I : J^1) \subseteq \ldots \subseteq (I : J^k) = (I : J^{k+1}) =: (I : J^\infty)$.

b) Calculate the quotient ideals $(I : J)$, $(J : I)$, $(I : J^\infty)$, and $(J : I^\infty)$ for $I = (x^2-1)$ and $J = (x-1)^2$ in the ring $\mathbb{C}[x]$.

c) Let $J = (f_1, \ldots, f_r)$. Show that $(I : J^\infty) = \{x \in R \,|\, \exists n : xJ^n \subseteq I\} = \{x \in R \,|\, \forall y \in J \; \exists n \colon xy^n \in I\} = \{x \in R \,|\, \exists n \forall \nu \colon x f_\nu^n \in I\} \subseteq (\sqrt{I} : J)$.

d) In $\operatorname{Spec} R$ it holds true that $V(I) \setminus V(J) = V(I : J^\infty) \setminus V(J)$ and $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$. (*Hint:* W.l.o.g. $I = 0$ and $(0 : J) = (0)$.)

Problem 58. Let $M = \oplus_{i \in \mathbb{Z}} M_i$ be a graded module over a graded ring $S = \oplus_{i \in \mathbb{N}} S_i$. For an $m = \sum_i m_i$ with $m_i \in M_i$ we call m_i the “homogeneous component” of degree i of m. Let $N \subseteq M$ be an S-submodule. Show that $N = \oplus_{i \in \mathbb{Z}} (M_i \cap N)$ (“N is a *graded* submodule of M”) $\Leftrightarrow$ for all $m \in N \subseteq M$ the homogeneous components $m_i \in M$ are contained in N, too $\Leftrightarrow$ N is generated by homogeneous elements of M, i.e. by certain elements from $\bigcup_i M_i$.

Problem 59. Let $S = \oplus_{d \in \mathbb{N}} S_d$ be an $\mathbb{N}$-graded ring. Note that for homogeneous ideals $I \subseteq S$ (i.e. graded submodules of S) the ring S/I becomes graded, too.

a) Show that a homogenous ideal $P \subseteq S$ is prime $\Leftrightarrow$ for all homogenous $a, b \in S$ the membership $ab \in P$ implies that $a \in P$ or $b \in P$.

b) Does Statement (a) remain true for gradings over more general groups like $\mathbb{N}^2$ or $\mathbb{Z}^2$ or $\mathbb{Z}/2\mathbb{Z}$ instead of just $\mathbb{N}$?

Problem 60. Let $M = \oplus_{e \in \mathbb{Z}} M_e$ be a $\mathbb{Z}$-graded module over the $\mathbb{N}$-graded ring $S = \oplus_{d \in \mathbb{N}} S_d$ which is supposed to be finitely generated as an algebra over S_0. Show that the finite generation of M implies that all M_e are finitely generated S_0-modules. Give an example that the opposite implication fails.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

2. Aufgabenblatt zum 4.5.2022

Problem 61. In class, see (11.3), we have claimed that $\widetilde{R} := k[t, \mathbf{x}]/I^h$ is flat over $k[t]$. For this, we have used that $k[t]$-modules M are flat if and only they are torsion free. On the other hand, we had just checked that $(\cdot t) : \widetilde{R} \to \widetilde{R}$ was injective (which was equivalent to the t-saturation of the ideal I^h). Conclude the proof.

(Hint: Exploit the knowledge $p_X^{-1}(\mathbb{A}^1 \setminus 0) \cong X \times (\mathbb{A}^1 \setminus 0)$ over $\mathbb{A}^1 \setminus 0$ where $X = \operatorname{Spec} R$ with $R = k[\mathbf{x}]/I$. While the LHS corresponds to $\widetilde{R}_t = \widetilde{R} \otimes_{k[t]} k[t, t^{-1}]$, try to write the RHS as a tensor product, too.)

Problem 62. For a fixed ideal $\mathfrak{m} \subseteq A$ in a ring, e.g. if $(A, \mathfrak{m})$ is a local ring, we define $\operatorname{Gr}_{\mathfrak{m}}(A) := \oplus_{\nu \geq 0} \mathfrak{m}^\nu / \mathfrak{m}^{\nu+1} = \oplus_{\nu \geq 0} \mathfrak{m}^\nu / \mathfrak{m}^{\nu+1} \cdot t^\nu$. Check that this is a graded $A/\mathfrak{m}$-algebra.

For an element $f \in \mathfrak{m}^\nu \setminus \mathfrak{m}^{\nu+1}$, we set $\operatorname{in}(f) := \overline{f} \in \mathfrak{m}^\nu / \mathfrak{m}^{\nu+1} = \operatorname{Gr}_{\mathfrak{m}}(A)_\nu$. And, since $\bigcap_{\nu \geq 0} \mathfrak{m}^\nu = 0$, there is a (unique) $\nu = \nu(f)$ for every $f \in A \setminus 0$. For an ideal $I \subseteq A$ we define $\operatorname{in}(I) := (\operatorname{in}(f) \,|\, f \in I \setminus 0) \subseteq \operatorname{Gr}_{\mathfrak{m}}(A)$.

If $I = (f_1, \ldots, f_k)$, then compare $\operatorname{in}(I)$ with $(\operatorname{in}(f_1, \ldots, \operatorname{in}(f_k))$ and give an example where they do not coincide.

Problem 63. Show that the family $V(f_1, f_2, f_3) \stackrel{t}{\to} \mathbb{A}^1$ with $f_i = x_i x_{i+1} - t$ ($i \in \mathbb{Z}/3\mathbb{Z}$) is not flat in a neighborhood of $t = 0$, i.e. check that, with $R := \mathbb{C}[t]_{(t)}$, the R-algebra $A := R[x_1, x_2, x_3]/(f_1, f_2, f_3)$ is not a flat one. Can you visualize what is going on when $t \to 0$?

Problem 64. Give an example for an ideal $I \subseteq R$ and a pair of R-modules $M' \subseteq M$ such that $I(I^k M \cap M') \subsetneq I^{k+1} M \cap M'$ for some k.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

VL "Algebra II"
FU Berlin, Summer 2022

3. Aufgabenblatt zum 11.5.2022

Problem 65. a) Let $I \subseteq A$ be an ideal with $\bigcap_\nu I^\nu = 0$, e.g. $I \neq (1)$ in a noetherian local ring. Show that the lack of zero divisors in the graded ring $\mathrm{Gr}_I(A) := \oplus_{d\geq 0} I^d/I^{d+1}$ implies that the original ring A was an integral domain, too.

b) Present an example indicating the necessity of the assumption $\bigcap_\nu I^\nu = 0$.

c) Give an example of an integral domain A and an ideal $I \subseteq A$ with $\bigcap_\nu I^\nu = 0$ such that $\mathrm{Gr}_I(A)$ has zero divisores.

Problem 66. Let $A = \mathbb{C}[x, y]$ and consider the ideal $I := (x, y)$. Write the ring $\widetilde{A} := \oplus_{\nu\geq 0} I^\nu$ as a polynomial ring over $\mathbb{C}$ modulo some ideal. Moreover, express the embedding $A \hookrightarrow \widetilde{A}$ within this language.

Problem 67. In Subsection 16.1.1 we took a fan Σ and have interpreted the affine toric varieties corresponding to the cones $\sigma \in \Sigma$ and their mutual open embeddings coming from the face relation among the cones of Σ.

Now, play the same game with $\Sigma := \{\sigma_1, \sigma_2, \tau\}$ where the σ_i are the 2-dimensional cones

$$\sigma_1 := \langle (1,0),\ (1,1)\rangle \qquad \sigma_2 := \langle (1,1),\ (0,1)\rangle \qquad \tau := \mathbb{R}_{\geq 0}\cdot (1,1).$$

(Strictly speaking, this is not a fan, since $\tau = \sigma_1 \cap \sigma_2$ is only one face of the cones σ_i. The other 1-dimensional faces and the 0-cone is missing – but they are not important, hence we simplify everything by just forgetting about them.)

a) Draw these three cones.

b) Determine and draw the dual cones $\sigma_1^\vee$, $\sigma_2^\vee$, and $\tau^\vee$. Determine the semigroups obtained by intersecting with $\mathbb{Z}^2$. What is their isomorphy type understood as abstract semigroups?

c) Describe the associated semigroup rings by generators and relations.

d) Describe the ring homomorphisms among these rings and express what this means for the associated affine varieties $X_i = \mathbb{TV}(\sigma_i)$ and $U = \mathbb{TV}(\tau)$.

e) Show that there exist (natural) morphisms of schemes $X_i \to \mathbb{A}^2 = \mathbb{C}^2$ which coincide on U.

To become familiar with the Spec notation, let us repeat Problem 13 from Algebra I. I recommend *not* using the published solution if you still have them somewhere in your notes.

Problem 68. Let $\varphi : A \to B$ be a ring homomorphism. Show that

a) the associated $(f = \operatorname{Spec}\varphi) : \operatorname{Spec} B \to \operatorname{Spec} A$ (defined via $f : Q \mapsto \varphi^{-1}Q$) is continous. That is, with respect to the Zariski topology on both sides, show that the pre-images of closed subsets are closed.

b) Give an alternative proof of (a) by showing that the pre-image of the so-called elementary open subsets $D(f) \subseteq \operatorname{Spec} A$ (for $f \in A$) are open in $\operatorname{Spec} B$. Why does it suffice to consider these special open subsets instead of all ones?

c) Recall that, for every $P \in \operatorname{Spec} A$, we denote by $K(P) := \operatorname{Quot}(A/P)$ the associated residue field of P. Show that φ and f from (a) provide a natural embedding $\overline{\varphi} : K\big(f(Q)\big) \hookrightarrow K(Q)$ for each $Q \in \operatorname{Spec} B$.

d) Recall that elements $a \in A$ can be understood as functions on $\operatorname{Spec} A$ via assigning each P its residue class $\overline{a} \in K(P)$. Show that, in this context, the map $\varphi : A \to B$ can be understood as the pull back map (along f) for functions, i.e. that, under use of (c), $\varphi(a) \,\widehat{=}\, a \circ f$.
(A maybe confusing remark: Making the last correspondence more explicit – but maybe less user friendly – one is tempted to write $\varphi(a) = \overline{\varphi} \circ a \circ f$. However, this is even less correct, since there is no "general map" $\overline{\varphi}$; even the domain and the target of $\overline{\varphi}$ depend on Q.)

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

4. Aufgabenblatt zum 18.5.2022

Problem 69. Let k be a field, and let $P_1, \ldots, P_5 \in \mathbb{P}_k^2$ be five points with no three of them being on a common line. Show that there is then exactly one conic in $\mathbb{P}^2$ containing these points, i.e. there is (up to a constant factor) exactly one homogenous polynomial $Q(z_0, z_1, z_2)$ of degree 2 such that $P_1, \ldots, P_5 \in V(Q)$.

(*Hint:* First, use linear coordinate changes to assume that all points are in the affine chart $\mathbb{A}_k^2 \subseteq \mathbb{P}_k^2$ and, moreover, that $P_3 = (0,0)$, $P_4 = (1,0)$, $P_5 = (0,1)$ in affine coordinates. Then, we may deal with $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$.)

Problem 70. a) Let $E = V(y^2 - x^3 + x) \subseteq \mathbb{A}^2$, and denote by $\overline{E} \subseteq \mathbb{P}^2$ the projective closure obtained by homogenizing the equation. Show that $\overline{E} \setminus E$ consist of a single point P.

b) Denote by $(\mathbb{A}^2)' \subseteq \mathbb{P}^2$ one of the standard charts containing P. Describe the affine coordinate ring of $\overline{E} \cap (\mathbb{A}^2)'$.

Problem 71. a) The map $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ sending $(c_0, \ldots, c_n) \mapsto (c_0 : \ldots : c_n)$ looks locally like $\pi^{-1}\big(D_+(z_i)\big) = D(z_i) \to D_+(z_i)$. Within the Spec language, this could be understood as

$$k[\mathbf{z}]_{(z_i)} \hookrightarrow k[\mathbf{z}]_{z_i} = k[\mathbf{z}]_{(z_i)}[z_i, z_i^{-1}] = k[\mathbf{z}]_{(z_i)} \otimes_k k[z_i, z_i^{-1}].$$

Geometrically, this means that $D(z_i) \cong D_+(z_i) \times k^*$. Show this directly at the level of (closed) points.

b) Let $v_2 : \mathbb{P}^1 \to \mathbb{P}^2$ be the Veronese embedding $v_2 : (z_0 : z_1) \mapsto (w_0 : w_1 : w_2) := (z_0^2 : z_0 z_1 : z_1^2)$. Describe the ring homomorphism corresponding to the restriction $v_2|_{D_+(z_0)} : D_+(z_0) \to D_+(w_0)$.

Problem 72. a) Let $J \subseteq k[\mathbf{z}] := k[z_0, \ldots, z_n]$ be an ideal. Show that $(J : (\mathbf{z})^\infty)$ is the largest ideal J' containing J but still satisfying $J'_{z_i} = J_{z_i}$ for all $i = 0, \ldots, n$.

b) Let $J \subseteq k[\mathbf{z}] := k[z_0, \ldots, z_n]$ be a *homogeneous* ideal. Show that $(J : (\mathbf{z})^\infty)$ is the largest homogeneous ideal J' containing J but still satisfying $J'_{(z_i)} = J_{(z_i)}$ for all $i = 0, \ldots, n$ where these expressions mean the homogeneous localizations.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

5. Aufgabenblatt zum 25.5.2022

Problem 73. For an ideal $I \subseteq k[\mathbf{x}]$ with $\mathbf{x} = (x_1, \ldots, x_n)$ denote by $I^h := (f^h \mid f \in I) \subseteq k[\mathbf{z}]$ with $\mathbf{z} = (z_0, \ldots, z_n)$ and $x_i = z_i/z_0$ its homogenization. On the contrary, for a homogeneous ideal $J \subseteq k[\mathbf{z}]$ we denote by $J^0 \subseteq k[\mathbf{x}]$ its dehomogenization obtained by $z_0 \mapsto 1$ and $z_i \mapsto x_i$ for $i \geq 1$. It equals the homogenous localization $J_{(z_0)}$. Eventually, we denote by $V_{\mathbb{P}}(J) \subseteq \mathbb{P}^n$ and $V_{\mathbb{A}}(I) \subseteq \mathbb{A}^n = D_+(z_0) \subset \mathbb{P}^n$ the respective vanishing loci.

a) Recall that $V_{\mathbb{A}}(J^0) = V_{\mathbb{P}}(J) \cap D_+(z_0)$ inside $\mathbb{A}^n = D_+(z_0)$. Assume that $k = \overline{k}$, and use the Hilbert Nullstellensatz to show that then $V_{\mathbb{P}}(I^h) = \overline{V_{\mathbb{A}}(I)}$ inside $\mathbb{P}^n_k$.

b) Show by presenting a suitable example that the equality of (a) fails for $k = \mathbb{R}$.

c) In Subsection (11.2) we had considered $\mathbb{A}' := \mathbb{A}^{n+1}$ instead of $\mathbb{P} := \mathbb{P}^n$. In particular, we denote $V_{\mathbb{A}'}(J) \subseteq \mathbb{A}'$ for the affine subsets induced by homogeneous ideals $J \subseteq k[\mathbf{z}]$. Comparing both situations via $\pi : \mathbb{A}' \setminus 0 \to \mathbb{P}$ we have now open subsets $D(z_0) \subset \mathbb{A}'$ and $D_+(z_0) \subset \mathbb{P}$ with $D(z_0) = \pi^{-1}\big(D_+(z_0)\big)$, see Problem 71.

