Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

VL "Algebra III"
FU Berlin, Winter 2022/23

13. Aufgabenblatt zum 8.2.2023

Problem 132. a) Let $N = \mathbb{Z}^2$ and denote by Σ the smooth fan in $N_\mathbb{R} = \mathbb{R}^2$ that is generated by the two full-dimensional cones

$$\sigma_1 = \langle(-1,1),\ (0,1)\rangle \quad \text{and} \quad \sigma_2 = \langle(0,1),\ (1,1)\rangle.$$

Since $X = \mathbb{TV}(\Sigma)$ is smooth, it is automatically Gorenstein. Show that X is even Calabi-Yau ("CY"), i.e., that $K_X = 0$, that is $\omega_X \cong \mathcal{O}_X$.

b) Recall from Problem 107 (or from our seminar) the construction of $\pi : \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) \to \mathbb{P}^1$ for $\ell \in \mathbb{Z}$. For which ℓ do we have $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) = X$ from (a)? For which (other) values of ℓ is $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell))$ a CY-variety?

c) What is the combinatorial characterization (in terms of Σ) of "CY" for general toric varieties $X = \mathbb{TV}(\Sigma)$ in arbitrary dimensions? (You may assume that all maximal cones are full-dimensional.)

d) Show that $X = \mathbb{TV}(\Sigma)$ cannot be CY whenever $|\Sigma| := \cup_{\sigma\in\Sigma}\,\sigma = N_\mathbb{R}$.

Solution: (a) The dual cones of the σ_i $(i = 1, 2)$ are

$$\sigma_1^\vee = \langle[1,1],\ [-1,0]\rangle \quad \text{and} \quad \sigma_2^\vee = \langle[1,0],\ [-1,1]\rangle.$$

The local generators of $\omega_X \subset K(X)$ are given by the generators of the "semigroup-modules" $\operatorname{int}\sigma_i^\vee \cap M$ $(i = 1, 2)$. In the smooth case, this equals the sum of the fundamental generators of $\sigma_i^\vee$, and this is $[0,1]$ for both cases. The fact that these two generators coincide means that we have a single global generator, i.e., it encodes the CY property.

(b) From Problem 107(e) we know that the toric variety $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell))$ can be built from the two cones

$$\sigma_0 := \langle(1,0),(0,1)\rangle \quad \text{and} \quad \sigma_\infty := \langle(0,1),(-1,-\ell)\rangle.$$

This fan is isomorphic to Σ from (a) if and only if $\ell = -2$. This case is characterized by the property that the three fundamental generators $(1,0)$, $(0,1)$, and $(-1,-\ell)$ are collinear (as it is the case for $(-1,1)$, $(0,1)$, and $(1,1)$).

The sum of the ray generators of $\sigma_0^\vee = \langle[1,0],\ [0,1]\rangle$ and $\sigma_\infty^\vee = \langle[-1,0],\ [-\ell,1]\rangle$ is $[1,1]$ and $[-\ell-1,1]$, respectively. Both results coincide iff $\ell = -2$.

(c) For a full-dimensional cone $\sigma = \langle\rho^1,\ldots,\rho^n\rangle$, the d-dimensional affine toric variety $\mathbb{TV}(\sigma)$ (thus $n \geq d$) is Gorenstein (CY) if and only if $\operatorname{int}\sigma^\vee \cap M$ is generated (as a $(\sigma^\vee \cap M)$-module) by a single element u_σ. This element is uniquely characterized by the property $\langle u, \rho^i\rangle = 1$ for all $i = 1,\ldots,n$.

Thus, a general toric variety $\mathbb{TV}(\Sigma)$ is CY if and only if there is a (unique) $u \in M$ such that $\langle u, \rho\rangle = 1$ for all $\rho \in \Sigma(1)$.

(d) In particular, the condition from (c) implies that all elements of $\Sigma(1)$, hence the entire fan Σ, is contained in some half space of $N_{\mathbb{R}}$.