We have seen in Subsection (16.6) that $V_{\mathbb{A}'}(J) \cap D(z_0) = \pi^{-1}\big(V_{\mathbb{A}}(J^0)\big)$ with $V_{\mathbb{A}}(J^0) \subseteq \mathbb{A} = D_+(z_0) \subset \mathbb{P}$. Or, with other symbols, and $V_{\mathbb{A}'}(I^h) \cap D(z_0) = \pi^{-1}\big(V_{\mathbb{A}}(I)\big)$. Using this, we have got in Subsection (11.2) that $V_{\mathbb{A}'}(I^h) = \overline{V_{\mathbb{A}'}(I^h) \cap D(z_0)}$ inside $\mathbb{A}'$. Now, use this to derive $V_{\mathbb{P}}(I^h) = \overline{V_{\mathbb{P}}(I^h) \cap D_+(z_0)}$ inside $\mathbb{P}$.

Problem 74. a) Let H denote the hexagon with the vertices $v_1 = [0,0]$, $v_2 = [1,0]$, $v_3 = [2,1]$, $v_4 = [2,2]$, $v_5 = [1,2]$, $v_6 = [0,1]$. Describe the corresponding embedding $\mathbb{P}(H) \hookrightarrow \mathbb{P}^6$ by giving some homogeneous equations by hand and, afterwards, "all" homogeneous equations by using Singular or Macauly2.

b) If $\Delta_1, \Delta_2 \subseteq M_{\mathbb{Q}}$ are lattice polyhedra, then we define their *Minkowski sum* as $\Delta_1 + \Delta_2 := \{a + b \mid a \in \Delta_1,\, b \in \Delta_2\}$. Show that this is again a lattice polyhedron and that its vertices are sums of the vertices of Δ_1 and Δ_2, respectively. Does every such sum provide a vertex of $\Delta_1 + \Delta_2$?

c) Calculate the Minkowski sum Δ of the triangles $\Delta_1 = \operatorname{conv}\{[0,0];\, [1,0];\, [1,1]\}$ and $\Delta_2 = \operatorname{conv}\{[0,0];\, [1,1];\, [0,1]\}$. Can you find a Minkowski decomposition of Δ into one-dimensional sums?

d) Show that there is always a regular map $\mathbb{P}(\Delta_1 + \Delta_2) \to \mathbb{P}(\Delta_1 \times \Delta_2)$. Describe this map explicitely for $\Delta_1 = \Delta_2 = [0,1] \subseteq \mathbb{Q}^1$.

e) Show how the two-dimensional $\mathbb{P}(H)$ of part (a) becomes a closed subset of both $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Describe its equations in both instances.

Problem 75. Let $\Delta \subseteq M_{\mathbb{R}}$ be a lattice polytope and $\Sigma = \mathcal{N}(\Delta)$ the associated (inner) normal fan in the dual space $N_{\mathbb{R}}$. Denote by $n : \mathbb{TV}(\Sigma) \to \mathbb{P}(\Delta) \subseteq \mathbb{P}^{\#(\Delta \cap M)-1}$ the map being glued from the following local pieces:

For each $\sigma \in \Sigma$ there is some (maybe non-unique) $w = w(\sigma) \in \Delta \cap M$ such that $\min\langle \Delta, a\rangle = \langle w, a\rangle$ for all $a \in \sigma$. On the other hand, $w \in \Delta \cap M$ gives rise to a homogeneous coordinate z_w of $\mathbb{P}^{\#(\Delta\cap M)-1}$; denote $U_w := D_+(z_w)$. Now, there is an inclusion $(\Delta - w) \subseteq \sigma^\vee$.

a) Check this inclusion.

b) Derive a morphism between the affine varieties $p_\sigma : \mathbb{TV}(\sigma) \to U_w \cap \mathbb{P}(\Delta)$ with $w = w(\sigma)$.

c) Under which conditions do we have $\mathbb{R}_{\geq 0} \cdot (\Delta - w) = \sigma^\vee$? And, if both cones are different, what is the true relation between them (or, between their respective duals)?
Maybe, instead of checking this formally, it would be more helpful to explain and digest this via pictures and examples. At least as a first step.

d) Show that these local maps glue, i.e., that for faces $\tau \leq \sigma$ the restiction $p_{\sigma|\mathbb{TV}(\tau)}$ equals p_τ when considered as maps towards $\mathbb{P}(\Delta)$.

e) Assuming equality in (c), show that $p_\sigma : \mathbb{TV}(\sigma) \to U_w \cap \mathbb{P}(\Delta)$ is an isomorphism after replacing Δ by some dilation $\Delta_N := N \cdot \Delta$ with $N \gg 0$. Can you give an exact condition for the minimal possible Δ_N?

Problem 76. A lattice polyhedron $\Delta \subseteq M_{\mathbb{Q}}$ is called *normal* if $d\Delta \cap M = d(\Delta \cap M)$ where the latter means the set of all sums obtained by exactly d summands from $\Delta \cap M$.

a) Show that $\nabla := \operatorname{conv}\{[000], [100], [010], [112], [113]\}$ is not normal – namely, $[d-1, 1, 1] \in dP \cap M$, but not in $d(P \cap M)$ for any $d \geq 2$.

b) Let $\Delta \subseteq M_{\mathbb{Q}}$ be a lattice polytope and $d \in \mathbb{N}$. Show that the homogeneous coordinates z_v of $\mathbb{P}(\Delta)$ corresponding to a vertex $v \in \Delta$ cannot simultaneously vanish. Use this to construct the natural map $\varphi : \mathbb{P}(d\Delta) \to \mathbb{P}(\Delta)$.

c) If Δ is additionally normal, then define the d-th Veronese map $\nu_d : \mathbb{P}(\Delta) \to \mathbb{P}(d\Delta)$ and check that it is the inverse of φ of (b).

d) Show that (two-dimensional) lattice polygons are always normal.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

6. Aufgabenblatt zum 1.6.2022

Problem 77. Show that the local blowing up map $\varphi : k[x,y] \hookrightarrow k[x, \frac{y}{x}]$ is not flat – do this by calculating all modules $\operatorname{Tor}_i^{k[x,y]}(k[x, \frac{y}{x}], k)$ with $k = k[x,y]/(x,y)$. What is the geometric meaning of $\operatorname{Tor}_0^{k[x,y]}(k[x, \frac{y}{x}], k) = k[x, \frac{y}{x}] \otimes_{k[x,y]} k = k[x, \frac{y}{x}]/(x,y) = k[\frac{y}{x}]$? Likewise, with $t = \frac{y}{x}$, the map φ can be denoted by $k[x, xt] \hookrightarrow k[x,t]$.

Having done this – can you find now an injection $M \hookrightarrow N$ of $k[x,y]$-modules that does not stay injective after being tensorized with $k[x, \frac{y}{x}]$?

Problem 78. a) Let $\ell \subset \mathbb{A}^2_k$ be a line through the origin. Describe both, the total and the strict transform $\pi^{-1}(\ell)$ and $\pi^{\#}(\ell)$ inside the blowing up $\widetilde{\mathbb{A}}^2_k$. Try both the ("naive") geometric description and the algebraic one via the covering with affine charts.

b) Use the example of blowing up the origin $f = \pi : \widetilde{\mathbb{A}}^2_k \to \mathbb{A}^2_k$ to show that it might happen that $\overline{f^{-1}(y)} \neq f^{-1}(\overline{y})$ (for points $y \in \mathbb{A}^2_k$). Is, however, one of the two sides always contained in the other?

Problem 79. Let $I \subseteq A$ be an ideal in a ring A.

a) Show that $\pi : \operatorname{Proj} \oplus_{d \geq 0} I^d \to \operatorname{Spec} A$ is an isomorphism outside $V(I)$.

b) Assume that $I = (f)$ with a non-zero divisor $f \in A$. Show that the blowing up of I is an isomorphism everywhere.

Problem 80. Let Σ be the two-dimensional fan in $\mathbb{Q}^2$ that is spanned by the six rays

$$a^0 = (1,0),\ b^2 = (1,1),\ a^1 = (0,1),\ b^0 = (-1,0),\ a^2 = (-1,-1),\ b^1 = (0,-1),$$

i.e. it consists of six two-dimensional cones, six rays, and the origin.

a) Show that the three fans induce two different morphisms $\varphi_a : \mathbb{TV}(\Sigma) \to \mathbb{P}^2$ and $\varphi_b : \mathbb{TV}(\Sigma) \to \mathbb{P}^2$. Can you comment the relation between, e.g., φ_a and the blowing up of $0 \in \mathbb{A}^2$?

b) Show that φ_a is birational, i.e. it provides an isomorphism between certain non-empty, open (hence dense) subsets $U \subseteq \mathbb{TV}(\Sigma)$ and $V \subseteq \mathbb{P}^2$. Can you spot those U, V (as large as possible) explicitely?

c) Describe the rational (i.e., not everywhere defined) map $(\varphi_b \circ \varphi_a^{-1}) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by explicit coordinates. In which points is this map not defined?

Aufgabenblätter und Nicht-Skript: http://www.math.fu-berlin.de/altmann

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

7. Aufgabenblatt zum 8.6.2022

Problem 81. Let $I \subseteq A$ be an ideal in a ring A. Then, we denote by

$$\widetilde{X} := \operatorname{Proj} \oplus_{d\geq 0} I^d \cdot t^d \stackrel{\pi}{\to} \operatorname{Spec} A =: X$$

the blowing up of X in in $Z := V(I) = \operatorname{Spec} A/I$, cf. Problem 79. If $J \subseteq I$, then $Y := \operatorname{Spec} A/J$ is sandwiched between Z and X, i.e., $Z \subseteq Y \subseteq X$. We define the strict transform of Y as

$$\pi^{\#}(Y) := \overline{\pi^{-1}\big(Y\big) \setminus \pi^{-1}(Z)}$$

where $E = \pi^{-1}(Z)$ was the so-called exceptional divisor in $\widetilde{X}$. Show that $\pi^{\#}(Y)$ equals (is isomorphic) to the blowing up of Y in Z.

Problem 82. Recall Problems 31, 32, 35 from Algebra I; they deal with the notion of direct limits. You can find them with their proposed solutions earlier in this text. I have, additionally, added them (without solutions) at the end of this sheet.

Problem 83. Let $\mathcal{O}$ be a presheaf of rings, let $\mathcal{F}, \mathcal{G}$ be presheaves of abelian groups or, for the second problem, of $\mathcal{O}$-modules on a topological space X. Let $p \in X$. Show that there are natural isomorphisms $\varphi : (\mathcal{F} \oplus \mathcal{G})_p \stackrel{\sim}{\to} \mathcal{F}_p \oplus \mathcal{G}_p$ and $\psi : (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p \stackrel{\sim}{\to} \mathcal{F}_p \otimes_{\mathcal{O}_p} \mathcal{G}_p$.

Problem 84. Let $f : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of abelian groups. Show that the map f is

a) zero (i.e. $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is zero for all open $U \subseteq X$), or

b) injective (i.e. $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subseteq X$), or

c) an isomorphism

if and only if for all $p \in X$ the corresponding maps $f_p : \mathcal{F}_p \to \mathcal{G}_p$ are zero, injective, or an isomorphism, respectively.

Here are the old problems from Algebra I dealing with direct limits:

Problem 31. a) Let $I := (I, \leq)$ be a poset. It turns into a category via objects $:= I$ and $\mathrm{Hom}_I(a,b) := \begin{cases} \{(a,b)\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise} . \end{cases}$ A "*directed system* on I with values in a category $\mathcal{C}$" is a (covariant) functor $I \to \mathcal{C}$; the "direct limit" $\varinjlim X_i$ of such a system $X = (X_i \mid i \in I)$ is defined via the following universal property: $\mathrm{Hom}_{\mathcal{C}}(\varinjlim X_i, Z) = \{\varphi \in \prod_i \mathrm{Hom}(X_i, Z) \mid i \leq j \Rightarrow \varphi_i = \varphi_j \circ [X_i \to X_j]\}$. In particular, there are canonical maps $X_j \to \varinjlim X_i$ (as the image of $\mathrm{id} \in \mathrm{Hom}_{\mathcal{C}}(\varinjlim X_i, \varinjlim X_i)$). Translate the notion of the direct limit into that of an initial object in some category.

b) What is $\varinjlim X_i$ if I contains a maximum? What is $\varinjlim X_i$ if all elements of I are mutually non-comparable, i.e. if $i \leq j \Leftrightarrow i = j$?

c) Let $\mathcal{C} = \mathcal{M}od_R$ be the category of modules over some ring R. For an element $m_j \in M_j$ we will use the same symbol m_j for its canonical image in $M := \oplus_{i \in I} M_i$, too. Using this notation, show that $\varinjlim M_i = M/N$ where the submodule $N \subseteq M$ is generated by all differences $m_j - \varphi_{jk}(m_j)$ with $m_j \in M_j$, $j \leq k$, and $\varphi_{jk} : M_j \to M_k$ being the associated R-linear map.

d) Assume $(I, \leq)$ to be *filtered*, i.e. for $i, j \in I$ there is always a $k = k(i,j) \in I$ with $i, j \leq k$. If $\mathcal{C} = \mathcal{M}od_R$, then $\varinjlim M_i = \coprod_i M_i / \sim$, where $\coprod$ means the disjoint union (as sets) and "$\sim$" is the equivalence relation generated by $[\varphi_{ij}(m_i) \sim m_i$ for $i \leq j]$ (with $\varphi_{ij} : M_i \to M_j$). (*Hint:* First, define an R-module structure of the right hand side. Then check that an element $x \in M_i$ turns into $0 \in \varinjlim M_i$ if and only if there is a $j \geq i$ with $\varphi_{ij}(x) = 0 \in M_j$.)

Problem 32. a) Let $P \in \mathrm{Spec}\, R$ be a prime ideal and M an R-module. Show that the localisation M_P is the direct limes of modules M_f with distinguished elements $f \in R$. What is the associated poset $(I, \leq)$? Is it filtered?

b) Show that infinite direct sums are filtered direct limits of finite direct sums. What is the underlying poset?

Problem 35. In the category of directed systems of R-modules on a poset $I := (I, \leq)$ (cf. Problem 31) define kernels, images, and cokernels in a natural way, e.g.

$$\ker\big(\varphi : (M_i \mid i \in I) \to (N_i \mid i \in I)\big) := \big(\ker[\varphi_i : M_i \to N_i] \mid i \in I\big).$$

This leads to the notion of exact sequences of directed systems.

a) Show that $\varinjlim$ is right exact (by constructing a right adjoint functor).

b) Show that *filtered* direct limits with values in $\mathcal{M}od_R$ are even exact.

c) Consider the set $I := \{m, a, b\}$ with $m < a$ and $m < b$. Show that the direct limit over this I (even with values in $\mathcal{M}od_R$) is not left exact.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

8. Aufgabenblatt zum 15.6.2022

Problem 85. In class we had defined the so-called constant presheaf $\mathcal{F} := \underline{A}^{\text{pre}}$ via $\mathcal{F}(U) := A$. Afterwards, assuming that X is a locally connected topological space, we define another presheaf $\mathcal{G}$ via $\mathcal{G}(U) := A^{\pi_0(U)}$. Show that $\mathcal{G}$ is actually a sheaf, namely $\mathcal{G} = \mathcal{F}^a$. It is called the "constant sheaf" $\mathcal{G} = \underline{A}$.

Problem 86. Let A be a ring and $X := \operatorname{Spec} A$. Show that the functor $M \mapsto \widetilde{M}$ from the category of A-modules into the category of $\mathcal{O}_X$-modules is fully faithful, i.e. that it induces isomorphisms on the sets $\operatorname{Hom}(\bullet, \bullet)$.

Problem 87. Let S be a graded ring and $f \in S_1$. Show that, for every $k \in \mathbb{Z}$, the $S_{(f)}$- modules $S_{(f)}$ and $S(k)_{(f)}$ are isomorphic where $S(k)$ denotes the degree shift by k. Find a counterexample for $\deg f = 2$.

Problem 88. Define $\mathcal{F}$ as the sheaf of regular sections of the map $h : \widetilde{\mathbb{A}^n} \to \mathbb{P}^{n-1}$ arising from blowing up of the origin $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$. Recall that such a section s assignes to each $\ell \in \mathbb{P}^{n-1}$ a point $c \in \ell \subseteq \mathbb{A}^n$.

On the other hand, we define $\mathcal{G} := \widetilde{S(-1)}$ with $S := k[\mathbf{z}] := k[z_1, \ldots, z_n]$. It is a sheaf of $\mathcal{O}_{\mathbb{P}^{n-1}}$-modules where $\mathcal{O}_{\mathbb{P}^{n-1}} = \widetilde{S}$.

a) Show that the sheaves $\mathcal{F}$ and $\mathcal{G}$ are isomorphic by investigating (and glueing) their local pieces on the open subsets $D_+(z_i)$. The sheaf $\mathcal{F} = \mathcal{G}$ is usually called $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.

b) What is its global sections? That is, determine $\Gamma\big(\mathbb{P}^{n-1}, \mathcal{O}(-1)\big) = \mathcal{O}(-1)(\mathbb{P}^{n-1})$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

9. Aufgabenblatt zum 22.6.2022

Problem 89. a) Let A be a ring and $f, g \in A$ with $D(f) \subseteq D(g)$ within $\operatorname{Spec} A$. Show that $g \in A_f^*$.
(*Hint:* Reduce the problem w.l.o.g. to the case $f = 1$.)

b) Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a *non-local* homomorphism of local rings. Show that, for every S-module N, the modules $\operatorname{Tor}_i^R(N, R/\mathfrak{m})$ vanish for every $i \in \mathbb{Z}$.

Problem 90. Let $X = [0, 1] \subset \mathbb{R}$ with the classical, i.e., Euclidean topology.

a) We define $\mathcal{F}$ as the so-called *skyscraper sheaf* on $0 \in X$ (with value $\mathbb{Z}$): For each open $U \subseteq X$ we define

$$\mathcal{F}(U) := \begin{cases} \mathbb{Z} & \text{if } 0 \in U \\ 0 & \text{otherwise} \end{cases}$$

with the canonical restriction maps (always $\mathrm{id}_{\mathbb{Z}}$, whenever this makes sense). Show that $\mathcal{F}$ is really a sheaf and calculate all its stalks.

b) Let $\mathcal{G} = \underline{\mathbb{Z}}$ be the constant sheaf (the sheafification of the constant pre-sheaf). Then, show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})_0 = 0$. Compare this with $\operatorname{Hom}(\mathcal{F}_0, \mathcal{G}_0)$.

Problem 91. Let $f : X \to Y$ and $g : Y \to Z$ be continous maps between topological spaces. Show that for a sheaf $\mathcal{H}$ on Z we have that $(gf)^{-1}(\mathcal{H}) = f^{-1}g^{-1}\mathcal{H}$.

Problem 92. a) Let $\mathcal{R}$ be a sheaf of rings on some space X. A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules is called locally free if there is an open covering $X = \bigcup_i U_i$ such that all restrictions $\mathcal{F}|_{U_i}$ are isomorphic to direct sums of copies of $\mathcal{R}|_{U_i}$. Show that tensorizing with locally free sheaves is an exact functor.

b) Let $S = \mathbb{C}[z_0, z_1]$ and take $X := \operatorname{Proj} S$. It becomes a locally ringed space via $\mathcal{O}_X := \widetilde{S}$. Show that the sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X(k) := \widetilde{S(k)}$ (for $k \in \mathbb{Z}$) are locally free.

c) Show that $\mathcal{O}_X$ and $\mathcal{O}_X(-1)$ are not isomorphic to each other.

d) Show that $\mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{O}_X(k') \cong \mathcal{O}_X(k + k')$.

e) Show that $\mathcal{O}_X(k) \cong \mathcal{O}_X(k') \Leftrightarrow k = k'$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

10. Aufgabenblatt zum 29.6.2022

Problem 93. a) Let $\mathcal{F} := \underline{A}$ be the constant sheaf on $U := (-2,0) \cup (0,2) \subseteq \mathbb{R}$; denote by $j : U \hookrightarrow \mathbb{R}$ the natural embedding. What are the germs of $j_*\mathcal{F}$ in the points 0, 1, 2, and 3, respectively?

b) I mentioned in class, as a general philosophy, that in a series of constructions with sheaves it suffices to sheafify only once at the end. Recall this philosophy in the construction of, say $(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H}$.
Demonstrate that this philosophy does not work in the context of (a) – if we start with $\mathcal{F} := \underline{A}^{\mathrm{pre}}$, show that $(j_*\mathcal{F}^a)^a \neq (j_*\mathcal{F})^a$. Why does the usual stalk argument does not work anymore?

c) Let $j : U \to X$ be an open embedding and $\mathcal{F}, \mathcal{G}$ be sheaves on U and X, respectively. Are there natural maps between $\mathcal{F}$ and $(j_*\mathcal{F})|_U$ or between $\mathcal{G}$ and $j_*(\mathcal{G}|_U)$?

d) Let $i : K \to X$ be a closed embedding of topological spaces, i.e. $K \subseteq X$ is a closed subset with the induced toplogy. If $\mathcal{F}$ is a sheaf on K, then calculate the stalks of $i_*\mathcal{F}$ in terms of those of $\mathcal{F}$.

e) Let $f : X \to Y$ be a closed (and continous) map, i.e. images of closed subsets are closed in Y. Show that $(f_*\mathcal{F})_y = \varinjlim_{U \supseteq f^{-1}y} \mathcal{F}(U)$ for sheaves $\mathcal{F}|X$ and $y \in Y$.

Problem 94. a) Let $\varphi : A \to B$ be a ring homomorphism and denote by $f : \operatorname{Spec} B \to \operatorname{Spec} A$ the associated map between the associated affine schemes. Assume that M and N are A- and B-modules, respectively. For the corresponding sheaves show that

$$f^*\widetilde{M} = \widetilde{M \otimes_A B}$$

on $\operatorname{Spec} B$ and

$$f_*\widetilde{N} = \widetilde{N_A}$$

on $\operatorname{Spec} A$.

b) Let $j : D(a) \hookrightarrow \operatorname{Spec} A$ be the "nice" open embedding obtained for an $a \in A$. Does (a) say something about $\widetilde{M}|_{D(a)}$?

Problem 95. a) Denote by R^* the group of units in a ring R. Similarly, if $\mathcal{O}_X$ is a sheaf of rings on X, then we define $\mathcal{O}_X^*(U) := \mathcal{O}_X(U)^*$. Show that this defines a sheaf of abelian groups. Moreover, for a section $s \in \Gamma(U, \mathcal{O}_X)$ it satisfies $s \in \Gamma(U, \mathcal{O}_X^*) \Leftrightarrow s_P \in \mathcal{O}_{X,P}^*$ for all $P \in U$.

b) We call locally free sheaves of rank one on a ringed space $(X, \mathcal{O}_X)$ *invertible sheaves*. Let L be such an invertible sheaf, i.e. there is an open cover $\{U_i \subseteq X\}_{i \in I}$ with isomorphisms $\varphi_i : L|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$. Show that the composition maps $\varphi_j \circ \varphi_i^{-1}$ are given by elements $h_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ having the property $h_{ji} = h_{ij}^{-1}$ and

$h_{ij} \cdot h_{jk} \cdot h_{ki} = 1$ on $U_i \cap U_j \cap U_k$ ("cocycle condition").

c) How do the h_{ij} change if the isomorphisms φ_i are altered?

d) Show that $L \cong \mathcal{O}_X \Leftrightarrow$ there are elements $g_i \in \Gamma(U_i, \mathcal{O}_X^*)$ such that $h_{ij} = g_i \cdot g_j^{-1}$ ("$h_{\bullet\bullet}$ is a coboundary").

e) How does one obtain the cocycle $\{H_{ij}\}$ for the sheaves $L \otimes L'$ and $L^\vee := \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$ out of the cocycles $\{h_{ij}\}$ and $\{h'_{ij}\}$ of L and L', respectively?

f) let $\{U_i \subseteq X\}_{i \in I}$ be an open cover, and let $h_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ be elements satisfying the cocycle condition. Show that there is an, up to isomorphism unique, invertible sheaf L on X inducing the given h_{ij} via (b).

g) How do the cocycles of the sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$ on $\mathbb{P}^n$ look like? (Consider at least $\ell = -1, 0, 1$ and $n = 1$ or $n = 2$.)

h) Show that the set of isomorphism classes of invertible sheves forms a group; it is called the "*Picard group*" $\operatorname{Pic} X$. Note that the group operation is $\otimes$ and that the inverse of L is given by $L^\vee := \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$.

Problem 96. a) Let $\varphi : A \to B$ be an injective ring homomorphism. Show (without using (b)) that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is dominant, i.e., that $f(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A$, i.e., that its closure equals the whole $\operatorname{Spec} A$.
(Hint: A set $X \subseteq \operatorname{Spec} A$ is contained in a proper closed subset $F \subsetneq \operatorname{Spec} A$ iff there is a non-empty (nice?) open subset $U \subseteq \operatorname{Spec} A$ being disjoint to X.)

b) Let $\varphi : A \to B$ be a ring homomorphism leading to $f : \operatorname{Spec} B \to \operatorname{Spec} A$. Assume that $f(\operatorname{Spec} B) \subseteq V(I)$ for some ideal $I \subseteq A$. Show that then exists another ideal $I' \subseteq I$ with $V(I') = V(I)$ such that φ factorizes via A/I'. Moreover, give an example where $I' = I$ cannot be achieved.

For the glueing of schemes, please have a look at Problem [Hart, II/2.12].

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

11. Aufgabenblatt zum 6.7.2022

Problem 97. Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism that is induced from a ring homomorphism $\varphi : A \to B$. Assume that there is an open covering $\operatorname{Spec} A = \bigcup_i U_i$ such that all maps $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are closed embeddings, i.e., locally (on the target) of the form $\operatorname{Spec} A_i/J_i \hookrightarrow \operatorname{Spec} A_i$. Show that this implies that $\varphi : A \to B$ is surjective. (I.e. f is a closed embedding "on the direct way", namely not just using some open covering.)

Problem 98. a) Let X be a scheme with an open, affine cover $\{U_i = \operatorname{Spec} A_i\}$. Show that the affine schemes $\operatorname{Spec}(A_i)_{\text{red}}$ (with $(A_i)_{\text{red}} := A_i/\sqrt{0}$) can be glued to become a reduced scheme X_{red}. Are there maps between X and X_{red}? Are they finite? How do they look like on the topological level?

b) Let X be an irreducible scheme. Show that there is a unique "generic point" $\eta \in X$, i.e. it is characterized by the property $\overline{\eta} = X$. How can one obtain open affine subsets $\operatorname{Spec} A \subseteq X$ containing η?

c) Let X be an integral (irreducible and reduced) scheme. Show that $\mathcal{O}_{X.\eta}$ is a field (the "function field" of X). How does it look like for $X = \mathbb{A}^2_{\mathbb{C}}$, or $X = \mathbb{P}^2_{\mathbb{C}}$?

Problem 99. Let $f : X \to \operatorname{Spec} B$ be a morphism of schemes. If $X = \bigcup_{i \in I} \operatorname{Spec} A_i$, then we had defined in class the scheme theoretic image of f as $\operatorname{Spec} B/J$ with $J := \bigcap_i \ker(B \to A_i) \subseteq B$. It was the the smallest closed subscheme of $\operatorname{Spec} B$ such that f factorizes through.

a) Assume that the index set I is finite. Show that $V(J) = \overline{f(X)}$.

b) f corresponds to a ring homomorphism $f^* : B \to \Gamma(X, \mathcal{O}_X)$. What is the relation between $\ker(f^*)$ and the ideal J from (a)?

Problem 100. Let X be a scheme and $x \in X$ be a closed point. This gives rise to the local ring $A := \mathcal{O}_{X,x}$. We denote its maximal ideal by $\mathfrak{m} \subset A$. We call $T^*_x X := \mathfrak{m}/\mathfrak{m}^2$ the cotangent space of X in x.

a) Show that an open embedding $U \hookrightarrow X$ and a closed embedding $Z \hookrightarrow X$ give rise to isomorphisms $T^*_x X \xrightarrow{\sim} T^*_x U$ and surjections $T^*_x X \twoheadrightarrow T^*_x Z$, respectively (if $x \in U$ and $x \in Z$).

b) Determine these maps explicitely for the origin $x = (0, 0)$ with respect to the closed embeddings
(i) $Z_1 = \operatorname{Spec} \mathbb{C}[x, y]/(y) \hookrightarrow \operatorname{Spec} \mathbb{C}[x, y]$,
(ii) $Z_2 = \operatorname{Spec} \mathbb{C}[x, y]/(y^2) \hookrightarrow \operatorname{Spec} \mathbb{C}[x, y]$, and

(iii) $Z_3 = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^3) \hookrightarrow \operatorname{Spec} \mathbb{C}[x,y]$.
Moreover, draw a rough picture of the three situations (i)-(iii).

c) Choose some concrete tangent vector $0 \neq t \in T_{(0,0)} Z_3$ and describe the associated morphism $\operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2 \to Z_3$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

12. Aufgabenblatt zum 13.7.2022

Problem 101. Let F be a locally free sheaf on an integral, i.e. irreducible and reduced scheme X. Show that, for open subsets $U \subseteq X$, the restriction map $\Gamma(X, F) \to \Gamma(U, F)$ is injective.
Give counter examples for the cases when one of the assumptions is violated.