Problem 133. Which of the two-dimensional cyclic quotient singularities $X_{n,q} = \frac{1}{n}(1,q) = \mathbb{TV}(\sigma)$ with $\sigma = \langle (1,0),\ (-q,n)\rangle$ is Gorenstein?

Solution: Just the A_n-singularities, i.e. those with $q = -1$, i.e. the matrix describing the $(\mathbb{Z}/n\mathbb{Z})$-action has $\det = 1$.

Klaus Altmann
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75428
altmann@math.fu-berlin.de

Anna-Lena Winz
Mathematisches Institut
Freie Universität Berlin
Tel.: (030) 838 - 75426
anelanna@math.fu-berlin.de

VL "Algebra III"
FU Berlin, Winter 2022/23

14. Aufgabenblatt zum 15.2.2023

Problem 134. Let Σ be the fan in $\mathbb{Q}^3$ built from the rays

$$\Sigma(1) = \{e^i, a^i, (-1,-1,-1) \mid i = \mathbb{Z}/3\mathbb{Z}\}$$

(with e^i denoting the canonical basis vectors and $a^i := (1,1,1) + e^i$) and being spanned by the three-dimensional cones $\langle (-1,-1,-1), e^i, e^{i+1}\rangle$, $\langle e^i, e^{i+1}, a^{i+1}\rangle$, $\langle e^i, a^i, a^{i+1}\rangle$, and $\langle a^1, a^2, a^3\rangle$ for $i = \mathbb{Z}/3\mathbb{Z}$. Show that Σ is not the normal fan of a polytope, i.e. that $\mathbb{TV}(\Sigma)$ is complete, but not projective.

Solution: If Δ has Σ as its inner normal fan, then we denote by $A_{i+1} \in \Delta$ the vertices corresponding to the cones $\langle e^i, e^{i+1}, a^{i+1}\rangle \in \Sigma$ and $E_i \in \Delta$ the vertices corresponding to the cones $\langle e^i, a^i, a^{i+1}\rangle$ $(i = \mathbb{Z}/3\mathbb{Z})$. This implies the equations

$$\langle E_i, e^i\rangle = \langle A_i, e^i\rangle = \langle A_{i+1}, e^i\rangle \quad \text{and} \quad \langle E_{i-1}, a^i\rangle = \langle E_i, a^i\rangle = \langle A_i, a^i\rangle$$

and the inequalities

$$\langle A_i, e^i\rangle < \langle E_{i-1}, e^i\rangle, \quad \langle A_i, e^{i-1}\rangle < \langle E_i, e^{i-1}\rangle$$

and

$$\langle E_i, a^i\rangle < \langle A_{i+1}, a^i\rangle, \quad \langle E_i, a^{i+1}\rangle < \langle A_i, a^{i+1}\rangle.$$

From these data one should be able to derive a contradiction by some cyclic addition... But, so far, I did not get it.

Problem 135. Let $f : X \dashrightarrow Y$ be a rational map between k-varieties; let X be smooth and Y be complete. In the next class it will be shown that $\operatorname{codim}_X(X \setminus U) \geq 2$ if $U \subseteq X$ is the maximal open subset such that f can be represented as morphism $U \to Y$.

a) Provide counter examples for the cases that X is not smooth or Y is not complete.

b) Let C, C' be smooth, complete curves, i.e. k-varieties of dimension one, that are birationally isomorphic over k, i.e. $K(C)$ and $K(C')$ are isomorphic field extensions of k. Show that $C \cong C'$.

Solution: (a) $\mathbb{P}^1 \overset{\text{id}}{\dashrightarrow} \mathbb{A}^1$. The other example arises from $\operatorname{Spec}\mathbb{C}[t^2,t^3] \dashrightarrow \operatorname{Spec}\mathbb{C}[t]$. Outside the origin this is again the identiy, hence both sides can be simultaneously compactified.

(b) Since codimension 2 subsets of curves are empty, the maximal domains of definition of rational maps $C \dashrightarrow C'$ or $C' \dashrightarrow C$ are the whole C and C', respectively.