Problem 102. a) Show directly that the diagonal $\Delta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is a closed embedding. What is the homogeneous ideal of $\Delta(\mathbb{P}^1_{\mathbb{C}}) \subseteq \mathbb{P}^3_{C}$ after additionally using the Segre embedding? Do you see the Veronese embedding within this picture?

b) Let $X := \mathbb{A}^1_{\mathbb{C}} \cup \mathbb{A}^1_{\mathbb{C}}$ glued along the common $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. Show directly that there are affine open $U_1, U_2 \subseteq X$ such that either $U_1 \cap U_2$ is not affine or that $U_1 \cap U_2 = U$ is affine with $U_i = \operatorname{Spec} A_i$ and $U = \operatorname{Spec} B$ such that $A_1 \otimes_{\mathbb{C}} A_2 \to B$ is not surjective.

c) In the situation of (b) show that $\Delta(X) \subseteq X \times_{\operatorname{Spec} \mathbb{C}} X$ is not a closed subset.

Problem 103. a) Show that d-dimensional k-varieties (with a perfect field k) are birational equivalent to hypersurfaces in $\mathbb{P}^{d+1}$.
(*Hint:* Use the theorem of the primitive element.)

b) Let $f, g \in k[x]$ be two different polynomials with simple roots. Construct a hypersurface of $\mathbb{C}^2$ that is birational equivalent to $V(y^2 - f(x),\, z^2 - g(x)) \subseteq \mathbb{C}^3$.

Problem 104. Assume that the ring A is factorial. Show that this implies $\operatorname{Pic}(\operatorname{Spec} A) = 0$, i.e. every invertible sheaf on $\operatorname{Spec} A$ is isomorphic to $\mathcal{O}_{\operatorname{Spec} A}$.
(*Hint:* For invertible sheaves $\mathcal{L}$ one is supposed to use the cocycle description on an open covering $\{D(g_i)\}$ with $\mathcal{L}|_{D(g_i)} \cong \mathcal{O}_{D(g_i)}$, cf. Problem 95. Via induction by the overall number of prime factors of the g_i, one can reduce the claim to the special case that all elements g_i are prime. Now, using again Problem 95, one can attain that $h_{ij} \in A^*$ for all i, j.)

Problem 105. Show (by using the toric language via polytopes in $M_{\mathbb{R}}$) that the blowing up of $\mathbb{P}^2$ in two points is isomorphic to the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ in one single point.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

1. Aufgabenblatt zum 26.10.2022

Problem 106. Let $f : X \to Y$ be a morphism of schemes, and let $F|Y$ be an $\mathcal{O}_Y$-module.

a) Show that there is a natural $\Gamma(Y, \mathcal{O}_Y)$-linear map $f^* : \Gamma(Y, F) \to \Gamma(X, f^*F)$.

b) A subset $S \subseteq \Gamma(Y, F)$ is said to "generate F" if S generates all stalks F_y as $\mathcal{O}_{Y,y}$-modules. Show that this implies that $f^*(S) \subseteq \Gamma(X, f^*F)$ generates f^*F.

c) Prove the so-called projection formula: Let E be an $\mathcal{O}_X$-module and suppose that F is a locally free sheaf on Y. Then, $f_*(E \otimes_{\mathcal{O}_X} f^*F) = f_*E \otimes_{\mathcal{O}_Y} F$.

d) Give a counter example for (c) with a sheaf F which is not locally free.

Solution: (a) $\Gamma(Y, F) \to F(Y) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(X) \to \Gamma(X, f^{*\text{prä}}F) \to \Gamma(X, f^*F)$. Alternatively, one may use $f^* \dashv f_*$: $\Gamma(Y, F) \to \Gamma(Y, f_*f^*F) = \Gamma(X, f^*F)$.

(b) For every $x \in X$ we have a commutative diagram

$$\begin{array}{ccccc} \Gamma(Y,F) & \longrightarrow & \Gamma(X, f^*F) & & \\ \downarrow & & \downarrow & & \\ F_{f(x)} & \longrightarrow & (f^*F)_x & = & F_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}. \end{array}$$

(c) First, we have a natural map $f_*E \otimes_{\mathcal{O}_Y} F \to f_*E \otimes_{\mathcal{O}_Y} f_*f^*F \to f_*(E \otimes_{\mathcal{O}_X} f^*F)$. The isomorphism property can be checked locally – in particular, we may assume that $F = \mathcal{O}_Y$. But then, the map turns into $\mathrm{id} : f_*E \to f_*E$.

(d) Take $f := j$ with $j : (\mathbb{A}^2 \setminus \{0\}) \hookrightarrow \mathbb{A}^2$, $E := \mathcal{O}_{\mathbb{A}^2 \setminus \{0\}}$, and $F := i_*\mathcal{O}_0$ being the skyscraper sheaf on $0 \in \mathbb{A}^2$ arising from the closed embedding $i : \{0\} \hookrightarrow \mathbb{A}^2$. Then $j^*F = j^*i_*\mathcal{O}_0 = 0$, but on the RHS we have $j_*E = j_*\mathcal{O}_{\mathbb{A}^2 \setminus \{0\}} = \mathcal{O}_{\mathbb{A}^2}$ (this was already used in the proof that $\mathbb{A}^2 \setminus \{0\}$ is not affine). Thus, $j_*E \otimes_{\mathcal{O}_Y} F = F \neq 0$.

Problem 107. Let E be a locally free $\mathcal{O}_X$-module of rank r on a scheme X, i.e., there exists an affine, open covering $\{U_i\}_{i \in I}$ of X together with isomorphisms $\phi_i : E|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^r$.

a) Show that $E^\vee := \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ is locally free of rank r, too. Moreover, it satisfies $E^{\vee\vee} = E$.

b) Analogously to the same construction on modules over rings, we define

$$(\mathrm{Sym}^d E)(U \subseteq X) := \mathrm{Sym}^d E(U).$$

Thus, we obtain via $\mathcal{A} := \oplus_{d \geq 0} \mathrm{Sym}^d E$ a ring sheaf on X. How does $\mathcal{A}$ look like for the special case $E = \mathcal{O}_X \cdot s_1 \oplus \ldots \oplus \mathcal{O}_X \cdot s_r$ being a free $\mathcal{O}_X$-module?

c) Let $\pi : \mathrm{Spec}_X \mathcal{A} \to X$ be the gluing of the schemes and morphisms $\mathrm{Spec}\, \mathcal{A}(U_i) \to U_i = \mathrm{Spec}\, B_i$ where $\{U_i\}_{i \in I}$ is like in (a). Show that π is a *vector bundle*, i.e., it is *locally* isomorphic to $X \times_{\mathrm{Spec}\, \mathbb{Z}} \mathbb{A}^r_{\mathbb{Z}} \to X$, an the transition maps $U_i \times \mathbb{A}^r \hookleftarrow \pi^{-1}(U_i \cap U_j) \hookrightarrow U_j \times \mathbb{A}^r$ are linear in the fibers (on $U_i \cap U_j$).

d) The sets of sections of π – in the original meaning of this word, i.e., $s_U : U \to \pi^{-1}(U)$ with $\pi \circ s_U = \mathrm{id}_U$) form a sheaf of $\mathcal{O}_X$-modules on X. Accordingly, we denote $\mathrm{Spec}_X \mathcal{A}$ as $\mathbb{A}$(name of this sheaf).

e) For $X = \mathbb{P}^1_k$ and $E = \mathcal{O}_{\mathbb{P}^1}(\ell)$ describe $\pi : \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) \to \mathbb{P}^1$ in the toric language, i.e., via fans. Can you spot the toric among the global sections of π (again, in the original, literal meaning of the word)?

(*Hint:* For the bundle $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{P}^1$ we do already know the result – it has to be the blowing up $\widetilde{\mathbb{A}}^2 \to \mathbb{P}^1$.)

Solution: (a) The first question is local, i.e., it follows from $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^r, \mathcal{O}_X) \cong \mathcal{O}_X^r$. For the second, one starts with the natural $\mathcal{O}_X$-module homomorphism

$$E \to \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X), \mathcal{O}_X\big)$$

and checks locally that it is an isomorphism.A

(b) In the special case of $E = \mathcal{O}_X \cdot s_1 \oplus \ldots \oplus \mathcal{O}_X \cdot s_r$ we obtain $\mathcal{A} = \mathbb{C}[s_1, \ldots, s_r] \otimes_{\mathbb{C}} \mathcal{O}_X$.

(c) The local triviality follows from (b). Moreover, if $s_1, \ldots, s_r$ and $t_1, \ldots, t_r$ are bases for E on U_i and U_j, respectively, then they are related by a regular, i.e., invertible, $(r \times r)$-base change matrix with entries in $\mathcal{O}_{U_i \cap U_j}$. Regularity is equivalent to the determinant being contained in $\mathcal{O}^*_{U_i \cap U_j}$.

(d) The sheaf of sections is isomorphic to $E^\vee$; hence $\mathrm{Spec}_X \mathcal{A} =: \mathbb{A}(E^\vee)$. The reason for this is the fact that sections $s_U : \mathrm{Spec}\, B = U \to \pi^{-1}(U) = \mathrm{Spec}\,\mathrm{Sym}^\bullet\big(E(U)\big)$ correspond to B-algebra homomorphisms $s_U^* : \mathrm{Sym}^\bullet\big(E(U)\big) \to B$ – recall that $E(U) \cong B^r$ is a B-module. Those maps are completely determined by their behavior on degree 1, i.e., by the B-module homomorphisms $(s_U^*)_1 : E(U) \to B$. Thus, sections s correspond to $\mathcal{O}_X$-linear maps $s_1^* : E \to \mathcal{O}_X$.

(e) The result depends on the choice of coordinates on $\mathbb{P}^1$. One version of the desired description is the following: The fan of $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell))$ is spanned from the two maximal cones

$$\sigma_0 := \langle (1,0), (0,1) \rangle \quad \text{and} \quad \sigma_\infty := \langle (0,1), (-1,-\ell) \rangle.$$

This reflects the fact that it was glued from two affine pieces over $U_0, U_\infty \subset \mathbb{P}^1$. The map $\pi : \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) \to \mathbb{P}^1$ is given by the first projection $\mathrm{pr}_1 : \mathbb{Z}^2 \to \mathbb{Z}$. The embeddings $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$, $1 \mapsto (1, i)$ mit $0 \le i \le \ell$ display the toric among all sections of π.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

2. Aufgabenblatt zum 2.11.2022

Problem 108. a) Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ be a (not necessarily closed) point with residue field $K(y) := \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$. Show that the underlying topological space of the scheme $X_y := X \times_Y \operatorname{Spec} K(y)$ equals the fiber $f^{-1}(y)$.

(This should be understood as being in contrast to facts like that the underlying topological space of $\mathbb{A}^2_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$ is not equal to the product of those of $\mathbb{A}^1_{\mathbb{C}}$.)

(*Hint:* As usual, try to reduce everything to the case that both X and Y are affine.)

Solution: The question is obviously local on the target. Moreover, also the equality of two topological spaces on the source can be checked locally. Hence, we may and will assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with y corresponding to a prime ideal $Q \subset B$.

Denote $\varphi : B \to A$ the ring homomorphism corresponding to f. Now, X_y equals the spectrum of $(B \setminus Q)^{-1}A \otimes_B A \otimes_B B/Q$. Hence, the localization kills the prime ideals $P \subset A$ with $\varphi(B \setminus Q) \cap P \neq \emptyset$. On the other hand, modding out Q kills P unless $\varphi(Q) \subseteq P$. Altogether, this means that the points of X_y consist exactly of the survivers $P \in \operatorname{Spec} A$, i.e., of those satisfying $\varphi^{-1}(P) = Q$.

Problem 109. a) Let $\sigma \subseteq N_{\mathbb{Q}}$ be a polyedral cone with $N \cong \mathbb{Z}^n$; let $\tau \leq \sigma$ be a face. Show that

$$k[\sigma^\vee \cap M] \to k[\sigma^\vee \cap \tau^\perp \cap M], \quad x^r \mapsto \begin{cases} x^r & \text{if } r \in \tau^\perp \\ 0 & \text{otherwise} \end{cases}$$

defines a closed embedding $\mathbb{TV}(\overline{\sigma}, N/\operatorname{span}(\tau)) \hookrightarrow \mathbb{TV}(\sigma, N)$ where $\overline{\sigma}$ denotes the image of σ under the real version of the projection $N \twoheadrightarrow N/\operatorname{span}(\tau)$.

Caution: The closed embedding $\mathbb{TV}(\overline{\sigma}, N/\operatorname{span}(\tau)) \hookrightarrow \mathbb{TV}(\sigma, N)$ is *not* the result of applying the $\mathbb{TV}$ functor to some map $(\overline{\sigma}, N/\operatorname{span}(\tau)) \to (\sigma, N)$. An immediate indication for this is that the image is disjoint to the torus.

b) Let Σ be a fan in $N_{\mathbb{Q}}$; let $\tau \in \Sigma$. Show that all $\mathbb{TV}(\overline{\sigma}, N/\operatorname{span}(\tau))$ glue to a closed subvariety of $\mathbb{TV}(\Sigma, N)$. (What do you do with the cones σ *not* containing τ as a face?) This variety will be denoted by $\overline{\operatorname{orb}}(\tau)$.

c) $\overline{\operatorname{orb}}(\tau)$ is toric, too – how does the associated fan look like? What is the dimension of $\overline{\operatorname{orb}}(\tau)$? What is $\overline{\operatorname{orb}}(\tau) \cap \overline{\operatorname{orb}}(\tau')$?

d) Let $\Delta \subseteq M_{\mathbb{Q}}$ be a lattice polytope. It gives rise to a morphism $\mathbb{TV}(\mathcal{N}(\Delta), N) \to \mathbb{P}(\Delta) \subseteq \mathbb{P}^{\#(\Delta \cap M)-1}$ which becomes an isomorphism for sufficiently large Δ, e.g., for $\Delta := (\ell \gg 0) \cdot \Delta$. For faces $F \leq \Delta$ show that $\mathbb{P}(\Delta) \cap V(z_r \mid r \notin F)$ (z_r denotes the homogeneous coordinate associated to the lattice point r) coincides with a suitable $\overline{\operatorname{orb}}(\tau)$ inside $\mathbb{TV}(\mathcal{N}(\Delta), N)$.

Solution: (a) First, since $\tau \leq \sigma$, also $(\sigma^\vee \cap \tau^\perp) \leq \sigma^\vee$ is a face ("dual to τ"). Thus, the proposed surjection is indeed a ring homomorphism. Moreover,

$$(\sigma^\vee \cap \tau^\perp)^\vee = \sigma^{\vee\vee} + (\tau^\perp)^\vee = \sigma + \operatorname{span}(\tau) = \sigma - \tau.$$

Since we build toric varieties only from cones in $N_{\mathbb{R}}$ admitting a vertex, we are supposed to modd out the linear subspace $\operatorname{span}(\tau)$. This does not alter the result of dualization, and we obtain $\overline{\sigma} = (\sigma - \tau)/\operatorname{span}(\tau)$.

(b) If $\delta \le \sigma$ is a face containing τ, then we have just to check the commutativity of the diagram

$$\begin{array}{ccc} k[\sigma^\vee \cap M] & \twoheadrightarrow & k[\sigma^\vee \cap \tau^\perp \cap M] \\ \downarrow & & \downarrow \\ k[\delta^\vee \cap M] & \twoheadrightarrow & k[\delta^\vee \cap \tau^\perp \cap M] \end{array}$$

which is obvious. In case that a cone, e.g., δ, does not contain τ, then the target of the horizontal map will be replaced by the ring 0. That is, the kernel is the ideal $(1) = k[\delta^\vee \cap M]$.

(c) The fan results from the subfan $\Sigma' := \{\sigma \in \Sigma \,|\, \sigma \supseteq \tau\}$ of Σ by modding out $\operatorname{span}\tau$. That is, we obtain a sequence of embeddings

$$\overline{\operatorname{orb}}(\tau) \hookrightarrow \mathbb{TV}(\Sigma') \hookrightarrow \mathbb{TV}(\Sigma)$$

where the latter is an open, but the first and the composed embedding are closed ones. The dimension of $\overline{\operatorname{orb}}(\tau)$ is that of $N/\operatorname{span}(\tau)$, i.e., $\operatorname{rank} N - \dim\tau = \operatorname{codim}\tau$. Finally, $\overline{\operatorname{orb}}(\tau) \cap \overline{\operatorname{orb}}(\tau') = \overline{\operatorname{orb}}(\langle \tau, \tau' \rangle)$ where $\langle \tau, \tau' \rangle$ denotes the unique minimal face of σ containing both τ and τ'.

(d) The face $F \le \Delta$ has an associated normal cone inside the normal fan $\mathcal{N}(\Delta)$, namely

$$\tau := \mathcal{N}(\Delta, F) := \{a \in N_{\mathbb{R}} \,|\, \langle f, a \rangle \le \langle r, a \rangle \text{ for all } f \in F,\ r \in \Delta\}.$$

Note that $a \in \mathcal{N}(\Delta, F)$ implies that the value of $\langle f, a \rangle$ does not depend on $f \in F$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

VL “Algebra III”
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3. Aufgabenblatt zum 9.11.2022

Problem 110. a) Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on an integral scheme X. Show that $\mathcal{F} \otimes_{\mathcal{O}_X} K(X) = \mathcal{F}_\eta$ where $\eta \in X$ denotes the generic point and both $K(X)$ and $\mathcal{F}_\eta$ mean the constant sheafs with these values.

b) Give an example for a non-coherent subsheaf $\mathcal{F} \subseteq K(X)$ where the claim of Part(a) fails.

Solution: (a) Recall that $K(X) = \mathcal{O}_{X,\eta} = \widetilde{\mathrm{Quot}(A)}$. Moreover, we have a canonical map $\mathcal{F} \otimes_{\mathcal{O}_X} K(X) \to \mathcal{F}_\eta$ which arises from the restriction maps $\mathcal{F}(U) \to \mathcal{F}_\eta$ and the fact that $\mathcal{F}_\eta$ is a $\mathcal{O}_{X,\eta}$-module, i.e., a $K(X)$-vector space.

Let us check this map locally: If $\mathcal{F} = \widetilde{M}$ on $\mathrm{Spec}\, A$, then

$$\mathcal{F} \otimes_{\mathcal{O}_X} K(X) = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{\mathrm{Quot}(A)} = M \otimes_A \widetilde{\mathrm{Quot}(A)} = \widetilde{M_{(0)}}.$$

Note that $M_{(0)}$ does not denote any homogeneous localization (which would not make any sense at all) – but it is the localization by the ideal (0) which equals the stalk $\mathcal{F}_\eta$. Moreover, since the A-module $\mathcal{F}_\eta$ is already a $\mathrm{Quot}(A)$-vector space, none of the localizations will change this module. Hence, the associated sheaf is constant.

(b) Take $X = \mathrm{Spec}\, A$ with A being a DVR, e.g., $A = k[x]_{(x)}$. Its spectrum consists of $\eta = (0)$ and the maximal ideal $\mathfrak{m} = (x)$. Let $\mathcal{F} \subset K(\mathrm{Spec}\, A) = k(x)$ be the sheaf defined as $\mathcal{F}(\{\eta\}) = k(x)$ and $\mathcal{F}(X) = 0$. Then, $\mathcal{F} \otimes k(x) = \mathcal{F} \neq k(x)$.

Problem 111. Let $f : (N, \Sigma) \to (N', \Sigma')$; this gives rise to a morphism $F := \mathbb{TV}(f) : \mathbb{TV}(N, \Sigma) \to \mathbb{TV}(N', \Sigma')$. If $\sigma \in \Sigma$, which orbit $\mathrm{orb}(\sigma' \in \Sigma')$ does contain $F(\mathrm{orb}(\sigma))$?

Solution: $\sigma' \in \Sigma'$ is the smallest cone such that $f(\sigma) \subseteq \sigma'$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

VL "Algebra III"
FU Berlin, Winter 2022/23

4. Aufgabenblatt zum 16.11.2022

Problem 112. Let $\widetilde{X} = \operatorname{Proj} \oplus_{d \geq 0} I^d \xrightarrow{\pi} \operatorname{Spec} A = X$ be the blowing up of X in the ideal $I \subseteq A$. Then, the so-called exceptional divisor $E = \pi^{-1}(V(I)) \subseteq \widetilde{X}$ is given by the ideal sheaf $\pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}$. Show that this is isomorphic to one of the sheaves $\mathcal{O}_{\widetilde{X}}(\ell)$. What is ℓ?

Solution: Writing $I = (g_1, \ldots, g_n) \subseteq A$, we can describe $\pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\widetilde{X}}$ locally on $D_+(g_i) = \operatorname{Spec} A[\mathbf{g}/g_i]$ by the ideal (g_i). The sheaves $g_i \cdot \mathcal{O}_{D_+(g_i)}$ glue to $\mathcal{O}_{\widetilde{X}}(1)$. Note that this is in contrast to the ideal sheaves of $V(F_d) \subset \mathbb{P}^n$ where F_d is a homogeneous polynomial of degree d. These ideal sheaves had been $\mathcal{O}(-d)$.

Problem 113. a) Recall that we had seen in class that $D = \sum_i \lambda_i\, p_i \in \operatorname{Div} \mathbb{P}^1_{\mathbb{C}}$ (with $\lambda_i \in \mathbb{Z}$ and closed points $p_i \in \mathbb{P}^1_{\mathbb{C}}$) is a principal divisor $\Leftrightarrow \deg D := \sum_i \lambda_i = 0$.
b) Let $I := V(z_0^2 + z_1^2) \in \mathbb{P}^1_{\mathbb{R}}$. Is $D = 1 \cdot [0] + 1 \cdot [\infty] - 1 \cdot I$ a principal divisor in $\mathbb{P}^1_{\mathbb{R}}$? (Here, we used the notation $0 := (1:0) \in \mathbb{P}^1_{\mathbb{R}}$ and $\infty := (0:1) \in \mathbb{P}^1_{\mathbb{R}}$.) Is there a general concept so that this becomes compatible with (a)?

Solution: a) With $p_i = (a_i : b_i)$ take $f := \prod_i (b_i x_0 - a_i x_1)^{\lambda_i}$. Moreover, for any principal divisor $\operatorname{div}(f)$, the numerator and denominator of the homogeneous f have the same number of zeros.
b) Let $f := x_0 x_1 / (x_0^2 + x_1^2)$. define $\deg(\sum_i \lambda_i p_i) := \sum_i \lambda_i \deg p_i$ and $\deg p := [K(p) : \mathbb{R}]$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

5. Aufgabenblatt zum 23.11.2022

Problem 114. Let $\Sigma_a \subseteq \mathbb{Q}^2$ ($a \in \mathbb{Z}_{\geq 1}$) be the complete fan spanned by the rays $\Sigma_a(1) = \{(0,-1),\ (1,0),\ (0,1),\ (-1,a)\}$. The corresponding toric variety $F_a := \mathbb{TV}(\Sigma_a)$ is called the a-th Hirzebruch surface.

a) Find all non-trivial toric contractions of these surfaces F_a, i.e. surjective toric maps $f: F_a \to X$ where X is some other toric variety. What is X in all the cases; what kind of singularities has it? Which prime divisors are contracted to points? Which of the contractions are birational? Are they blowing ups?

b) Determine the groups $\mathrm{Cl}(F_a)$, $\mathrm{Pic}(F_a)$ and the cone $\mathrm{Eff}(F_a) \subseteq \mathrm{Cl}(F_a)_{\mathbb{Q}}$.

c) Show that $\mathrm{Cl}(\mathbb{TV}(\Sigma))$ is torsion free if the the rays $\rho \in \Sigma(1) \subset N$ generate N as an abelian group. Give a counter example when this assumption is violated.

Solution: (a) Denote by $D_1, \ldots, D_4$ the toric prime divisors corresponding to the four rays (in the given order). The vertical projection $(1,0): \mathbb{Z}^2 \to \mathbb{Z}$ defines a contraction $F_a \to \mathbb{P}^1$ with D_2 and D_4 being fibers. The birational contractions are obtained by contracting D_3. The resulting surface is the weighted projective space $\mathbb{P}(a,1,1)$. It has exactly one singular point (if $a \geq 2$) being the cone overt the Veronese $\nu_a: \mathbb{P}^1 \hookrightarrow \mathbb{P}^a$. The contraction $F_a \to \mathbb{P}(a,1,1)$ is exactly the blowing up of this point, and D_3 becomes the exceptional divisor.

(b) The fan Σ_a is obtained from $\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & a \end{pmatrix}: \mathbb{Z}^4 \to \mathbb{Z}^2$; the Gale transform into the M-level yields $\begin{pmatrix} 1 & 0 & 1 & 0 \\ a & 1 & 0 & 1 \end{pmatrix}: \mathbb{Z}^4 \to \mathbb{Z}^2 = \mathrm{Cl}(F_a) = \mathrm{Pic}(F_a)$. In particular, we can observe the fiber class $e_2 = [D_2] = [D_4]$. Moreover, $\mathrm{Eff}(F_a) = \langle e_1, e_2 \rangle = \langle [D_3], [D_2] \rangle$ and $\mathrm{Amp}(F_a) = \mathrm{int}\langle e_1 + ae_2,\ e_2 \rangle = \mathrm{int}\langle [D_1], [D_2] \rangle$.

(c) Under thie assumption, we obtain a split exact sequence $0 \to K \to \mathbb{Z}^{\Sigma(1)} \to N \to 0$ and K has to be torsion free. Dualizing, it stays a split sequence. On the other hand, $\Sigma = \sigma = \langle (1,0),\ (1,2) \rangle$ yields $\mathrm{Cl} = \mathbb{Z}/2\mathbb{Z}$.

Problem 115. If D is a divisor on some X, then $x \in X$ is called a base point of D if it is contained in the support of all $D' \in |D| := \{D' \geq 0 \mid D' \sim D\}$. We denote by $\mathrm{Bp}(D)$ the set of all these points; it is called the base locus of D.

a) Is this notion depending on D or on its class $\overline{D} \in \mathrm{Cl}(X)$?

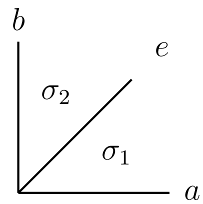
b) Let $X := \widetilde{\mathbb{A}^2}$ be the blowing up of $\mathbb{A}^2$ in the origin and denote by $E \subset X$ the exceptional (prime) divisor. Draw this situation via some fan and identify the ray corresponding to E.

c) Draw the two cones in $M_{\mathbb{R}}$ that represent the sections of $\mathcal{O}(E)$ on the two affine charts of X. Determine $\Gamma(X, \mathcal{O}(E))$ as the intersection of these two regions. What are the base points of E?

d) Draw the two cones for $\mathcal{O}(-E)$ and determine $\Gamma(X, \mathcal{O}(-E))$ as the intersection of these two regions. For each vertex of this region (representing some global section of $\mathcal{O}(E)$) determine the associated effective divisor being equivalent to $-E$. What is their intersection? What is $\mathrm{Bp}(-E)$?

Solution: (a) The linear system $|D|$ depends only on the class of D – it consists of all effective divisors within this class.

(b) The fan in $N_{\mathbb{R}} = \mathbb{R}^2$ looks as follows:



The central ray e corresponds to the divisor E. The horizontal and vertical rays a and b encode the strict transforms of the prime divisors $V(x), V(y) \subset \mathbb{A}^2$, respectively.

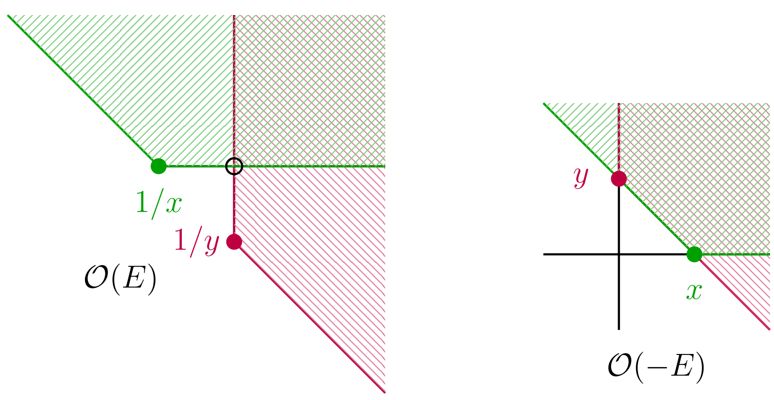
(c) The dual cones of the σ_i are generated by

$$\sigma_1^\vee = \langle [[0,1],\ [1,-1] \rangle \quad \text{and} \quad \sigma_2^\vee = \langle [[1,0],\ [-1,1] \rangle.$$

The associated charts are represented by

$$\mathbb{C}[\sigma_1^\vee \cap M] = \mathbb{C}[y, x/y] \quad \text{and} \quad \mathbb{C}[\sigma_2^\vee \cap M] = \mathbb{C}[x, y/x],$$

respectively. The equations of E are $y \in \mathbb{C}[\sigma_1^\vee \cap M]$ and $x \in \mathbb{C}[\sigma_2^\vee \cap M]$. The local generators of the sheaf are their inverses, i.e., $1/y$ and $1/x$, respectively. See the left hand side of the following figure:



The intersection of both regions gives $\Gamma(X, \mathcal{O}(E)) = \mathbb{C}[x,y] = \Gamma(\mathbb{A}^2, \mathcal{O}) = \Gamma(\widetilde{\mathbb{A}^2}, \mathcal{O})$. The origin 0 corresponds to the global section $1 = \chi^0$, and this provides the effective diviasor $E = E + \mathrm{div}(1)$. All other divisors inside $|E|$ are obtained by adding further effective divisors to E, i.e., E is always contained in it. Thus, $\mathrm{Bp}(E) = E$.

(d) The drawing is done in the right hand side of the previous figure. We consider the vertex $[1,0]$ encoding $x \in \Gamma(X, \mathcal{O}(-E))$. We obtain

$$\mathrm{div}(x) = 1 \cdot \overline{\mathrm{orb}}(a) + 1 \cdot \overline{\mathrm{orb}}(e),$$

thus $(-E) + \operatorname{div}(x) = \overline{\operatorname{orb}}(a)$. Similarily, $(-E) + \operatorname{div}(y) = \overline{\operatorname{orb}}(b)$. In particular, these two effective divisors are disjoint (look at the orbits they are containing). Thus, $\operatorname{Bp}(-E) = \emptyset$, i.e., $(-E)$ is basepoint free.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

6. Aufgabenblatt zum 30.11.2022

Problem 116. Let X be a normal k-variety ($\overline{k} = k$) with function field $K := K(X) \supseteq k$.

a) Show that elements $f \in K^* \setminus k$ correspond to dominant rational maps $f : X \dashrightarrow \mathbb{P}^1_k$.

b) Let $U \subseteq X$ be an open subset such that $\operatorname{div}(f)_{|U} \geq 0$. Show that $f : X \dashrightarrow \mathbb{P}^1_k$ is then truely defined on U. What about $V \subseteq X$ with $(-\operatorname{div}(f))_{|V} \geq 0$? Conclude that $f : X \dashrightarrow \mathbb{P}^1_k$ is always defined on the whole X, i.e., leading to a regular $f : X \to \mathbb{P}^1_k$, whenever X is a curve.

c) Give two examples where one cannot extend $f : X \dashrightarrow \mathbb{P}^1_k$ to a globally defined $X \to \mathbb{P}^1_k$. One with X being a non-normal curve, the second with X being a normal surface.

Solution: (a) $f \in K^*$ induces $k[t] \to K$ via $t \mapsto f$. Since $f \notin k$ and $\overline{k} = k$, the element f is transcendental over k, hence this map is injective – inducing an embedding $K(\mathbb{P}^1) = k(t) \hookrightarrow K$.

(b) Assume that $U = \operatorname{Spec} A$. Since $\operatorname{ord}_P(f) \geq 0$ for all $P \in \operatorname{Spec} A$ of height one, we obtain that $f \in A$. Hence, $k(t) \to K(X) = \operatorname{Quot}(A)$ with $t \mapsto f$ factors via $k[t] \to A$. Second, for $V \subseteq X$ this works similarily; we just look at $k[\frac{1}{t}] \subset k(t) \to K(X)$. Finally, if X is a curve, then each point has an open neighborhood satisfying either $\operatorname{div}(f) \geq 0$ or $\operatorname{div}(f) \leq 0$.

(c) First, one can see how normality is important: $\mathbb{P}^2 \supseteq V(y^2z - x^3) \dashrightarrow \mathbb{P}^1$ given by $f := y/x$. Then, we consider the same $f := y/x$ as a rational function $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. It becomes the linear projection $(x : y : z) \mapsto (x : y)$ out of $(0 : 0 : 1)$ – and it cannot be extended to the whole $\mathbb{P}^2$. To obtain a decent map one is either forced to remove this point, i.e., considering $\mathbb{P}^2 \setminus \{((0 : 0 : 1))\} \to \mathbb{P}^1$, or one replaces $\mathbb{P}^2$ by the blowing up $\mathbb{F}_1 := \widetilde{\mathbb{P}^2}$ of $\mathbb{P}^2$ in $(0 : 0 : 1)$.
Recall from class that this can be observeed within the toric language, too.

Problem 117. Let $E := V(y^2z - x^3 + xz^2) \subseteq \mathbb{P}^2_{\mathbb{C}}$; it is the usually first example of a smooth elliptic curve.

a) Show that for two (closed, maybe assume distinct) points $p, q \in E$ the line $\overline{pq}$ intersects E in exactly one further point $\ell(p, q)$.

b) Show that the divisor $D := [p] + [q] - [r] - [s]$ (with closed points $p, q, r, s \in E$) is a principal one if $\ell(p, q) = \ell(r, s)$.

c) Consider the map $\Phi : E \to \operatorname{Cl}_0(E) := \ker(\deg) \subseteq \operatorname{Cl}(E)$, $p \mapsto [p] - [(0 : 1 : 0)]$. For points $p, q \in E$ find a third one $r \in E$ such that $\Phi(p) + \Phi(q) = \Phi(r)$.

Solution: (a) Let $L = L(x, y, z) = ax + by + c$ be the affine equation (with $z = 1$) for the line $\overline{pq}$; assume, w.l.o.g., $b \neq 0$. Then, substituting $y = -a/bx - c/b$, the affine E-equation $y^2 = x^3 - x = x(x^2 - 1)$ becomes an x-polynomial of degree 3. Besides

$x(p)$ and $x(q)$ it has exactly one further root.

(b) Let $L_{p,q}, L_{r,s} \in \mathbb{C}[x,y,z]$ the homogeneous linear equations of the projective lines $\overline{pq}$ and $\overline{rs}$, respectively. Then, $f := L_{p,q}/L_{r,s} \in K(E)$ is a rational function with $\operatorname{div}(f) = [p] + [q] - [r] - [s]$.

(c) Let $r \in E$ be the point obtained by reflecting $\ell(p,q)$ at the x-axis, i.e., switching the sign of the y-coordinate, Then, the line connecting r and $(0 : 1 : 0)$ passes through $\ell(p,q)$, too.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

7. Aufgabenblatt zum 14.12.2022

Problem 118. Let $\mathcal{F}$ be a coherent sheaf on some scheme X. The fact that $\mathcal{F}$ is generated by finitely many global sections is equivalent to the existence of a sheaf homomorphism $f : \mathcal{O}_X^n \to \mathcal{F}$ such that (a) f is surjective, or (b) $\Gamma(X, f)$ is surjective? What is the right answer – (a) or (b)? Give a proof of your answer and a counterexample for the wrong one: Is the condition too strong or too weak?

Solution: (a) is true. The condition (b) is neither too strong, nor too weak. It just says that $\Gamma(X, \mathcal{F})$ is a finitely generated $\Gamma(X, \mathcal{O}_X)$-module. Example: On $\mathbb{P}^1$ is $\mathcal{O} \xrightarrow{0} \mathcal{O}(-1)$ surjective on the global sections.

Problem 119. In class we have shown that $(E^2) = (E \cdot E) = -1$ holds true for the exceptional divisor $E \subset \widetilde{A^2} \to \mathbb{A}^2$ of the blowing up of $0 \in \mathbb{A}^2$. Now, calculate the self intersection number $(H^2) = (H \cdot H)$ for $H \subset \mathbb{P}^2_{\mathbb{C}}$ being some projective line, e.g., $H = V(z_2)$.

Solution: There is two ways to calculate this self intersection:
First, H is linearily equivalent to any other projective line $H' \subset \mathbb{P}^2$. Hence,

$$(H \cdot H) = (H \cdot H') = 1.$$

The latter equality does, e.g., follows from considering $\iota : \mathbb{P}^1 = L \hookrightarrow \mathbb{P}^2$, $(z_0 : z_1) \mapsto (z_0 : z_1 : 0)$. The pull back $\iota^* H'$ is well defined. For instance, if $H' = V(z_0) \subset \mathbb{P}^2$, then $\iota^* H' = V(z_0) \subset \mathbb{P}^1$. This can also be checked locally. Now, the result follows from $\deg V(z_0) = 1$; this divisor cosists just of the single point $(0 : 1)$.
For the second method, we still use ι and consider the local generators of $\mathcal{O}(H)$: They are $\frac{z_0}{z_2}$, $\frac{z_1}{z_2}$, and 1 on $D_+(z_0)$, $D_+(z_1)$, and $D_+(z_2)$, respectively. Since z_2/z_i generates the kernel of

$$\iota^* : k[\tfrac{z_0}{z_i}, \tfrac{z_1}{z_i}, \tfrac{z_2}{z_i}] \twoheadrightarrow k[\tfrac{z_0}{z_i}, \tfrac{z_1}{z_i}]$$

on the i-th chart $D_+(z_i)$, we cannot pull them back. However, the 1-cocycle $\frac{z_0}{z_1} = \frac{z_0}{z_2} / \frac{z_1}{z_2}$ on the charts $D_+(z_0)$ and $D_+(z_1)$ can. It is the same as for the sheaf $\mathcal{O}(1)$ on $\mathbb{P}^1$: This sheaf is locally generated by, e.g., $z_i \cdot k[\frac{z_0}{z_i}, \frac{z_1}{z_i}]$ with $i = 0, 1$. The associated 1-cocycle is z_0/z_1.

Problem 120. Let $f : Y \to X$ be a morphism of schemes. Show that, for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, there is a natural map

$$\Gamma(X, \mathcal{F}) \to \Gamma(Y, f^*\mathcal{F}).$$

This we call the pulling back of sections.

Solution: The easiest way to see this is to recall that $f^* \dashv f_*$. This induces the adjunction map $\mathrm{id} \to f_* f^*$, hence,

$$\Gamma(X, \mathcal{F}) \to \Gamma(X, f_* f^* \mathcal{F}) = \Gamma(Y, f^* \mathcal{F}).$$

On the other hand, this can be done directly, too. We start with

$$\Gamma(Y, f^{-1}\mathcal{F}) = \varinjlim_{U \supset f(Y)} \mathcal{F}(U) \leftarrow \Gamma(X, \mathcal{F}),$$

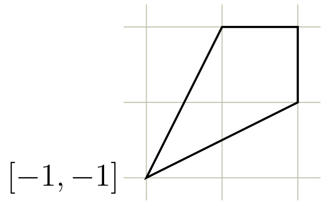
and this is followed by the map $f^{-1}\mathcal{F} \to (f^{-1}\mathcal{F}) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Y,\ f \mapsto f \otimes 1$ plus sheafification.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

8. Aufgabenblatt zum 4.1.2023

Problem 121. Let $\Delta \subset \mathbb{R}^2$ be the quadrangle with the vertices $v_1 = [1,0]$, $v_2 = [1,1]$, $v_3 = [0,1]$, $v_4 = [-1,-1]$.

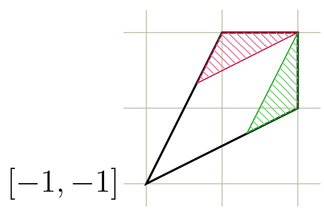


a) Show that Δ cannot be written as a Minkowski sum $\Delta = \nabla^1 + \nabla^2$ with lattice polygons $0 \neq \nabla^i \subset \mathbb{R}^2$ $(i = 1,2)$.
b) Give an example of a decomposition $\ell \cdot \Delta = \nabla^1 + \nabla^2$ with $\ell \in \mathbb{N}$ and $\nabla^i \subset \mathbb{R}^2$ being two lattice triangles $(i = 1,2)$.
c) Construct the polyhedral cone $C(\Delta)$ (of Minkowski summands of Δ) and explain how the subsemigroup of lattice points within $C(\Delta)$ reflects (a) and (b).

Solution: (a) There is the decomposition

$$\Delta = \operatorname{conv}\{[1,1],\ [0,1],\ [-\tfrac{1}{3}, \tfrac{1}{3}]\} + \operatorname{conv}\{[0,-1],\ [0,0],\ [-\tfrac{2}{3}, -\tfrac{4}{3}]\}$$

which is, up to opposite shifts of the two summands, unique. In the figure below,



the red triangle displays the first summand, but the green one is just an integral shift of the second one. Since the summands are non-lattice triangles, we are done.
(b) Multiplying the decomposition of (a) with 3 gives the result.
(c) We have four edges

$$d^1 = \overrightarrow{v_1 v_2} = [0,1],\ \ d^2 = \overrightarrow{v_2 v_3} = [-1,0],\ \ d^3 = \overrightarrow{v_3 v_4} = [-1,-2],\ \ d^4 = \overrightarrow{v_4 v_1} = [2,1],$$

i.e., $C(\Delta) \subset \mathbb{R}^4_{\geq 0} = \{(t_1,t_2,t_3,t_4) \mid t_i \geq 0\}$ is obtained from the two linear closing conditions

$$t_1[0,1] + t_2[-1,0] + t_3[-1,-2] + t_4[2,1] = [0,0].$$

That is, $2t_4 = t_2 + t_3$ and $2t_3 = t_1 + t_4$. Using just the coordinates (t_3, t_4), the non-negaitivity conditions yield

$$t_3, t_4 \geq 0, \quad \text{and} \quad 2t_3 - t_4 = t_1 \geq 0,\ 2t_4 - t_3 = t_2 \geq 0.$$

Hence, $C(\Delta)^\vee = \langle [2,-1];\ [-1,2] \rangle$, yielding $C(\Delta) = \langle (1,2),\ (2,1) \rangle$.

The point $(1,1) \in C(\Delta)$ stands for $(t_1, t_2, t_3, t_4) = (1,1,1,1)$ (i.e., no edge dilation at all); it corresponds to the original Δ. The generator $(1,2)$ means $(t_1, t_2, t_3, t_4) = (0,3,1,2)$; the zero-entry encodes the disappearence of an edge – leading to a triangle. The second generator is $(2,1)$, i.e., $(t_1, t_2, t_3, t_4) = (3,0,2,1)$. Their sum is $(3,3)$, i.e., $3 \cdot \Delta$. The original Δ, however, corresponds to $(1,1) \in C(Q)$ and cannot be written a sum of two integral points on the rays of $C(\Delta)$. Instead, we may write $(1,1) = \frac{1}{3} \cdot (1,2) + \frac{1}{3} \cdot (2,1)$. This reflects the non-integral decomposition we had started with.

Problem 122. Show that $\operatorname{Pic}(\mathbb{TV}(\sigma)) = 0$ for a rational polyhedral cone σ.

(*Hint*: For any T-invariant Cartier divisor D consider $\mathcal{O}(\pm D)$; they are $k[\sigma^\vee \cap M]$-submodules of $k[M]$ which are generated by monomials.)

Solution: Assume that $\mathcal{O}(D) = \langle \chi^{r^1}, \ldots, \chi^{r^m} \rangle$ and $\mathcal{O}(-D) = \langle \chi^{s^1}, \ldots, \chi^{s^n} \rangle$. They are $k[\sigma^\vee \cap M]$-modules, and we have indicated the generators. Note that the generators can be chosen as monomials – the reason is that $\pm D$ are T-invariant, hence $\mathcal{O}(\pm D)$ are M-graded modules. In particular, $\mathcal{O}(\pm D) \subseteq k[M]$.

For Cartier divisors we know that $\mathcal{O}(D) \otimes \mathcal{O}(-D) = \mathcal{O}(D) \cdot \mathcal{O}(-D) = \mathcal{O}$ where the product in understood inside $K(X)$ or, in our toric case, even in $k[M]$. Down to earth, this product is generated by the monomials $\chi^{r^i} \chi^{s^j} = \chi^{r^i + s^j}$ ($i = 1, \ldots m$, $j = 1, \ldots, n$). Hence, $r^i + s^j \in \sigma^\vee \cap M$ (for all i,j) and, moreover, they generate the whole semigroup $\sigma^\vee \cap M$ as an "$(\sigma^\vee \cap M)$-module". This means that, w.l.o.g., $r^1 + s^1 = 0$.

From this we obtain that $r^i - r^1 = r^i + s^1 \in \sigma^\vee \cap M$ (and similarily $s^j - s^1$). Thus, $\mathcal{O}(D) = \langle \chi^{r^1} \rangle$, i.e., it represents a principal divisor.

Problem 123. Let $E := V(Y^2 Z - X^3 + XZ^2) \subseteq \mathbb{P}^2$ be the most famous elliptic curve. Show directly, by describing explicit generators in the affine charts that $(\omega_E =)\ \Omega_E \cong \mathcal{O}_E$.

Solution: We start with the chart $Z \neq 0$, i.e., the affine coordinates are $x = X/Z$ and $y = Y/Z$. The equation $y^2 = x^3 - x$ yields $2y\,dy = (3x^2 - 1) \cdot dx$. Thus,

$$dx/(2y) =: \xi := dy/(3x^2 - 1).$$

On the one hand, dx and dy together generate Ω on $D_+(Z)$. On the other, ξ is regular: Indeed, in each point, at least one of the two denominators has to be different from zero. Thus, $\Omega|_{D_+(Z)} = \xi \cdot \mathcal{O}_{D_+(Z)}$.

We do the same thing for $D_+(Y)$ to catch the missing point $(0:1:0)$. Our coordinates are $a = X/Y$ and $b = Z/Y$. We obtain

$$da/(1 + 2ab) = db/(3a^2 - b^2)$$

as the (almost unique) generator of $\Omega|_{D_+(Y)}$.

At last, we use the transition $b = 1/y$ and $a = x/y$ to show that both generators coincide (up to sign).

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

9. Aufgabenblatt zum 11.1.2023

Problem 124. a) Let k be a field and $R = k[x_1, \ldots, x_n]$. Show that $\Omega_{R|k} = \oplus_{i=1}^n R\, dx_i$.

b) Let $\varphi : A \to B$ be an algebra and let $a \in A$ with $\varphi(a) \in B^*$ (inducing a ring homomorphism $A_a \to B$) and $b \in B$. Show that $\Omega_{B|A_a} = \Omega_{B|A}$ and $\Omega_{B_b|A} = \Omega_{B|A} \otimes_B B_b$.

Solution: (a) We check the universal property. If M is an R-module, then a k-derivation $D : R \to M$ is determined by the images $D(x_i) \in M$ with $i = 1, \ldots, n$. Indeed, for any $f(x_1, \ldots, x_n) \in R = k[x_1, \ldots, x_n]$ we obtain $D(f) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i$. This formula can be checked by, w.l.o.g., assuming that $f = x_1^{r_1} \cdot \ldots \cdot x_n^{r_n}$ is a monomial.

On the other hand, any choice of elements $m_1, \ldots, m_n \in M$ leads to a derivation D satisfying $D(x_i) = m_i$, It is just defined by the previous formula.

Altogether, this means that $\mathrm{Der}_k(R, M) = M^n = \mathrm{Hom}_R(\oplus_{i=1}^n R\, dx_i, M)$.

(b) For B-modules M, we have that $\mathrm{Der}_A(B, M) = \mathrm{Der}_{A_a}(B, M)$. The reason is that derivations $D : B \to M$ which vanish on A do also kill $\frac{1}{a}$: $D(\frac{1}{a}) = -\frac{1}{a^2}\, da = 0$.

For the other claim, we consider B_b-modules N. Here we use that $\mathrm{Der}_A(B_b, N) \xrightarrow{\sim} \mathrm{Der}_A(B, N)$ is an isomorphism, where N is considered as a B-module on the right hand side. Indeed, if $D : B_b \to N$ is an A-derivation, then $D(\frac{1}{b})$ can be recovered as $D(\frac{1}{b}) = -\frac{1}{b^2}\, D(b)$. And, similaily, if $D : B \to M$ is given, then D can be extended to B_b this way.

For the Ω-modules, this equality between the Der-s means

$$\mathrm{Hom}_{B_b}(\Omega_{B_b|A}, N) = \mathrm{Hom}_B(\Omega_{B|A}, N) = \mathrm{Hom}_{B_b}(\Omega_{B|A} \otimes_B B_b, N).$$

Alternatively, both equalities of (b) can be checked via the exact sequence of the Ω-modules on $A \to B \to C$; just specialize this to $A \to A_a \to B$ and $A \to B \to B_b$.

Problem 125. Let $A \to B$ be an algebra, denote $I := \ker(B \otimes_A B \twoheadrightarrow B)$. and consider $B \otimes_A B$ (and thus I) as B-modules via the multiplication on the left hand factors.

a) Show that $D : B \to I/I^2$, $b \mapsto b \otimes 1 - 1 \otimes b$ is an A-derivation.

b) Show that the induced B-linear map $\Omega_{B|A} \to I/I^2$ is an isomorphism.

Solution: (a) The key equation is $D(bc) = bD(c) + cD(b) - (b \otimes 1 - 1 \otimes b) \cdot (c \otimes 1 - 1 \otimes c)$.

(b) Denote $\Phi : \Omega_{B|A} \to I/I^2$; for every B-module M it induces, via $\mathrm{Hom}_B(\bullet, M)$, the B-linear map $\Phi_M : \mathrm{Hom}_B(I/I^2, M) \to \mathrm{Der}_A(B, M)$ sending $\varphi \mapsto \varphi \circ D$.

The elements $D(b) = b \otimes 1 - 1 \otimes b$ generate I: Indeed, $cD(b) = (bc) \otimes 1 - c \otimes b$, hence modulo those element, we can modify any $\sum_i b_i \otimes c_i \in I$ to $\sum_i (b_i c_i) \otimes 1$. On the other hand, the membership with I means $\sum_i b_i c_i = 0$.

Thus, Φ_M is injective. For the surjectivity of Φ_M, assume that $f : B \to M$ is an A-derivation. We define $F : B \otimes_A B \to M$ via the A-bilinear map $(b, c) \mapsto b \cdot f(c)$. This map is even B-linear (recall that the B-action on the source happens via the first factor). Eventually, we consider the restriction $F|_I$ and it remains to check that $F|_{I^2} = 0$. This follows from the derivation properties of f and the key equation from Part (a).

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

10. Aufgabenblatt zum 18.1.2023

Problem 126. a) Let D be a prime and Cartier divisor on a normal variety X. It gives rise to the invertible sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(-D)$. Since $1 \in \Gamma(X, \mathcal{O}_X(D)) \subset K(X)$, the inclusion provides an injection $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$. Dualizing, this displays the inclusion $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$, i.e., $\mathcal{O}_X(-D)$ is an ideal sheaf. Show that this is exactly the ideal sheaf corresponding to D understood as a closed subvariety of X. In other words, we have the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

b) Consider $\mathbb{P}^n$ as a toric variety with the usual fan $\Sigma = \partial \mathbb{R}^{n+1}_{\geq 0} / \mathbb{R} \cdot (1, 1, \ldots, 1)$. Show that the closed orbits $\overline{\mathrm{orb}}(\rho)$ with $\rho \in \Sigma(1)$ are exactly the hyperplanes $H_i := V(z_i)$ when $z_0, \ldots, z_n$ denote the homogeneous coordinates of $\mathbb{P}^n$.

c) Show that, in (b), $\mathcal{O}(H_i) \cong \mathcal{O}(1)$ and $\mathcal{O}(-H_i) \cong \mathcal{O}(-1)$.

Solution: I give a very detailed description of both solutions. While everything is easy, one has, nevertheless, to be carefull with all the details and indices and signs. Thus, sorry for the long text – skip it if you feel bored.

(a) Let $U \subseteq X$ open such that $D \geq 0$ is represented by some $f \in K(X)$, i.e., (U, f) is part of the Cartier data. Then, on U, we know that $\mathcal{O}_U(D) = \frac{1}{f} \mathcal{O}_U$ and $\mathcal{O}_U(-D) = f \cdot \mathcal{O}_U$. Moreover, since D is effective, we know that $f \in \Gamma(U, \mathcal{O}_U)$.

On the other hand, if $U = \operatorname{Spec} A$ is additionally affine, then $D \cap U = V(f) \subset U$, i.e., its embedding corresponds to the surjection $A \twoheadrightarrow A/(f)$. That is, $(f) \subset A$ is the ideal (sheaf) of D on $U = \operatorname{Spec} A$. We see that this coincides with $\mathcal{O}_U(-D)$.

(b) Within an affine chart $\mathbb{TV}(\sigma)$ $(\sigma \in \Sigma)$, the closure $\overline{\mathrm{orb}(\rho)}$ (with $\rho \in \sigma(1) \subseteq \Sigma(1)$) is given by the surjection

$$k[\sigma^\vee \cap M] \twoheadrightarrow k[\sigma^\vee \cap \rho^\perp \cap M], \quad \chi^r \mapsto \begin{cases} \chi^r & \text{if } \langle r, \rho \rangle = 0 \\ 0 & \text{otherwise, i.e., } \langle r, \rho \rangle > 0. \end{cases}$$

Let σ be one of the full-dimensional $\mathbb{P}^n$-cones. After choosing coordinates $N \xrightarrow{\sim} \mathbb{Z}^n$, we may write $\sigma = \langle e^1, \ldots, e^n \rangle$ where $\{e^1, \ldots, e^n\}$ is the standard basis of $\mathbb{Z}^n$. Assume that $\rho = e^1$. Then, the above surjection becomes

$$k[x_1, \ldots, x_n] \twoheadrightarrow k[x_2, \ldots, x_n], \quad x_1 \mapsto 0.$$

That is, the (local) ideal of $\overline{\mathrm{orb}(e^1)}$ is (x_1). The associated $V(x_1)$ is the hyperplane in question.

Understanding $\mathcal{O}_{\mathbb{P}^n}(H)$ is a global question – hence we switch to global, i.e., the homogeneous coordinates $z_0, \ldots, z_n$ of $\mathbb{P}^n$. Assume that $H = H_0 = V_+(z_0) \subset \mathbb{P}^n$. Locally, on

$$U_i = \operatorname{Spec} k[z_0, \ldots, z_n]_{(z_i)} = \operatorname{Spec} k[\tfrac{z_0}{z_i}, \ldots, \tfrac{z_n}{z_i}],$$

the equation of H_0 becomes $f = z_0/z_i$. Thus, by (a), the sheaf $\mathcal{O}(-H_0)$ equals $\mathcal{O}(-H_0)|_{U_i} = \frac{z_0}{z_i} \cdot \mathcal{O}_{U_i}$.

On the other hand, $\mathcal{O}(-1)$ is glued from the local pieces $\mathcal{O}(-1)|_{U_i} = \frac{1}{z_i} \cdot \mathcal{O}_{U_i}$; this ring $\frac{1}{z_i} \cdot k[\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}]$ equals the homogeneous localization $k[z_0, \ldots, z_n](-1)_{(z_i)}$ of the degree-shifted homogeneous coordinate ring.

Finally, both sheaves become isomorphic after dividing by z_0. This happens on all charts U_i simultaneously – and it does not depend on i. Hence, these operations glue.

Problem 127. In comparison to Problem 123 we consider the singular elliptic curve $E := V(y^2z - x^3) \subseteq \mathbb{P}^2_{\mathbb{C}}$. Show that Ω_E is not locally free. What about $\mathcal{H}om_{\mathcal{O}_E}(\Omega_E, \mathcal{O}_E)$? Calculate all $\operatorname{Ext}^i_{\mathcal{O}_{E,p}}(\Omega_{E,p}, \mathcal{O}_{E,p})$ for $p \in E$.

Solution: Only the singular affine chart $E \supset V(y^2 - x^3) \subseteq \mathbb{A}^2_{\mathbb{C}}$ matters. Let $R := \mathbb{C}[x,y]/(y^2 - x^3) = \mathbb{C}[t^2, t^3]$ and denote $I := (y^2 - x^3) \subset \mathbb{C}[x,y]$. Then, the short exact sequence associated to $\mathbb{C} \to \mathbb{C}[x,y] \to \mathbb{C}[x,y]/I = R$ yields

$$0 \longrightarrow \left[I/I^2 = R\right] \xrightarrow{(-3x^2, 2y)^T} \left[\Omega_{\mathbb{C}[x,y]} = R^2\right] \longrightarrow \Omega_R \longrightarrow 0$$

where the injectivity on the left hand side is an additional feature which has to be checked by hand. Dualizing yields

$$0 \longrightarrow \operatorname{Hom}_R(\Omega_R, R) \longrightarrow R^2 \xrightarrow{(-3x^2, 2y)} R \longrightarrow \operatorname{Ext}^1_R(\Omega_R, R) \longrightarrow 0.$$

Hence $\operatorname{Hom}_R(\Omega_R, R) = \{(a,b) \in R^2 \mid 3x^2 a = 2yb\}$. We can take a closer look by using $R = \mathbb{C}[t^2, t^3]$. Then, the previous condition for $a = a(t)$, $b = b(t) \in R$ becomes

$$3\, t^4 a = 2\, t^3 b, \text{ i.e., } b = \frac{3}{2} a \cdot t$$

inside $\mathbb{C}[t]$. Thus, b is determined by a, and a has to be an element of $(t^2, t^3)\, \mathbb{C}[t^2, t^3] = (x, y) \subsetneq R$ (in particular, cannot be a constant). Altogether we obtain

$$\operatorname{Hom}_R(\Omega_R, R) \cong (x, y) \subsetneq R$$

which is not free in the origin, i.e., in the point $(x, y) \in \operatorname{Spec} R$.

Similarily, we obtain $\operatorname{Ext}^1_R(\Omega_R, R) = R/(x^2, y) = \mathbb{C}[x,y]/(x^2, y) = \mathbb{C}[x]/(x^2) \neq 0$. This module is supported at the origin, i.e., Ω_R is not free at this point.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

11. Aufgabenblatt zum 25.1.2023

Problem 128. a) Let $0 \to A \to B \to C \to 0$ be an exact sequence among free R-modules of ranks a, b, and c, respectively. In particular, we have $b = a + c$. For any $n \in \mathbb{N}$ we define a decreasing filtration $F^\bullet(\Lambda^n B)$ as follows:

$$F^k(\Lambda^n B) := \langle a_1 \wedge \ldots \wedge a_k \wedge b_{k+1} \wedge \ldots \wedge b_n \,|\, a_i \in A, b_j \in B\rangle.$$

In particular, $F^0(\Lambda^n B) = \Lambda^n B$ and $F^n(\Lambda^n B) = \Lambda^n A$ and $F^{n+1} = 0$.

Show that there are natural isomorphisms $F^k/F^{k+1} \cong (\Lambda^k A) \otimes_R (\Lambda^{n-k} C)$. That means that they should not depend on the choice of special bases, they should commute with localizations of R and, hence, yield a corresponding result for locally free $\mathcal{O}_X$-modules on some ringed space $(X, \mathcal{O}_X)$.

b) Consider the special case of $n = b$.

Solution: (a) We will define an R-linear map

$$\Phi_k : (\Lambda^k A) \otimes_R (\Lambda^{n-k} C) \to F^k(\Lambda^n B)/F^{k+1}(\Lambda^n B);$$

there seems to be no good way to define a natural inverse. Thus, aiming at Φ_k, we set

$$(a_1 \wedge \ldots \wedge a_k) \otimes (c_{k+1} \wedge \ldots \wedge c_n) \mapsto (a_1 \wedge \ldots \wedge a_k) \wedge (b_{k+1} \wedge \ldots \wedge b_n)$$

where $b_j \in B$ are some preimages of $c_j \in C$. First, this assignment is well-defined: If we replace some b_j by another b'_j representing c_j, then $a_j := b'_j - b_j \in A$, and the RHS is contained in $F^{k+1}(\Lambda^n B)$.

Second, the assignment is multilinear and alternating in both factors. Hence, Φ_k does indeed define an R-linear map as being announced. Moreover, it is obviously surjective. To check injectivity one just chooses compatible bases of A, B, C which, in particular, fixes some splitting of the given exact sequence.

While the last step does leave the canonical setup, we should emphasize that the isomorphism Φ_k had been defined in a natural way. Thus, it is compatible with localizations and glues to the setup of locally free sheaves.

(b) If $n = b$, then asking for $k \le a$ and $(n-k) \le c$ ensuring that $(\Lambda^k A) \otimes_R (\Lambda^{n-k} C) \neq 0$ implies that $k = a$ and $n - k = c$. In particular, we obtain that

$$F^k(\Lambda^b B)/F^{k+1}(\Lambda^b B) = \begin{cases} (\Lambda^a A) \otimes_R (\Lambda^c C) & \text{if } k = a \\ 0 & \text{otherwise.} \end{cases}$$

Hence, since the filtration consists of a single jump only, $\Lambda^b B = (\Lambda^a A) \otimes_R (\Lambda^c C)$.

Problem 129. a) Let $k = \mathbb{F}_3(u)$. We define C as the affine curve $C := V(y^2 - x^3 - u) \subseteq \mathbb{A}^2_k$. Show that the prime ideal $P := (\overline{y})$ with $\overline{y} \in k[x, y]/(y^2 - x^3 - u)$ being the class of y, is a closed point $P \in C$. Moreover, check that the local ring $\mathcal{O}_{C,P}$ is regular, but that Ω_C is not even free at this point.

b) What happens, if we consider the same C over $\overline{k}$ instead?

Solution: (a) Denote $A := k[x,y]/(y^2 - x^3 - u)$. Then $A/P = k[x]/(x^3+u) = k(\sqrt[3]{u})$ is a field. Since P is principal, the one-dimensional local ring A_P is regular. On the other hand, $\Omega_{C,P} = (A\,dx \oplus A\,dy/2y\,dy)_P = A_P\,dx \oplus A_P/(y)\,dy$, i.e. this module does even contain torsion elements.

(b) In $\overline{k}$ there is an element v with $v^3 + u = 0$. The ideal (y) is not prime anymore, but it can be replaced by $P := (y, x - v)$. But then, this new P is not principal anymore (not even in A_P).

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

12. Aufgabenblatt zum 1.2.2023

Problem 130. a) Let R be a noetherian ring and M a finitely generated R-module. Show that for $L := M^\vee := \mathrm{Hom}_R(M,R)$ the natural map $L \to L^{\vee\vee}$ is an isomorphism. (Those R-modules L are called *reflexive*.)

b) Let $R := k[x,y]$ and $I := (x,y)$. Show that the map $R = \mathrm{Hom}_R(R,R) \to \mathrm{Hom}_R(I,R)$ is an isomorphism.
Thus, the embedding $I \hookrightarrow R$ becomes an isomorphism after dualization, and even more after double dualization. In particular, I cannot be reflexive.

Solution: (a) Dualizing the natural map $M \to M^{\vee\vee}$ yields $L^{\vee\vee} = M^{\vee\vee\vee} \to M^\vee = L$. Moreover, from the general compatibility properties of adjoint functors, it follows that the latter is the inverse of $L \to L^{\vee\vee}$.

Indeed, dualization $D : M \mapsto M^\vee$ is self-adjoint, i.e., $D \dashv D^{\mathrm{opp}}$ and, in general, if $F \dashv G$ are adjoint (covariant) functors, then we have the natural adjunction maps $\mathrm{id} \to GF$ and $FG \to \mathrm{id}$ satisfying that $F \to F(GF) = (FG)F \to F$ is the identity map (and similarily with $G \to (GF)G = G(FG) \to G$). (This was already discussed in Problem 24 in Algebra I.)

(b) Injectivity is clear; it remains to show that each R-linear $\varphi : (x,y) \to R$ can be extended to some (then uniquely determined) $\widetilde{\varphi} : R \to R$. If φ is given, then R-linearity implies

$$x\,\varphi(y) = \varphi(xy) = y\,\varphi(x) \quad \text{within } R = k[x,y].$$

In particular, $x|\varphi(x)$ and $y|\varphi(y)$ and $\varphi(x)/x = \varphi(y)/y =: r \in R$. Thus, $\widetilde{\varphi}(1) := r$ yields the extension we were looking for.

Problem 131. a) Let $C = V(F_d) \subseteq \mathbb{P}^2_k$ be a smooth, plane curve defined by a homogeneous polynomial of degree $d \geq 1$. Show that $\omega_C \cong \mathcal{O}_C(\ell) := \mathcal{O}_{\mathbb{P}^2}(\ell) \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_C$ for some $\ell \in \mathbb{Z}$. What is ℓ in terms of d?

b) What is its geometric genus $p_g(C) := \dim_k \Gamma(C, \omega_C)$ in terms of d? You may use that $\mathrm{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) = 0$ for all $k \in \mathbb{Z}$, i.e. whenever there is a short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2}(k) \to \mathcal{F} \to \mathcal{G} \to 0$ of sheaves on $\mathbb{P}^2$, then the derived sequence of global sections remains excact.

Solution: (a) The adjunction formula yields $\omega_C = \omega_{\mathbb{P}^2} \otimes \mathcal{N}_{C|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_C(d) = \mathcal{O}_C(d-3)$.

(b) Tensorizing $0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$ with $\mathcal{O}_{\mathbb{P}^2}(d-3)$ yields the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(d-3) \to \mathcal{O}_C(d-3) \to 0$$

Since $\Gamma(\mathbb{P}^2, \mathcal{O}(-3)) = \mathrm{H}^1(\mathbb{P}^2, \mathcal{O}(-3)) = 0$, this yields an isomorphism of k-vector spaces $\Gamma(\mathbb{P}^2, \mathcal{O}(d-3)) \xrightarrow{\sim} \Gamma(C, \mathcal{O}(d-3)) = \Gamma(C, \omega_C)$. Thus, $p_g(C) = \binom{d-1}{2}$.

Aufgabenblätter und Nicht-Skript: http://www.math.fu-berlin.de/altmann

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

13. Aufgabenblatt zum 8.2.2023

Problem 132. a) Let $N = \mathbb{Z}^2$ and denote by Σ the smooth fan in $N_\mathbb{R} = \mathbb{R}^2$ that is generated by the two full-dimensional cones

$$\sigma_1 = \langle(-1, 1),\ (0, 1)\rangle \quad \text{and} \quad \sigma_2 = \langle(0, 1),\ (1, 1)\rangle.$$

Since $X = \mathbb{TV}(\Sigma)$ is smooth, it is automatically Gorenstein. Show that X is even Calabi-Yau ("CY"), i.e., that $K_X = 0$, that is $\omega_X \cong \mathcal{O}_X$.

b) Recall from Problem 107 (or from our seminar) the construction of $\pi : \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) \to \mathbb{P}^1$ for $\ell \in \mathbb{Z}$. For which ℓ do we have $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) = X$ from (a)? For which (other) values of ℓ is $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell))$ a CY-variety?

c) What is the combinatorial characterization (in terms of Σ) of "CY" for general toric varieties $X = \mathbb{TV}(\Sigma)$ in arbitrary dimensions? (You may assume that all maximal cones are full-dimensional.)

d) Show that $X = \mathbb{TV}(\Sigma)$ cannot be CY whenever $|\Sigma| := \cup_{\sigma\in\Sigma}\, \sigma = N_\mathbb{R}$.

Solution: (a) The dual cones of the σ_i $(i = 1, 2)$ are

$$\sigma_1^\vee = \langle[1, 1],\ [-1, 0]\rangle \quad \text{and} \quad \sigma_2^\vee = \langle[1, 0],\ [-1, 1]\rangle.$$

The local generators of $\omega_X \subset K(X)$ are given by the generators of the "semigroup-modules" $\operatorname{int} \sigma_i^\vee \cap M$ $(i = 1, 2)$. In the smooth case, this equals the sum of the fundamental generators of $\sigma_i^\vee$, and this is $[0, 1]$ for both cases. The fact that these two generators coincide means that we have a single global generator, i.e., it encodes the CY property.

(b) From Problem 107(e) we know that the toric variety $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell))$ can be built from the two cones

$$\sigma_0 := \langle(1, 0), (0, 1)\rangle \quad \text{and} \quad \sigma_\infty := \langle(0, 1), (-1, -\ell)\rangle.$$

This fan is isomorphic to Σ from (a) if and only if $\ell = -2$. This case is characterized by the property that the three fundamental generators $(1, 0)$, $(0, 1)$, and $(-1, -\ell)$ are collinear (as it is the case for $(-1, 1)$, $(0, 1)$, and $(1, 1)$).

The sum of the ray generators of $\sigma_0^\vee = \langle[1, 0],\ [0, 1]\rangle$ and $\sigma_\infty^\vee = \langle[-1, 0],\ [-\ell, 1]\rangle$ is $[1, 1]$ and $[-\ell - 1, 1]$, respectively. Both results coincide iff $\ell = -2$.

(c) For a full-dimensional cone $\sigma = \langle\rho^1, \ldots, \rho^n\rangle$, the d-dimensional affine toric variety $\mathbb{TV}(\sigma)$ (thus $n \geq d$) is Gorenstein (CY) if and only if $\operatorname{int} \sigma^\vee \cap M$ is generated (as a $(\sigma^\vee \cap M)$-module) by a single element u_σ. This element is uniquely characterized by the property $\langle u, \rho^i\rangle = 1$ for all $i = 1, \ldots, n$.

Thus, a general toric variety $\mathbb{TV}(\Sigma)$ is CY if and only if there is a (unique) $u \in M$ such that $\langle u, \rho\rangle = 1$ for all $\rho \in \Sigma(1)$.

(d) In particular, the condition from (c) implies that all elements of $\Sigma(1)$, hence the entire fan Σ, is contained in some half space of $N_{\mathbb{R}}$.

Problem 133. Which of the two-dimensional cyclic quotient singularities $X_{n,q} = \frac{1}{n}(1,q) = \mathbb{TV}(\sigma)$ with $\sigma = \langle (1,0),\ (-q,n) \rangle$ is Gorenstein?

Solution: Just the A_n-singularities, i.e. those with $q = -1$, i.e. the matrix describing the $(\mathbb{Z}/n\mathbb{Z})$-action has $\det = 1$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

14. Aufgabenblatt zum 15.2.2023

Problem 134. Let Σ be the fan in $\mathbb{Q}^3$ built from the rays

$$\Sigma(1) = \{e^i, a^i, (-1,-1,-1) \mid i = \mathbb{Z}/3\mathbb{Z}\}$$

(with e^i denoting the canonical basis vectors and $a^i := (1,1,1) + e^i$) and being spanned by the three-dimensional cones $\langle(-1,-1,-1), e^i, e^{i+1}\rangle$, $\langle e^i, e^{i+1}, a^{i+1}\rangle$, $\langle e^i, a^i, a^{i+1}\rangle$, and $\langle a^1, a^2, a^3\rangle$ for $i = \mathbb{Z}/3\mathbb{Z}$. Show that Σ is not the normal fan of a polytope, i.e. that $\mathbb{TV}(\Sigma)$ is complete, but not projective.

Solution: If Δ has Σ as its inner normal fan, then we denote by $A_{i+1} \in \Delta$ the vertices corresponding to the cones $\langle e^i, e^{i+1}, a^{i+1}\rangle \in \Sigma$ and $E_i \in \Delta$ the vertices corresponding to the cones $\langle e^i, a^i, a^{i+1}\rangle$ $(i = \mathbb{Z}/3\mathbb{Z})$. This implies the equations

$$\langle E_i, e^i\rangle = \langle A_i, e^i\rangle = \langle A_{i+1}, e^i\rangle \quad \text{and} \quad \langle E_{i-1}, a^i\rangle = \langle E_i, a^i\rangle = \langle A_i, a^i\rangle$$

and the inequalities

$$\langle A_i, e^i\rangle < \langle E_{i-1}, e^i\rangle, \quad \langle A_i, e^{i-1}\rangle < \langle E_i, e^{i-1}\rangle$$

and

$$\langle E_i, a^i\rangle < \langle A_{i+1}, a^i\rangle, \quad \langle E_i, a^{i+1}\rangle < \langle A_i, a^{i+1}\rangle.$$

From these data one should be able to derive a contradiction by some cyclic addition... But, so far, I did not get it.

Problem 135. Let $f : X \dashrightarrow Y$ be a rational map between k-varieties; let X be smooth and Y be complete. In the next class it will be shown that $\operatorname{codim}_X(X \setminus U) \geq 2$ if $U \subseteq X$ is the maximal open subset such that f can be represented as morphism $U \to Y$.

a) Provide counter examples for the cases that X is not smooth or Y is not complete.

b) Let C, C' be smooth, complete curves, i.e. k-varieties of dimension one, that are birationally isomorphic over k, i.e. $K(C)$ and $K(C')$ are isomorphic field extensions of k. Show that $C \cong C'$.

Solution: (a) $\mathbb{P}^1 \stackrel{\text{id}}{\dashrightarrow} \mathbb{A}^1$. The other example arises from $\operatorname{Spec} \mathbb{C}[t^2, t^3] \dashrightarrow \operatorname{Spec} \mathbb{C}[t]$. Outside the origin this is again the identiy, hence both sides can be simultaneously compactified.

(b) Since codimension 2 subsets of curves are empty, the maximal domains of definition of rational maps $C \dashrightarrow C'$ or $C' \dashrightarrow C$ are the whole C and C', respectively.